DISCUSSION SECTION 5: INDUCTIVE FORMULATION

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Abstract. Below you'll find a collection of questions. You should think of these questions less as a homework assignment and more as a playground to run around in and spark your imagination. It's more fun playing on the swings with friends, so come to discussion section to talk with others about anything you find confusing or exciting!

1. Inductive Formulation Set-Up

Throughout, let X be a topological space. Let $n \geq 2$ and $\zeta \to X$ a rank n vector bundle with an inner product. Consider the principal $O(n)$-bundle $\tilde{X}$ of orthonormal frames of ζ.

Question 1. Show that $X_0 = \tilde{X}/O(n-1)$ can be given the structure of a fiber bundle over X with fiber S^{n-1}.

- How can you describe this bundle in terms of ζ?
- Let $p\colon X_0 \to X$ denote this S^{n-1}-bundle. Show that there is a universal rank $n-1$ vector bundle $\zeta_0 \to X_0$ with a map $p\colon X_0 \to X$ so that

$$p^*\zeta = \zeta_0 \oplus \underline{\mathbb{R}}$$

- Describe ζ_0 in terms of X_0 and ζ.
- Show that there is a functor

$$\mathbf{Bord}_{n-1}^{(X_0,\zeta_0)} \to \mathbf{Bord}_n^{(X,\zeta)}$$

  What is this a functor of? (∞, n)-categories? $(\infty, n-1)$-categories? How are you regarding both sides as that type of category?

You can read the top of page 53 of [1] for the answers to the above.

Question 2. Take $X = BG$ and ζ the rank n vector bundle $\zeta_\xi = (\mathbb{R}^n \times EG)/G$ on BG associated to a continuous homomorphism $\xi\colon G \to O(n)$. Describe X_0 (as defined in the previous problem) as a classifying space of some group.

See Remark 3.1.9 of [1] for the answer. More details about this pair (BG, ζ_ξ) can be found in Notation 2.4.21 of [1].

Question 3. Can you obtain $\mathbf{Bord}_{n-1}^{(X_0,\zeta_0)}$ from $\mathbf{Bord}_n^{(X,\zeta)}$ by removing the noninvertible n-morphisms? Why or why not?

See Remark 3.1.1 of [1] for the answer. The definition of a (X,ζ)-structure is Notation 2.4.16 of [1].

Question 4. Describe $\Omega^k\mathbf{Bord}_n^{(X_0,\zeta_0)}$ for $k < n, k = n, k > n$.

Question 5. Recall that the $(n-1)$-morphisms of $\mathbf{Bord}_n^{(X,\zeta)}$ are given intuitively by $(n-1)$-manifolds with corners.

- Show that when X is a point, there is a unique $(n-1)$-morphism in $\mathbf{Bord}_n^{(X,\zeta)}$ whose underlying manifold is S^{n-1}.

- For X this disjoint union of two points, how many such $(n-1)$-morphisms with underlying manifold S^{n-1} are there?
- What about $X = S^{n-1}$ and $\zeta = TS^{n-1} \oplus \underline{\mathbb{R}}$?
- For a general pair (X, ζ), give a description of the set of $(n-1)$-morphisms whose underlying manifold is S^{n-1}.
- Suppose $\mathcal{C}$ is a symmetric monoidal n-category with duals and suppose you have an $n-1$ dimensional TFT $Z_0 : \mathbf{Bord}_{n-1}^{(X,\zeta)} \to \mathcal{C}$. Consider the functor $\Phi : X \to \Omega^{n-1}\mathcal{C}$ sending $x \mapsto Z_0(S^{\zeta_x})$ where S^{ζ_x} is the fibre of the sphere bundle $\mathsf{Sph}(\zeta) \to X$. Show that if we have an n-dimensional TFT $Z : \mathbf{Bord}_n^{(X,\zeta)} \to \mathcal{C}$ extending Z_0, then we have a family of 1-morphisms $\eta : 1 \to \Phi(x) \in \Omega^{n-1}\mathcal{C}$ each exhibiting one half of the sphere as a right dual to the other.

Question 6. Read the statement of the inductive formulation, Theorem 3.1.8 of [1].

Question 7. Say $(X, \zeta) = (Bg, \zeta_\xi)$ as in Question 2. Reformulate the inductive formulation ([1, Thm. 3.1.8]) in terms of G actions and G equivariant maps.

See Remark 3.1.9 of [1] for an answer.

2. Cobordism Hypothesis from Inductive Formulation

Question 8. Assuming Theorems 2.4.6 and 2.4.18 in dimension $(n-1)$, what data is a symmetric monoidal functor

$$Z_0 \colon \mathbf{Bord}_{n-1}^{(X_0,\zeta_0)} \to \mathcal{C}$$

equivalent to?

See (a_1) on page 55 of [1] for an answer.

Question 9. Look over the Proof of Theorem 2.4.6 on page 55 of [1]. Give an outline of the proof of Theorem 2.4.6 (framed cobordism hypothesis) assuming Theorem 3.1.8 (inductive formulation).

References

[1] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.