Recall: Given a monoid M, there's a universal way of producing a gp out of M "formally adding inverses"

e.g. M comm:

$$G(M) = M \times M/\sim \qquad (m_1, m_2) \sim (m_1+n, m_2+n)$$

Goal: Generalise ~~[illegible] similar construction~~ to topological monoids.

$$\begin{tikzcd} \text{TopMonoid} \arrow[r, "\pi_0"] \arrow[d, dashed, "\Omega B(\cdot)"] & \text{Monoid} \arrow[d, "G"] \\ \text{H-Spaces} \arrow[r, "\pi_0"] & \text{Gp} \end{tikzcd}$$

"group completion"

Q: what does $\Omega B(\cdot)$ do on level of homology?

A: It just inverts $\pi_0(M) \subseteq H_*(M)$ $\qquad \pi = \pi_0(M)$

multiplicative subset $\qquad$ ring

(Group Completion)

Thm: Let M be top. monoid. If $\overbrace{\pi \text{ in centre of } H_*(M)}^{(*)}$, then $H_*(M)[\pi^{-1}] \xrightarrow{\cong} H_*(\Omega BM)$

Rmks: • If M htpy commutative then $(*)$ holds

Cor: M grouplike ($\pi_0(M)$ a group) $\Rightarrow$ $M \to \Omega BM$ w.e.

Applications: • Barratt–Priddy–Quillen thm (next week)

• Tillmann: $\mathbb{Z} \times B\Gamma_\infty^+$ is infinite loop space (need generalisation of GCT)

Proof: Strategy: ① Reduce to case $\pi_0(M) \cong \mathbb{N}$

② Construct a space M_∞ st $H_*(M_\infty) \cong H_*(M)[\pi^{-1}]$

③ Show H_*-equiv $M_\infty \to \Omega BM$.

Sketch: ① can assume $\pi = \pi_0(M)$ f.g., say by $\{s_1, \dots, s_k\}$. Let $s = s_1 s_2 \dots s_k$.

Then $H_*(M)[\pi^{-1}] = H_*(M)[s^{-1}]$ $\qquad$ so just need to invert single element in π. This is essentially the case $\pi \cong \mathbb{N}$.

② Choose $m_1 \in 1 \in \pi_0(M) \cong \mathbb{N}$.

Define $M_\infty := \operatorname{hocolim}(M \xrightarrow{\cdot m_1} M \xrightarrow{\cdot m_1} M \xrightarrow{\cdot m_1} \dots)$ i.e. mapping telescope.

then $H_*(M_\infty) = \lim(H_*(M) \xrightarrow{\cdot m_1} H_*(M) \xrightarrow{\cdot m_1} \dots) = H_*(M)[m_1^{-1}]$.

Note M acts on M_∞ by homology equivalences.

③ If $M \curvearrowright X$, $\rightsquigarrow$ Borel construction $X \times_M EM \xrightarrow{\pi} BM$

$$\boxed{\text{Prop}^n: \text{If } M \curvearrowright X \text{ by } H_*\text{-equivs then } X \times_M EM \xrightarrow{\pi} BM \text{ is } \underline{\text{homology fibration}} \text{ w/ fibre } X}$$

Def^n: $p: E \to B$ homology fibration if $\forall b \in B$, $p^{-1}(b) \to \mathrm{Fib}(p,b)$ H_*-equiv.

[Cf Prop^n: • M a group $\Rightarrow$ action by homeos $\Rightarrow$ π a fibration
• M acts by weak equivs $\Rightarrow$ $X \times_M EM \xrightarrow{\pi} BM$ quasifibration]

Now apply ~~this~~ Prop^n to $X = M_\infty$.

$EM = |B_\bullet(M, M, *)|$

$M \times_M BM = |B_\bullet(M, M, M)|$

Have $\underbrace{M_\infty \times_M EM}\xrightarrow{\pi} BM$

$= \mathrm{hocolim}(M \xrightarrow{\cdot m} M \xrightarrow{\cdot m} \dots) \times_M EM$

$\simeq \mathrm{hocolim}(M \times_M EM \xrightarrow{\cdot m} M \times_M EM \xrightarrow{\cdot m} \dots)$

$\simeq \mathrm{hocolim}(EM \xrightarrow{\cdot m} EM \xrightarrow{\cdot} EM) \simeq *$

$\therefore$ ~~$\pi: X \times_M E$~~ $\pi: M_\infty \times_M EM \to BM$ has htpical fibre $\simeq \Omega BM$
fibre $= M_\infty$

$\Longrightarrow$ $M_\infty \to \Omega BM$ H_*-equiv. $\square$

Generalisation to categories.

Top. monoid = top. category w/ one object $\rightsquigarrow$ topological category $\mathcal{M}$

$M \curvearrowright X$ a space $\rightsquigarrow F: \mathcal{M}^{op} \to$ Spaces

M acts by H_*-equiv $\rightsquigarrow$ $f: i \to j$ in $\mathcal{C}$ induces $H_*(F(j)) \xrightarrow{\cong} H_*(F(i))$

$X \times_M EM \xrightarrow{\pi} BM$ $\rightsquigarrow$ $\pi_{\mathcal{M}}: E_{\mathcal{M}} F \to B\mathcal{M}$ map of spaces

Prop^n': $\mathcal{M}$ topological cat, $F: \mathcal{M}^{op} \to$ Spaces sends morphisms to H_*-equivs.
Then $\forall i \in$ ~~ob~~ $ob(\mathcal{M})$, the map $F(i) \to \mathrm{Fib}(\pi_{\mathcal{M}}, i)$ is H_*-equiv.

Cor: Can form ~~a~~ a space $F_\infty(x)$ (analogous to M_∞) for $x \in X$.
Get H_*-equiv $F_\infty(x) \to$ ~~$\Omega B\mathcal{M}$~~ $\Omega B\mathcal{M}$.

Applications

Rmk: Obstruction to ~~ap~~ promoting H_*-equiv to htpy equiv is a perfect subgroup of π_1. This can be dealt with by Quillen's plus construction.

1. Have topological monoid $M = \coprod_{n \geq 0} B\Sigma_n$, mult^n from $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$

   $\rightsquigarrow M_\infty = \mathbb{Z} \times B\Sigma_\infty$ H_*-equiv to ΩBM.

   refinement is Barratt-Priddy-Quillen: $\mathbb{Z} \times B\Sigma_\infty^+ \simeq \Omega^\infty S^\infty$.

2. there's a surface category $\mathcal{S}$, and a stable mapping class group Γ_∞.

   GCT $\Rightarrow$ H_*-equiv ~~$\mathbb{Z} \times B\Gamma_\infty \to \Omega B$~~

   \+ homological stability for MCG $\mathbb{Z} \times B\Gamma_\infty \to \Omega B\mathcal{S}$

   $\Rightarrow$ ~~htpy equiv~~ $\mathbb{Z} \times B\Gamma_\infty^+ \xrightarrow{\simeq} \Omega B\mathcal{S}$ exhibits $B\Gamma_\infty^+$ as an infinite loop space.

   Madsen-Weiss ~~identify~~ identify ~~those~~ which infinite loop space.