DIFFERENTIAL INTEGRATION AND THE DELIGNE CUP PRODUCT

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Contents

0.1. **Review: Fiber Integration in Ordinary Cohomology.** Normally, we get a fiber integration map from combining the Thom isomorphism and the suspension isomorphism. Let $E \to S$ be an oriented fiber bundle with fiber a compact manifold of dimension k. Let $E \to \mathbb{R}^N$ be an embedding with normal bundle ν. Let E^ν denote the Thom space of the normal bundle. Then fiber integration is given by the composite

$$H^{q+k}(E) \xrightarrow{\sim} H^{q+N}(E^\nu) \xrightarrow{\mathrm{PT}} H^{q+N}(S_+ \wedge S^N) \simeq H^q(S_+)$$

where the first map is the Thom isomorphism, the second map is the Pontryagin-Thom collapse map, and the third map is the suspension isomorphism. Recall that the Thom isomorphism is given by taking the cup product with the Thom class.

To do all of this in differential cohomology, we need to differentialize the following things:

- the cup product
- Thom classes/orientations
- the suspension isomorphism

1. Cup Product

Let X be a manifold. Recall that the Deligne complex $\mathbb{Z}(k)$ is the homotopy pullback

$$\begin{tikzcd} \mathbb{Z}(k) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ \Sigma^k \Omega^k_{\mathrm{cl}} \arrow[r] & \mathbb{R} \end{tikzcd}$$

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Thus we can represent an element of $\mathbb{Z}(k)(X)$ as a triple (c, h, ω) where c is an integral degree k cocycle on X, ω is a closed k form on X, and h is a degree $k-1$ real cochain on X so that $dx = \omega - c$.

1.1. **Combining the Cup and Wedge Products.** We can form a ring structure on differential cohomology by combining the cup product on $H\mathbb{Z}$ and the wedge product on Ω_{dR}.

$$\begin{tikzcd}
\mathbb{Z}(n) \otimes \mathbb{Z}(m) \arrow[r] \arrow[d] & H\mathbb{Z}[n] \otimes H\mathbb{Z}[m] \arrow[r, "\cup"] \arrow[d] & H\mathbb{Z}[m+n] \arrow[dd] \\
\Omega^n_{\mathrm{cl}} \otimes \Omega^m_{\mathrm{cl}} \arrow[r] \arrow[d, "\wedge"] & H\mathbb{R}[n] \otimes H\mathbb{R}[m] \arrow[dr, "\cup"] & \\
\Omega^{n+m}_{\mathrm{cl}} \arrow[rr] & & H\mathbb{R}[m+n]
\end{tikzcd}$$

In particular, if we represent an element of $C^n(X; \mathbb{Z}(n)$ by a triple (c_1, h_1, ω_1) and an element of $C^m(X; \mathbb{Z}(m))$ by a triple (c_2, h_2, ω_2) we would like the product to be a triple

$$(c_1, h_1, \omega_1) \cup (c_2, h_2, \omega_2) = (c_3, h_3, \omega_3) \in C^{m+n}(X; \mathbb{Z}(m+n))$$

Saying that this product comes from combining the cup product and the wedge product, means that $c_3 = c_1 \cup c_2$ and $\omega_3 = \omega_1 \wedge \omega_2$. We are only left with figuring out what h_3 should be. Heuristically, h_3 should be a homotopy between c_3 and ω_3; i.e., a homotopy between the cup product and the wedge product.

Given forms $\omega \in \Omega^n(X)$ and $\eta \in \Omega^m(X)$, we can form the wedge product $\omega \wedge \eta \in \Omega^{n+m}(X)$ and view that as a real cochain under the map $\Omega^{n+m}(X) \to C^{n+m}(X; \mathbb{R}$. We could also map the forms ω, η to real cochains on X and then take their cup product. Let $B(\omega, \eta) \in C^{n+m-1}(X; \mathbb{R})$ be a choice of natural homotopy between these two cochains so that

$$dB(\omega, \eta) + B(d\omega, \eta) + (-1)^{|\omega|} B(\omega, d\eta) = \omega \wedge \eta - \omega \cup \eta$$

Note that we can take $B(\omega, 0) = 0$.

