# pre-Talbot seminar

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# 1   An intro to moduli spaces of manifolds

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**Overview:**   In this talk, Sander discussed the goal of the program and how it differs from classical programs.

All manifolds under consideration are smooth and compact.

## 1.1   Classification of manifolds:

dim 0, 1: All 0, 1 manifolds are, respectively, disjoint unions of points and circles – we study them by restricting to path-connected ones.

dim 2: use geometry to cut surfaces into simpler pieces by cutting along embedded curves with trivial normal bundle

$$[\text{picture of surface of genus 3} = \text{genus 2} \setminus D^2 + \text{genus 1} \setminus D^2]$$

∴ every smooth compact surface is a conected sum (= opposite surgery/operation to cutting along a curve) of the torus $=T^2$ and $\mathbb{R}P^2$

dim 3: similar idea to the 2-dimensional case – cut along spheres with trivial normal bundle to produce irreducible manifolds, then cut along tori, Thurston's program

dim $\geq$ 4: **Problem:** every finitely-presented group is $\pi_1$ of a smooth, compact 4-manifold - can't classify these (there's no algorithm to produce a list of these)

Instead, fix $\pi_1$. In fact, we go further by fixing the homotopy type. For $X$ to have the homotopy type of a manifold, it needs to have $H^*$ with Poincaré duality, stable spherical fibration, etc.

**Existence:**   Pick an arbitrary map $M \to X$. Try to improve it by modifying $M$ (to make it closer and closer to a homotopy equivalence) using surgeries.
[picture of $M \subset M \times I$, with $D^{i+1} \times D^n$ glued along $\partial(D^{i+1} \times D^n) \hookrightarrow M \times \{1\}$
new boundary: $(M \setminus int(\partial D^{i+1} \times D^n)) \cup (D^{i+1} \times \partial D^n)$ "result of surgery"]

*Question.* Can you make $M \to X$ a homotopy equivalence by surgeries?

There are two problems

- Algebraic problem: can arise when you reach middle dimension ("trading things off each other"): there is an obstruction in symmetric $L$-theory of $\mathbb{Z}[\pi_1 X]$ when $n$ is odd

– Geometric problem: only occurs in dimension 4 due to failure of the Whitney trick

Idea: these issues are governed by homotopy theory, and (as we'll see next) algebraic K-theory

**Uniqueness:** Suppose we have $M, M' \to X$ which are homotopy equivalences. Pick an arbitrary bordism $W$ between $M, M'$. Want to apply the same techniques in a relative (to $\partial W$) way. Eventually want $M \sqcup M' \subset \partial W$ such that the inclusions $M, M' \to W$ are homotopy equivalences. If $W \cong M \times I$ rel $M \times \{0\}$, then get $\partial_1 W \cong M \times \{1\}$. There is an algebraic K-theory obstruction to finding this ($W \cong M \times I$).

Classification of manifolds up to diffeomorphism (70s): dim $\leq 3$ were classified by geometric methods, dim $= 4$ we had no idea what was going on, and in dim $\geq 5$ these were classified by homotopy theory and algebra.

Nowadays, we think it's important to keep track of how things are diffeomorphic to each other (i.e. the types of symmetries each object has).

## 1.2 Classification of manifold bundles

A more refined classification also studies symmetries of isomorphism classes:

Why is studying diffeomorphisms the same as studying manifold bundles? (A: the former arise as clutching functions for latter)

**Definition.** A *manifold bundle* is $M \hookrightarrow E \twoheadrightarrow B$ such that for each $b \in B$ there is an open neighborhood $b \in U$ such that $p^{-1}(U) \cong U \times M$ (*local triviality*) and the transition functions $U \cap V \times M \to p^{-1}(U \cap V) \to U \cap V \times M$ are given by $(u, m) \mapsto (u, g(u)m)$ for some continuous map $g : U \cap V \to Diff(M)$. "locally trivial fiber bundle with fiber $M$ and structure group $Diff(M)$"

