DISCUSSION SECTION 10: FACTORIZATION HOMOLOGY

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Abstract. Below you'll find a collection of questions. You should think of these questions less as a homework assignment and more as a playground to run around in and spark your imagination. It's more fun playing on the swings with friends, so come to discussion section to talk with others about anything you find confusing or exciting!

Contents

1. $\mathcal{E}_n$-Algebras

Question 1. Recall the definition of an operad via symmetric sequences.

Question 2. Define the $\mathcal{E}_n$-operad in terms of configuration spaces of $\mathbb{R}^n$.

Question 3. Recall the definition of an algebra over an operad.

- What is the data of an $\mathcal{E}_0$-algebra?
- What is the data of an $\mathcal{E}_1$-algebra?
- What is the data of an $\mathcal{E}_\infty$-algebra?

Question 4. What is the difference between an $\mathcal{E}_n$-algebra in $(\mathrm{Ch}, \oplus)$ and an $\mathcal{E}_n$-algebra in $(\mathrm{Ch}, \otimes)$?

The following is known as May's Recognition Principle, [4].

Question 5. Let Y be a connected, grouplike topological space. Say Y is given the structure of an $\mathcal{E}_n$-algebra. Then there exists a pointed topological space X so that $Y \simeq \Omega^n X$. Check this in some easy cases, like $n = 1$ or when Y is a point.

Recall the symmetric monoidal category of framed n-disks Disk_n whose objects are

$$\coprod_k \mathbb{R}^n$$

and morphisms are framed embeddings.

Question 6. Let $\mathcal{C}$ be a symmetric monoidal category. Try to show that a Disk_n-algebra

$$A\colon \mathsf{Disk_n}^{\sqcup} \to \mathcal{C}^{\otimes}$$

is the same data as an $\mathcal{E}_n$-algebra.

2. Factorization Homology Basics

Let M be a framed n-manifold. Let A be a framed n-disk algebra. Recall the definition of factorization homology of M with coefficients in A, denoted $\int_M A$.

Question 7. Give descriptions of the following

- $\int_{\mathbb{R}^n} A$
- $\int_{\mathbb{R}^n \sqcup \mathbb{R}^n} A$
- Let W be a $(n-k)$-manifold. What is $\int_{\mathbb{R}^k \times W} A$ computing?

Ayala and Francis have proven the following theorem.

Theorem 2.1 (Excision)**.** *Let M be a smooth n-manifold that can be written as the union*

$$M = N_1 \cup_{\mathbb{R}\times W} N_2$$

where N_1, N_2 are also n-manifolds and W is an $(n-1)$-manifold. Then there is an equivalence

$$\int_M A \simeq \int_{N_1} A \bigotimes^{\mathbb{L}}_{\int_{\mathbb{R}\times W} A} \int_{N_2} A$$

You can find the precise formulation of this theorem here: [1, Lem. 3.18].

Question 8. Make sense of the tensor product in the excision theorem. How are $\int_{N_1} A$ and $\int_{N_2} A$ modules over $\int_{\mathbb{R}\times W} A$?

Recall that given an associative algebra A (i.e., an $\mathcal{E}_1$-algebra, i.e., a framed 1-disk algebra), the *Hochschild homology* of A is

$$HH_\bullet(A) \colon = A \bigotimes^{\mathbb{L}}_{A\otimes A^{\mathrm{op}}} A$$

Question 9. Show that

$$\int_{S^1} A \simeq HH_\bullet(A)$$

using excision.

This is [3, Ex. 4.1.22] and [1, Thm. 3.19].

3. Cobordism Hypothesis via Factorization Homology

This first question is just reviewing stuff from the talk.

Question 10. Given a framed n-disk algebra A, give a rough construction an n-dimensional TFT. In particular, you should have to check that something is functorial for cobordisms. Try to do that.

Question 11. Can you formulate the cobordism hypothesis in terms of factorization homology using your answer to Question 10?

See [3, Thm. 4.1.24] for an answer. See [2] for a fuller answer.

Question 12. Let $G \to O(n)$ be a morphism of topological groups. Then we have a notion of G-framed TFTs coming from the bordism category $\mathbf{Bord}_n^G$. Define a version of $\mathcal{E}_n$-algebras and a version of Disk_n-algebras that incorporate this G action. Using Question 10, your answers should recover the notion of a G-framed TFT.

See [3, Rmk. 4.1.26] for related stuff.

References

[1] David Ayala and John Francis. Factorization homology of topological manifolds. *J. Topol.*, 8(4):1045–1084, 2015.

[2] David Ayala and John Francis. The cobordism hypothesis. *arXiv e-prints*, page arXiv:1705.02240, May 2017.

[3] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.

[4] J. P. May. *The geometry of iterated loop spaces*. Springer-Verlag, Berlin-New York, 1972. Lectures Notes in Mathematics, Vol. 271.