Then the product of $(c_1, h_1, \omega_1) \in C^n(X; \mathbb{Z}(n))$ and $(c_2, h_2, \omega_2) \in C^m(X; \mathbb{Z}(m))$ is given by

$$(c_3, h_3, \omega_3) = (c_1 \cup c_2, (-1)^{|c_1|} c_1 \cup h_2 + h_1 \cup \omega_2 + B(\omega_1, \omega_2), \omega_1 \wedge \omega_2)$$

For this to be a differential cocyle, we need to have

$$d((-1)^{|c_1|} c_1 \cup h_2 + h_1 \cup \omega_2 + B(\omega_1, \omega_2) = \omega_1 \wedge \omega_2 - c_1 \cup c_2$$

This will only work if (c_1, h_1, ω_1) and (c_2, h_2, ω_2) are themselves cocycles; i.e., $dc_i = 0 = d\omega_i$. In this case, we have

$$\omega_1 \wedge \omega_2 - \omega_1 \cup \omega_2 = dB(\omega_1, \omega_2) = B(0, \omega_2) + (-1)^{|\omega_1|} B(\omega_1, 0) = dB(\omega_1, \omega_2)$$

Thus

$$\begin{aligned}
&d((-1)^{|c_1|} c_1 \cup h_2 + h_1 \cup \omega_2 + B(\omega_1, \omega_2) \\
&\qquad = (-1)^{|c_1|} d(c_1 \cup h_2) + d(h_1 \cup \omega_2) + dB(\omega_1, \omega_2) \\
&\qquad = (-1)^{|c_1|} \Big(dc_1 \cup h_2 + (-1)^{|c_1|} c_1 \cup dh_2 \Big) + dh_1 \cup \omega_2 + (-1)^{|h_1|} h_1 \cup d\omega_2 + dB(\omega_1, \omega_2) \\
&\qquad = c_1 \cup dh_2 + dh_1 \cup \omega_2 + dB(\omega_1, \omega_2) \\
&\qquad = c_1 \cup (\omega_2 - c_2) + (\omega_1 - c_1) \cup \omega_2 + \omega_1 \wedge \omega_2 - \omega_1 \cup \omega_2 \\
&\qquad = c_1 \cup \omega_2 - c_1 \cup c_2 + \omega_1 \cup \omega_2 - c_1 \cup \omega_2 + \omega_1 \wedge \omega_2 - \omega_1 \cup \omega_2 \\
&\qquad = \omega_1 \wedge \omega_2 - c_1 \cup c_2
\end{aligned}$$

Remark. In fact we can get $\mathcal{E}_\infty$-structure from the homotopy pullback diagram. View $H\mathbb{Z}$ as a (stupidly) filtered $\mathcal{E}_\infty$-algebra. View Ω as a filtered $\mathcal{E}_\infty$-algebra with filtration by $\Omega^{\geq k}$. Then the homotopy pullback of two $\mathcal{E}_\infty$-algebras is again an $\mathcal{E}_\infty$-algebra.

1.2. **Deligne Cup Product.** Recall that we have an identification of the homotopy pullback $\widehat{H\mathbb{Z}}(k)$ with the complex of sheaves $\mathbb{Z}(k)$,

$$\mathbb{Z}(k) = \left(\Gamma^*\mathbb{Z} \xrightarrow{\iota} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \Omega^{k-1}\right)$$

Under this identifcation, we can describe the product in differential cohomology more explicitly. This is sometimes called the "Deligne cup product."

Let X be a manifold and $U \subset X$ an open set. Then $\mathbb{Z}(k)(U)$ is a chain complex that is $C^0(U;\mathbb{Z})$ in degree 0 and $\Omega^p(U)$ in degree $p+1$.

Proposition 1.1. *The Deligne cup product* $\cup : \mathbb{Z}(k)(U) \otimes \mathbb{Z}(l)(U) \to \mathbb{Z}(k+l)(U)$ *is given by*

$$x \cup y = \begin{cases} x \cdot y & \deg(x) = 0 \\ x \wedge \iota y & \deg(x) > 0, \deg(y) = 0 \\ x \wedge dy & \deg(x) > 0, \deg(y) = l > 0 \\ 0 & \textit{otherwise} \end{cases}$$

Remark. This is only commutative up to homotopy.

