DIFFERENTIAL CHARACTERS AND GEOMETRIC INVARIANTS

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Abstract

This paper first appeared in a collection of lecture notes which were distributed at the A.M.S. Summer Institute on Differential Geometry, held at Stanford in 1973. Since then it has been (and remains) the authors' intention to make available a more detailed version. But, in the mean time, we continued to receive requests for the original notes. Moreover, the secondary invariants we discussed have recently arisen in some new contexts, e.g. in physics and in the work of Cheeger and Gromov on "collapse" (which was the subject of the first author's lectures at the Special Year). For these reasons we decided to finally publish the notes, albeit in their original form.

In this paper we sketch the study of a functor which assigns to a smooth manifold M a certain graded ring $\hat{H}^*(M)$, the ring of "differential characters" on M. Roughly speaking, if $\Lambda \subset R$ is a proper subgroup of the reals, a differential character (mod Λ) is a homomorphism f from the group of smooth singular k-cycles to R/Λ, whose coboundary is the mod Λ reduction of some (necessarily closed) differential form $\omega \in \Lambda^{k+1}(M)$. It is easily seen that f uniquely determines not only ω, but a class $u \in H^{k+1}(M,\Lambda)$ whose real image is cohomologous to the de Rahm class of ω. It turns out that ω and u both vanish if and only if f is an R/Λ cocycle the cohomology class of which is the mod Λ reduction of a real class. Thus, in general $\hat{H}^*$ contains more information than, Λ-cohomology and forms with Λ-periods.

Perhaps the main interest of our construction comes from the fact that the Weil homomorphism can be naturally factored through $\hat{H}^*$. As a consequence, we obtain a refinement of the theory of characteristic classes and characteristic forms. In appropriate contexts, this gives

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rise to obstructions to conformal immersion of Riemannian manifolds as well as R/Λ characteristic cohomology classes for flat bundles and foliations. Moreover, the calculus we develop, may be used to draw some conclusions from the recent "geometric index theorem" of Atiyah-Patodi-Singer.

We should mention that our invariants are closely related to the differential forms $TP(\theta)$ on the total space of a principle bundle with connection. These were considered by Chern and Simons in [9]. In fact, the present work arose out of the attempt to define objects in the base playing a role analogous to that of the $TP(\theta)$. Earlier results in this direction were formulated in [17]. The multiplication in $\hat{H}^*$ was already developed in [7].

The format of this paper will be as follows: In Section 1 we develop the general properties of the ring $\hat{H}^*$. In Section 2, we show how the Weil homomorphism can be factored through $\hat{H}^*$ and study the resulting invariants of bundles with connection. We show how these invariants change with connection and relate them to the forms $TP(\theta)$. Sections 3, 4 and 5 are concerned with more detailed consideration of the characters corresponding to the Euler, Chern and Pontrjagin classes. In particular we construct these characters intrinsically and give an analogue of the Whitney sum formula. In Section 6 we apply our previous results to give necessary conditions for a Riemannian manifold M^n to immerse conformally in $\mathbb{R}^{n+k}$. In Section 7 we take up foliations. The normal bundle of a foliation is equipped with a distinguished family of connections defined by Bott. In a suitable range the associate characters are independent of connection and as a consequence of Bott's vanishing theorem, are cohomology classes. In Section 8 we specialize to flat bundles in which case our invariants become R/Z cohomology classes. These are shown to come from R/Z Borel cohomology classes of the discretized structural group. We construct these classes explicitly in the bar resolution and relate their values to the volumes of geodesic simplicities on the sphere. Finally, in Section 9, we reformulate the geometric index theorem (mod Q) of Atiyah-Patodi-Singer in terms of our invariants, and use our previous computations to derive some special results in the case of flat bundles.

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§1. Differential Characters

Let M be a C^∞ manifold and let Λ^* denote the ring of differential forms on M. Let $C_k \supset Z_k \supset B_k$ denote the groups of normalized smooth singular cubic chains, cycles and boundaries, and $\partial : C_k \to C_{k-1}$ and $\delta : C^k \to C^{k+1}$ be the usual boundary and coboundary operators. If $\Lambda \subset R$ is a proper subring of the reals, we write Λ_0^k for the closed k-forms with periods lying in Λ. Let $R \xrightarrow{\sim} R/\Lambda$ be the natural homomorphism. If $\omega \in \Lambda^k$ then via integration, we may regard ω as a real cochain and write ω for the R/Λ-cochain obtained by reducing the values of ω mod Λ.