*Remark.* Can associate this to a principal bundle

Consider:

$$
\begin{array}{ccc}
I \times M \cong p^*E & \longrightarrow & E \\
\downarrow & & \downarrow \\
I & \longrightarrow & S^1
\end{array}
$$

The bundle on the RHS is determined by a diffeomorphism of $M$ identifying $M \times \{0\}$ with $M \times \{1\}$

Recall: vector bundles are "locally trivial fiber bundles with fiber $\mathbb{R}^n$ and structure group $GL_n(\mathbb{R})$"

When studying any bundle, should ask if there is a universal bundle:

**Definition.** $E_{univ} \to B_{uinv}$ $n$-dimensional vector bundle is *universal* if there is an isomorphism

$$\{n\text{-dim'l vector bundles over } B\}/\text{iso} \cong \{\text{maps } f : B \to B_{univ}\}/\text{htpy}$$
$$f^*E_{univ} \leftmapsto [f]$$

**Fact:** A principal $G$-bundle is universal $\iff$ its total space is contractible.

$\implies$ : $B_{univ}$ for $n$-dimensional (real) vector bundle fits in a fiber sequence

$$GL_n(\mathbb{R}) \to E_{univ} \cong * \to B_{univ}$$

By the LES on homotopy groups, have $\pi_i(B_{univ}) = \pi_{i+1}(GL_n(\mathbb{R}))$

*Example.* Let $V_n(\mathbb{R}^\infty) =$ Stiefel manifold $= \{$injective linear maps $\mathbb{R}^n \to \mathbb{R}^\infty\}$. This is contractible by a standard swindle (i.e. the map $(x_1, \ldots) \mapsto (0, x_1, \ldots)$). We have a map

$$V_n(\mathbb{R}^\infty) \to GL_n(\mathbb{R}^\infty) \cong \{\text{n-planes in } \mathbb{R}^\infty\}$$

. The latter is the classifying space for $GL_n(\mathbb{R}^\infty)/n$-dim'l vector bundles.

Fix $M$.

$$\{M \text{ bundles over } B\}/\text{iso} \longrightarrow \{ \text{ maps } f : B \to \text{ base of universal } M\text{-bundle}\}/\text{htpy}$$

$$\downarrow$$

$\{$principal $Diff(M)$-bundles over $B\}$

The universal such thing is given by *the moduli space of manifolds diffeomorphic to $M$*.

Consider $\{$embeddings $M \hookrightarrow \mathbb{R}^\infty\} \xrightarrow{\text{quotient by Diff}(M)} \{$submanifolds of $\mathbb{R}^\infty$ diffeo to $M\} = \mathcal{M}_M(\mathbb{R}^\infty)$.

$$\tilde{E} \times_{Diff(M)} M = \{\text{submanifolds of } \mathbb{R}^\infty \text{ diffeo to } M + \text{ point in } M\} \to \mathcal{M}_M(\mathbb{R}^\infty)$$

### 1.2.1 Conclusion

1. understanding $M$-bundles $\iff$ understanding homotopy type of $\mathcal{M}_M(\mathbb{R}^\infty)$

2. $\pi_i \mathcal{M}_M(\mathbb{R}^\infty) = \pi_{i-1} Diff(M)$

What was known about these things before people started studying moduli spaces of manifolds?

In the 80s: In dim $\leq 3$, $Diff(M)$ were homotopy discrete (i.e. given by $\pi_0$ - proven using hard group theory), in dim 4 ???. In dim $\geq 5$, "parametrized surgery theory" works in a range of dimensions $\leq \frac{n}{3}$. The goal of (the program of) moduli spaces of manifolds is to remove this last constraint and compute the homotopy type of $Diff(M)$ for all $M$. This is still impossible, but we can try to do this for certain classes of $M$ (e.g. simply connected) and/or express $\pi_i Diff(M)$ in terms of other things that are hard to compute (e.g. $\pi_*^{st}$ of spheres).