1.3. **Examples.** We analyze the Deligne cup product in detail n the lowest dimensions. Let X be a manifold. Recall the following computations.

- $\mathbb{Z}(0) = \Gamma^*\mathbb{Z}[0]$ is the complex with $\Gamma^*\mathbb{Z}$ in degree zero. Thus $\check{H}^0(X) = H^0(X;\mathbb{Z})$.
- $\check{H}^1(X) = \mathsf{Maps}_{\mathrm{sm}}(X, U(1))$.
- $\check{H}^2(X) = \{\text{line bundles on } X \text{ with connection}\}/\sim$.

Let $\mathbb{Z}(k)^l$ denote the degree l term of the complex $\mathbb{Z}(k)$. For example, $\mathbb{Z}(3)^2 = \Omega^1$. Let $\mathcal{U}$ be a good cover for X. Using Cech cohomology for this good cover, the Deligne cup product gives a map

$$\left(\bigoplus_{i+j=k} \check{C}^i(\mathcal{U};\mathbb{Z}(k)^j)\right) \bigotimes \left(\bigoplus_{i+j=l} \check{C}^i(\mathcal{U};\mathbb{Z}(l)^j)\right) \to \left(\bigoplus_{i+j=k+l} \check{C}^i(\mathcal{U};\mathbb{Z}(k+l)^j)\right)$$

Example 1.2. The Deligne cup product

$$\mathbb{Z}(0) \otimes \mathbb{Z}(0) \to \mathbb{Z}(0)$$

should give us a way of taking two connected locally constant functions of $X \to \mathbb{Z}$ and producing a third. By Proposition 1.1, the Deligne cup product of two elements in degree 0 agrees with the ordinary cup product in $H^0(X;\mathbb{Z})$; i.e., the product of the two locally constant functions.

Example 1.3. The Deligne cup product

$$\mathbb{Z}(0) \otimes \mathbb{Z}(1) \to \mathbb{Z}(1)$$

should give us a way of taking a locally constant function $X \to \mathbb{Z}$ and a smooth map $g \colon X \to U(1)$ and producing a new smooth map $X \to U(1)$. In the Cech complex, we are looking at a map

$$\check{C}^0(\mathcal{U};\mathbb{Z}(0)^0) \otimes \left(\check{C}^0(\mathcal{U};\mathbb{Z}(1)^1) \oplus \check{C}^1(\mathcal{U};\mathbb{Z}(1)^0)\right) \to \left(\check{C}^0(\mathcal{U};\mathbb{Z}(1)^1) \oplus \check{C}^1(\mathcal{U};\mathbb{Z}(1)^0)\right)$$

Identifying these terms, we have

$$\check{C}^0(\mathcal{U};\mathbb{Z}) \otimes \left(\check{C}^0(\mathcal{U};\Omega^0) \oplus \check{C}^1(\mathcal{U};\mathbb{Z})\right) \to \left(\check{C}^0(\mathcal{U};\Omega^0) \oplus \check{C}^1(\mathcal{U};\mathbb{Z})\right)$$

This sends $n \otimes (f, m)$ to $(n \cdot f, n \cdot m)$.

Example 1.4. The Deligne cup product

$$\mathbb{Z}(1) \otimes \mathbb{Z}(0) \to \mathbb{Z}(1)$$

should give us a way of taking a locally constant function $X \to \mathbb{Z}$ and a smooth map $g\colon X \to U(1)$ and producing a new smooth map $X \to U(1)$. In the Cech complex, we are looking at a map

$$\big(\check{C}^0(\mathcal{U};\Omega^0) \oplus \check{C}^1(\mathcal{U};\mathbb{Z})\big) \otimes \check{C}^0(\mathcal{U};\mathbb{Z}) \to \big(\check{C}^0(\mathcal{U};\Omega^0) \oplus \check{C}^1(\mathcal{U};\mathbb{Z})\big)$$

This map sends $(f,m) \otimes n)$ to $(f \cdot \iota n, m \cdot n)$.