Observe that a non-vanishing differential form never takes values lying only in a proper subring $\Lambda \subset R$. Therefore, the map $\omega \to \tilde{\omega} \in C^{k+1}(M,R/\Lambda)$ is an injection, and we may regard $\Lambda^{k+1} \overset{\sim}{\subseteq} C^{k+1}(M,R/\Lambda)$.

Definition.

$$\hat{H}^k(M,R/\Lambda) \quad = \quad \{f \in \mathrm{Hom}(Z_k,R/\Lambda) \,|\, f \circ \partial \in \Lambda^{k+1}\}.$$

The most interesting cases will be $\Lambda = Q,Z,0$.

A smooth map $\phi : M_1 \to M_2$ induces a homomorphism $\phi^* : \hat{H}^k(M_2,R/\Lambda) \to \hat{H}^k(M_1,R/\Lambda)$ with the obvious functorial properties. We set $\hat{H}^{-1}(M,\Lambda) = \Lambda$. $\hat{H}^*(M,R/\Lambda) = \oplus \hat{H}^k(M,R/\Lambda)$, is a graded Λ-module whose objects we will call <u>differential characters</u>. A ring structure on this module will be introduced presently.

We can measure the size of $\hat{H}$ by inserting it in some exact sequences. Set

$$R^k(M,\Lambda) = \{(\omega,u) \in \Lambda_0^k \times H^k(M,\Lambda) \,|\, r(u) = [\omega]\} \quad \text{and} \quad \oplus R^k(M,\Lambda) = R^*(M,\Lambda).$$

Here r is the natural map $r : H^k(M,\Lambda) \to H^k(M,R)$ and $[\omega]$ is the de Rahm class of ω. $R(M)$ has an obvious ring structure $(u,\omega) \cdot (v,\phi) = (u \cup v, \omega \wedge \phi)$.

Theorem 1.1. There are natural exact sequences

$$0 \to H^k(M,R/\Lambda) \to \hat{H}^k(M,R/\Lambda) \xrightarrow{\delta_1} \Lambda_0^{k+1}(m) \to 0$$

$$0 \to \Lambda^k(M)\ \Lambda_0^k(M) \to \hat{H}^k(M,R/\Lambda) \xrightarrow{\delta_2} H^{k+1}(M,\Lambda) \to 0$$

$$0 \to H^k(M,R)/r(H(M,\Lambda)) \to \hat{H}^k(M,R/\Lambda) \xrightarrow{(\delta_1,\delta_2)} R^{k+1}(M,\Lambda) \to 0.$$

<u>In particular if</u> $H^k(M,R) = 0$, <u>then</u> f <u>is determined uniquely by</u> $\delta_1(f), \delta_2(f)$.

<u>Proof</u>. Let $f \in \hat{H}^k$. Since R is divisible, there is a real cochain T with $\tilde{T}|Z_k = f$. Since $\widetilde{\delta T} = \delta\tilde{T} = f \circ \partial$, by assumption there exists $\omega \in \Lambda^{k+1}$ and $c \in C^{k+1}(M,\Lambda)$ such that $\delta T = \omega - c$. Then $0 = \delta^2 T = \delta\omega - \delta c = d\omega - \delta c$. Since, as we have mentioned, a nonvanishing differential form never takes values lying only in a proper subgroup $\Lambda \subset R$, we conclude $d\omega = \delta c = 0$. Since $\delta T = \omega - c$, it follows that $\omega \in \Lambda_0^{k+1}$, $[c] = u \in H^{k+1}(M,\Lambda)$ and $[\omega] = r(u)$. We claim that ω, u are independent of the choice of T. In fact if T' is another lift, then $T - T'|Z_k = 0$ so that $T' = T + d + \delta s$ for some $d \in C^k(M,\Lambda)$ and $s \in C^{k-1}(M,R)$. Then $\omega' - c' = \delta T' = \delta T + \delta d = \omega - c + \delta d$. Therefore $\omega - \omega' = c' - c + \delta d$ and as above it follows that $\omega = \omega'$ and $[c'] = u$. Set $\delta_1(f) = \omega$ and $\delta_2(f) = u$.