## 1.3 Stabilization

here's a simple example

$$W_g = \#_g S^n \times S^n$$
$$W_{g,1} = \#_g S^n \times S^n \setminus int(D^{2n})$$

Glue a copy of $W_{1,1}$ to $W_{g,1}$ to get $W_{g+1,1}$. This gives us a map

$$BDiff_\partial(W_{g,1}) \to BDiff_\partial(W_{g+1,1})$$

which has a geometric model

$$\mathcal{M}_{W_{g,1}}([0,1] \times [0,\infty) \times \mathbb{R}^\infty) \to \mathcal{M}_{W_{g+1,1}}([0,1] \times \mathbb{R} \times \mathbb{R}^\infty)$$

Why is Sander changing where the manifold lives? Because he wants the boundary to be fixed pointwise. $W_{g+1,1}$ contains a (many) copie(s) of $W_{g,1}$, but there isn't a canonical one, i.e. the map above has local inverses but no *global* ones.

[very colorful picture]

Consider the space of "inverses of stabilization" consisting of linear combinations of embedded $W_{g,1}$'s with weight in $[0,1]$, and if weight $= 0$ then delete it. It turns out that this space is highly connected.

**Theorem.** *(Galatius, Randall-Williams)*

$$H_*(BDiff_\partial(W_{g,1})) \to H_*(BDiff_\partial(W_{g+1,1}))$$

*is an isomorphism for* $* \leq \frac{g}{2}$

*Remark.* Think/intuition: all the choices above (of a copy of $W_{g,1}$ in $W_{g+1,1}$) are conjugate to each other, although this is not how the proof goes.

*Question.* What happens when $g \to \infty$? The *group completion theorem* says that

$$\varinjlim_{r \to \infty} \bigoplus_{g \geq 0} H_*(BDiff_\partial(W_{g+r,1})) \cong H_* \left( \bigsqcup_{g \geq 0} BDiff_\partial(W_{g,1}) \right) [stab^{-1}]$$

The RHS makes sense for $\mathcal{M}_{W_{g,1}}(-)$ (?)

Consider $\mathcal{M}_W(\mathbb{R} \times [0,\infty) \times \mathbb{R}^\infty)$, where we allow $W_{1,1}$'s to disappear at $\{\pm\infty\} \times [0,\infty) \times \mathbb{R}^\infty$

$$\varinjlim \bigoplus H(BDiff_\partial(W_{g+r,1})) \cong H_*(\Omega \mathcal{M}_W(\mathbb{R} \times [0,\infty) \times \mathbb{R}^\infty)$$

If you're allow to remove/add "bits at infinity," can implement surgeries:

[animated picture showing the joining of a handle with a disjoint $S^1 \times S^1$ at $\infty$]

In order to do surgery, we need to add extra data of tangent/normal bundle
$\mathcal{M}_W(\mathbb{R} \times [0,\infty) \times \mathbb{R}^\infty) \to \mathcal{M}^\theta(\mathbb{R} \times [0,\infty) \times \mathbb{R}^\infty)$ This map is a weak homotopy equivalence.
(?) Allowing things to disappear/reappear at infinity = adding more loops to the space

$$\varinjlim_{n \to \infty} \Omega^n \mathcal{M}^\theta(\mathbb{R} \times [0,\infty) \times \mathbb{R}^\infty) \sim \Omega^{\infty-1} MTO$$

(latter is some Thom spectrum).

**Theorem.** *(GRW)* $H_*(BDiff_\partial(W_{g,1})) \cong H_*(\Omega_0^\infty MTO)$ $* \leq \frac{g}{2}$ *where the RHS is very computable.*
$H_*(BDiff_\partial(W_{g,1}); \mathbb{Q}) \cong \mathbb{Q}[\kappa_c \mid c$ *monomial in* $e, p_i, n \leq 4i \leq 4n]$ *if* $* \leq \frac{g}{2}$.