More geometrically, we can describe the Deligne cup product as follows. Given a pair (n,f) where $n\colon X \to \mathbb{Z}$ is a locally constant function and $f\colon X \to S^1$ is a smooth map, the Deligne cup product of n with f is the smooth function $g\colon X \to S^1$ by $g(x) = e^{2\pi i n(x)} f(x)$.

Remark. We can note that the Deligne cup product commutes up to homotopy,

$$\begin{tikzcd}
\mathbb{Z}(1) \otimes \mathbb{Z}(0) \arrow[r] \arrow[d] & \mathbb{Z}(1) \\
\mathbb{Z}(0) \otimes \mathbb{Z}(1) \arrow[ur] &
\end{tikzcd}$$

since $(f \cdot \iota n = n \cdot f)$ as functions to $\mathbb{R}$.

Example 1.5. The Deligne cup product

$$\mathbb{Z}(1) \otimes \mathbb{Z}(1) \to \mathbb{Z}(2)$$

should give us a way of taking two smooth maps $X \to U(1)$ and producing a line bundle on X with connection. In the Cech complex, we are looking at a map

$$\big(\check{C}^0(\mathcal{U};\mathbb{Z}(1)^1) \oplus \check{C}^1(\mathcal{U};\mathbb{Z}(1)^0)\big)^{\otimes 2} \to \big(\check{C}^0(\mathcal{U};\mathbb{Z}(2)^2) \oplus \check{C}^1(\mathcal{U};\mathbb{Z}(2)^1) \oplus \check{C}^2(\mathcal{U};\mathbb{Z}(2)^0)\big)$$

Then the Deligne cup product sends

$$(f,n) \otimes (g,m) \mapsto (n_{\alpha\beta} \cdot m_{\beta\gamma}, n_{\alpha}\beta \cdot g_\beta + 0, f_\alpha dg_\alpha)$$

If we think of (f,n) and (g,m) as smooth maps $X \to U(1)$, then $(n_{\alpha\beta} \cdot m_{\beta\gamma}, n_{\alpha\beta} \cdot g_\beta, f_\alpha dg_\alpha)$ corresponds to the line bundle with transition function $n_{\alpha\beta} \cdot g_\beta$ and connection given by one form $(2\pi i) f_\alpha dg_\alpha$.

2. Integration

We explain how to combine fiber integration in ordinary cohomology with integration of forms to obtain a fiber integration map in ordinary differential cohomology.

The input will be a fiber bundle

$$M \to E \to X$$

where M is a closed, smooth manifold of dimension d and X is a simplicial manifold.The output will be a map of spectra

$$\mathbb{Z}(k)(E) \to \Sigma^d \mathbb{Z}(k-d)(X)$$

where $\mathbb{Z}(k)$ is the homotopy pull back

$$\begin{tikzcd}
\mathbb{Z}(k) \arrow[r] \arrow[d] & \Gamma^* H\mathbb{Z} \arrow[d] \\
\Sigma^{-k} H\Omega^k_{\mathrm{cl}} \arrow[r] & \Gamma^* H\mathbb{R}
\end{tikzcd}$$

in $\mathsf{Shv}(\mathsf{Man},\mathsf{Sp})$ and, similarly, $\mathbb{Z}(k-d)$ is the homotopy pullback

$$\begin{tikzcd}
\mathbb{Z}(k-d) \arrow[r] \arrow[d] & \Gamma^* H\mathbb{Z} \arrow[d] \\
\Sigma^{d-k} H\Omega^{k-d}_{\mathrm{cl}} \arrow[r] & \Gamma^* H\mathbb{R}
\end{tikzcd}$$

To produce a map $\mathbb{Z}(k) \to \Sigma^d \mathbb{Z}(k-d)$, it therefore suffices to produce maps $H\mathbb{Z} \to \Sigma^d H\mathbb{Z}$ and $\Omega^k_{\mathrm{cl}} \to \Omega^{k-d}_{\mathrm{cl}}$ together with a path between their images in $\Sigma^d \Gamma^* H\mathbb{R}$.

