DISCUSSION SECTION 2

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Abstract. Below you'll find a collection of questions. You should think of these questions less as a homework assignment and more as a playground to run around in and spark your imagination. It's more fun playing on the swings with friends, so come to discussion section to talk with others about anything you find confusing or exciting!

Contents

1. TFT in General

Question 1 (Kiran)**.** If you're given an n dimensional invertible TFT with target C by the data of the infinite loop map between the appropriate classifying spaces, is there a practical way to calculate what $Z(M^d)$? Is $d = n$ easier? Or maybe on the other end, $d = 0$? Is there an instructive example in low dimensions?

2. Segal Spaces

The following is Exercise 2.1.7 in [4].

Question 2. Let $X_\bullet$ be a simplicial set. Show that X is isomorphic to the nerve of a category $\mathcal{C}$ if and only if, for every pair of integers $m, n \geq 0$, the diagram

$$\begin{array}{ccc} X_{m+n} & \xrightarrow{p^*_{0,1,\ldots,m}} & X_m \\ {\scriptstyle p^*_{m,m+1,\ldots,m+n}}\downarrow & & \downarrow{\scriptstyle p^*_m} \\ X_n & \xrightarrow[p^*_0]{} & X_0 \end{array}$$

is a pullback square; in other words, if and only if the canonical map $X_{m+n} \to X_m \times_{X_0} X_n$ is bijective

Question 3. If X is a group like A_∞-space, show that the cyclic bar construction on X produces a Segal space. Is it complete? What is the corresponding $(\infty, 1)$-category?

Question 4 (Kiran)**.** When is the cyclic bar construction on a A_∞-space X a Segal space?

Question 5. Is the singular complex $\mathsf{Sing}_\bullet X$ of a topological space a Segal space? Is it complete?

Question 6. Find an example of a commuting square of spaces that is a pullback square but not a homotopy pullback square. Find an example that is a homotopy pullback but not an ordinary pullback.

The following is Example 2.1.21 of [4].

Question 7. Let $X_\bullet$ be a Segal space, and let $\delta\colon X_0 \to X_1$ be the degeneracy map induced by the functor $\{0,1\} \to \{0\}$. Show that for every $x \in X_0$, the morphism $[\delta(x)]$ in the homotopy category $hX_\bullet$ is the identity $\mathrm{id}\colon x \to x$.

Formulate the definition of a complete Segal space using this map δ.

Question 8. We defined $(\infty,1)$-categories to be complete Segal spaces. What other models of $(\infty,1)$-categories have you heard about? How do they compare to complete Segal spaces?

Question 9. Earlier, we noted that with our informal definition of $\mathbf{Cob}(n)$, composition was not well-defined. (There's issues of giving the glued bordism a smooth structure, see [4, Rmk 1.1.2].) How does the formalism of Segal spaces help with this problem?

See (a) on page 35 of [4] for the answer.

We wanted to define a fully-extended TFT to be a symmetric monoidal functor from an (∞,n)-category $\mathbf{Bord}_n$ to some other (∞,n)-category $\mathcal{C}$. To do this, we need a notion of symmetric monoidal (∞,n)-categories.

Question 10. How would you define a symmetric monoidal (∞,n)-category using the n-fold Segal space model?

See Part 1 of [2] for an answer. There's also a MathOverflow post about this here.

3. Cobordism Category as a Segal Space I

Reading through pages 34-38 of [4] will be helpful for understanding the following questions.

Question 11. Let M be a closed manifold. Show that as $n \to \infty$, the space

$$\mathsf{Emb}(M,\mathbb{R}^n)/\mathsf{Diff}(M)$$

becomes a good approximation for $B\mathsf{Diff}(M)$.

Question 12. Define a topological groupoid of composable chains of bordisms.

Question 13. How would you construct a classifying space for the topological groupoid of composable chains of bordisms you just defined?

Hint: Embed things in $\mathbb{R}^{\mathrm{big}}$ like in Question 11.

Question 14. Read the (1 page) introduction of [2]. Why is this a 75 page paper? What is so hard about defining a (∞,n)-category of cobordisms precisely?

Question 15. Compare Definition 2.2.6 of [4] and Definition 6.6 of [2]. Try it out in low-number cases.

- Take $V=\mathbb{R}$, $n=1$, and $k_1=0$. What is $(\mathrm{PBord}_1^{\mathbb{R},L})_0$?
- Take $V=\mathbb{R}$, $n=n$, and $k_1,\dots,k_n=0$. What is $(\mathrm{PBord}_n^{\mathbb{R},L})_{0,\dots,0}$?
- Take $V=\mathbb{R}$, $n=1$, and $k_1=k$. What is $(\mathrm{PBord}_1^{\mathbb{R},L})_k$?

Question 16. What is the topology on this "pre-bordism category"? Do you remember what the Whitney topology is?

Question 17. Lurie cites Galatius' paper [3] for putting a topology on $(\mathbf{PreBord}_n^V)_{k_1,\ldots,k_n}$. How does Galatius' notion of a "graph cobordism category" help here?

It turns out that Lurie's n-fold simplicial space PBord_n^L (see [2, Defn. 6.6]) is not an n-fold Segal space.

Question 18. Read Example 6.8 of [2] to see why PBord_n^L is not an n-fold Segal space.

Next we'll read the fixed definition.

Question 19. Compare Definition 6.6 and Definition 6.1 of [2]. Why is one a Segal space and one isn't?

Question 20. Why is the natural definition of the cobordism category as a Segal space not complete? How is this related to the s-cobordism theorem?

See Warning 2.2.8 of [4] for a discussion of this issue.

4. Cobordism Category as a Segal Space II

Question 21. Why are there no degeneracy maps in the "cartoon" Segal space Cob_d that Adrian defined?

Question 22. Look up the Baez-Dolan tangle hypothesis. In what sense is it an "unstable analogue" of the cobordism hypothesis?

See Remark 2.2.12 of [4] for an answer.

Question 23. Scroll through Ayala and Francis' *Flagged Infinity Categories*, [1], for what non-complete Segal spaces model. (You might want to read the review on mathscinet for a quick overview.)

References

[1] David Ayala and John Francis. Flagged higher categories. In *Topology and quantum theory in interaction*, volume 718 of *Contemp. Math.*, pages 137–173. Amer. Math. Soc., Providence, RI, 2018.

[2] Damien Calaque and Claudia Scheimbauer. A note on the (∞, n)-category of cobordisms. *Algebr. Geom. Topol.*, 19(2):533–655, 2019.

[3] Sø ren Galatius. Stable homology of automorphism groups of free groups. *Ann. of Math. (2)*, 173(2):705–768, 2011.

[4] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.