# DISCUSSION SECTION 3

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Abstract. Below you'll find a collection of questions. You should think of these questions less as a homework assignment and more as a playground to run around in and spark your imagination. It's more fun playing on the swings with friends, so come to discussion section to talk with others about anything you find confusing or exciting!

Contents



## 1. Dualizable Examples

The following is the first problem of Scheimbauer's MSRI Problem Session.

**Question 1.** Find the dualizable objects in the following monoidal categories:

- vector spaces and direct sum
- vector spaces and tensor product
- pointed vector spaces (a vector space together with a chosen vector in it), point-preserving linear maps, and tensor product
- sets and cartesian product
- Span, where objects are sets, a morphism from $X$ to $Y$ is an isomorphism class of spans $X \leftarrow S \rightarrow Y$, composition is pullback, and the monoidal product is the cartesian product
- $\mathsf{Alg}$, where objects are $\mathbb{C}$-algebras, a morphism from an algebra $A$ to an algebra $B$ is an iso-morphism class of bimodules, composition is relative tensor product, and tensor product over $\mathbb{C}$ as the monoidal structure.

**Question 2.** Consider the $(\infty, 2)$-category of $\mathbb{C}$-linear $\infty$-categories with monoidal structure $\otimes_{\mathbb{C}}$. Show that, for a smooth and proper complex variety, the dg-category $D^bCoh(Y)$ is fully dualizable.

Check out Example III.8.3.5 of [1], or Example 2.4.14 of [3], or Remark 4.2.4 of [3] to see a relationship between this and the Serre automorphism.

## 2. Cobordism Dualizability

The following is Example 2.3.23 of [3].

**Question 3.** Show that $\mathbf{Bord}_n$ has duals. What is the adjoint of a $k$-morphism represented by an oriented $k$-manifold $B$?

For $n = 1$, how does this relate to Zorro's lemma?

**Question 4.** Check in detail that the oriented point is fully dualizable in $\mathbf{Bord}_2$.

For a proof of the above, see Example III.6.0.3 of [1]. For a comment on the below, see Remark III.8.2.7 of [1].

**Question 5.** If we worked in with framed cobordisms, instead of oriented, what would change in your answer to Question 4? What about if we worked with $G$-strucutre?

**Question 6.** Check out [2]. They discuss the dualizability of a Morita category of $\mathcal{E}_n$-algebras. What is the main theorem and how does it relate to the cobordism hypothesis?

## 3. Adjoints

The following comes from Remark 2.3.14 of [3].

**Question 7.** Let $\mathcal{C}$ be an $(\infty, n)$-category. Show that if every $k$-morphism in $\mathcal{C}$ is invertible, then $\mathcal{C}$ admits adjoints for $k$-morphisms. What about the converse?

The following two questions come from problem 8 on the MSRI notes.

**Question 8.** Show that an $n$-dualizable object in an $(\infty, k)$-categroy, for $k < n$, is invertible. What does this imply for fully extended TFTs?

**Question 9.** Show that the image of an $n$-dualizable object under a symmetric monoidal functor is $n$-dualizable. What does this imply for fully extended TFTs?

**Question 10.** Can you rephrase Definition 2.3.13 of [3] in terms of the homotopy $n$-category $h_n\mathcal{C}$ of an $(\infty, n)$-category $\mathcal{C}$?

## 4. Picard $\infty$-Groupoids

**Question 11.** Recall from Example 2.3.18 of [3] the definition of a Picard $\infty$-groupoid. What is the relationship between Picard $\infty$-groupoids and $\mathcal{E}_\infty$-spaces? Can you rephrase this in terms of certain types of spectra?

**Question 12.** Let $\mathcal{C}$ be a Picard $\infty$-groupoid. Regard $\mathcal{C}$ as a $(\infty, n)$-category. For what $n$ does $\mathcal{C}$ have duals?

The above and below questions come from Example 2.3.18 of [3].

**Question 13.** Let $\mathcal{C}$ be a symmetric monoidal $(\infty, n)$-category. Regard $\mathcal{C}$ as a $(\infty, n+1)$-category. When does $\mathcal{C}$ have duals?

## 5. Classifying Categories

The following comes from Remark 2.3.17 of [3].

**Question 14.** Let $\mathcal{C}$ be a monoidal $(\infty, n)$-category. Define an $(\infty, n+1)$-category $B\mathcal{C}$ that generalizes the notion defined in [3, Example 2.3.7] for $n = 1$.

**Question 15.** What is the relationship between duals in $\mathcal{C}$ and adjoints in $B\mathcal{C}$?

**Question 16.** Check that $\Omega B\mathcal{C} \cong \mathcal{C}$.

## References


[1] Araminta Amabel, Artem Kalmykov, Lukas Müller, and Hiro Lee Tanaka. Lectures on Factorization Homology, Infinity-Categories, and Topological Field Theories. *arXiv e-prints*, page arXiv:1907.00066, June 2019.

[2] Owen Gwilliam and Claudia Scheimbauer. Duals and adjoints in higher Morita categories. *arXiv e-prints*, page arXiv:1804.10924, April 2018.

[3] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.