2.1. **Differential Thom Classes and Orientations.**

Definition 2.1. Let M be a smooth compact manifold and $V \to M$ a real vector bundle of dimension k. A *differential Thom cocycle* on V is a cocycle

$$U = (c, h, \omega) \in \check{Z}(k)^k_c(V)$$

such that, for each $m \in M$

$$\int_{V_m} \omega = \pm 1$$

Remark. A differential Thom class determines a ordinary Thom class in integral cohomology $H^k_c(V;\mathbb{Z})$.

The following is [1, Defn. 2.9].

Definition 2.2. An $\check{H}$-orientation of $p\colon E \to S$ consists of the following data
1) a smooth embedding $E \subset S \times \mathbb{R}^N$ for some N;
2) a tubular neighborhood $W \subset S \times \mathbb{R}^N$;
3) a differential Thom cocycle U on W.

2.2. **Differential Fiber Integration.** Our hope is to get an analogue of the suspension isomorphism

$$H^{q+N}_c(S \times \mathbb{R}^N) \simeq H^q(S)$$

Example 2.3. Consider the case when S is a point and $N = 1$. Then the ordinary suspension isomorphism says that

$$H^1(S^1;\mathbb{Z}) \cong H^0(\mathrm{pt};\mathbb{Z}) \simeq \mathbb{Z}$$

The calculation $H^1(S^1;\mathbb{Z}) \simeq \mathbb{Z}$ is by degree,

$$H^1(S^1;\mathbb{Z}) = [S^1, K(\mathbb{Z},1)] = [S^1, S^1] \simeq \mathbb{Z}$$

In differential cohomology, we have an equivalence

$$\check{H}^1(S^1) \simeq \mathsf{Maps}_{\mathrm{sm}}(S^1, S^1)$$

We still have a degree map

$$\mathsf{Maps}_{\mathrm{sm}}(S^1, S^1) \to \mathbb{Z}$$

but it is no longer an isomorphism.

Thus, we are looking for a suspension *map* not an isomorphism.

We start by working with the trivial bundle $S \times \mathbb{R}^N \to S$ and defining integration for compactly-supported forms. This is [1, §3.4]. Define the map

$$\int_{S\times\mathbb{R}^N/S} : \check{C}(p+N)^{q+N}_c(S \times \mathbb{R}^N) \to \check{C}(p)^q(S)$$

by the slant product with a fundamental cycle $Z_N \in C_N(\mathbb{R}^N; \mathbb{Z})$,

$$(c, h, \omega) \mapsto \left(c/Z_N, h/Z_N, \int_{S\times\mathbb{R}^N/S} \omega \right)$$

Note that this is simply a map *not an isomorphism.*

Remark. Checking that the slant product goes through to differential cohomology seems to require some work. See [1, §3.4].

The following is [1, Defn. 3.11]

Definition 2.4. Suppose that $p\colon E \to S$ is an $\check{H}$-oriented map of smooth manifolds with boundary of relative dimension k. The *integration map* is the map

$$\int_{E/S} : \check{C}(p+k)^{q+k}(E) \to \check{C}(p)^q(S)$$

given by the composite

$$\check{C}(p+k)^{q+k}(E) \xrightarrow{\cup U} \check{C}(p+N)^{q+N}_c(S\times\mathbb{R}^N) \xrightarrow{\int_{\mathbb{R}^N}(-)} \check{C}(p)^q_c(S)$$

2.3. **Example: S^1-bundle.** In one dimension, the only closed manifold is S^1. If $E \to S$ is an oriented S^1-bundle, then integration along the fibers defines a map

$$\int_{E/S} : \check{H}^2(E) \to \check{H}^1(E)$$

If $x \in \hat{H}^2(E)$ corresponds to a line bundle with connection, then

$$\int_{E/S} x$$

represents the function $S \to S^1$ sending $s \in S$ to the monodromy of x computed around the fiber E_s.

References

[1] M. J. Hopkins and I. M. Singer. Quadratic functions in geometry, topology, and M-theory. *J. Differential Geom.*, 70(3):329–452, 2005.