δ_1, δ_2 are surjective. In fact given $\omega \in \Lambda_0^{k+1}$ there exists $u \in H^{k+1}(\Lambda)$ with $[\omega] = r(u)$. Conversely given u we can find such an ω. Let $[c] = u$. Then $\omega - c$ is exact as a real cochain so there exists T with $\delta T = \omega - c$. Then $\tilde{T}/Z_k = f \in \hat{H}^k$ with $\delta_1(f) = \omega$, $\delta_2(f) = u$.

If $f \in \ker \delta_1$ then $\delta T = -c$, so that $f|B_k \equiv 0 \in R/\Lambda$. Thus $\tilde{T}$ defines an R/Λ co-cycle. $\tilde{T}' = \widetilde{T + \delta s} = \tilde{T} + \delta\tilde{s}$, so f defines an R/Λ cohomology class. Similarly if v is an R/Λ cohomology class represented by some cocycle s then $s|Z_k$ defines a differential character f. f is independent of the choice of s and $\delta_1(f) = 0$.

Finally, if $\delta_2(f) = 0$ then $\delta T = \omega - c$ with $c = \delta e$ for some $e \in C^k(M,\Lambda)$. Then $\delta(T-e) = \omega$. By the de Rahm theorem, we have also $d\theta = \omega$ for some $\theta \in \Lambda^k(M)$. Then, $\delta(T-e-\theta) = 0$ so that $T - e - \theta = z$ for some $z \in Z^k(M,R)$. Again, by the de Rahm theorem, there exists ϕ a closed form $\phi \in \Lambda^k(M)$ with $\phi|Z_k = z|Z_k$. So $T|Z_k = \theta + \phi + e$. Thus the map $\alpha \to \tilde{\alpha}|Z_k$ sends Λ^k onto $\ker \delta_2$ and its kernel is clearly Λ_0^k.

The third sequence follows immediately by combining the first two.

q.e.d.

<u>Corollary 1.2</u>. <u>Let</u> $B : H^k(M,R/\Lambda) \to H^{k+1}(M,\Lambda)$ <u>denote the Bockstein associated to the coefficient sequence</u> $0 \to \Lambda \to R \to R/\Lambda \to 0$. <u>Then</u>

1) $\delta_2|H^k(M,R/\Lambda) = -B$.

2) $\delta_1|\Lambda^k/\Lambda_0^k = d$.

Proof. This is straightforward to check from the arguments above.

We will often write $\delta_1(f) = \omega_f$ and $\delta_2(f) = u_f$.

Let $0 \to \Lambda_1 \xrightarrow{i} \Lambda_2 \to R$. Let Λ_i denote closed forms with periods in Λ_i. The inclusion i induces an obvious map $\hat{i} : \hat{H}^k(M,R/\Lambda_1) \to \hat{H}^k(M,R/\Lambda_2)$ as well as $i_* : H^k(M,R/\Lambda_1) \to H^k(m,R/\Lambda_2)$.

Corollary 1.3. We have the exact sequence

$$0 \to \ker i_* \to \hat{H}^k(M,R/\Lambda_1) \xrightarrow{\hat{i}} \hat{H}^k(M,R/\Lambda_2) \to \Lambda_2^{k+1}/\Lambda_1^{k+1} \to 0.$$

Example 1.4. $\hat{H}^0(M,R/Z) \cong C^\infty(M,S^1)$

$$\hat{H}^n(M,R/\Lambda) = H^n(M,R/\Lambda)$$

$$\hat{H}^k(M,R/\Lambda) = 0 \quad k > n = \dim M.$$

The following simple example illustrates how differential characters arise in geometry. In many ways it typifies the general case.

Example 1.5. Let $SO(2) \to E \xrightarrow{\pi} M$ be a circle bundle over M with connection θ. Let $\Omega \in \Lambda^2(M)$ denote its curvature form. Since $\frac{1}{2\pi}\Omega$ represents the real Euler class, $\frac{1}{2\pi}\Omega \in \Lambda_0^2$. For γ a closed curve let $H(\gamma) \in SO(2)$ be holonomy around γ, and define $\hat{\chi}(\gamma) \in R/Z$ by $H(\gamma) = e^{2\pi i \hat{\chi}(\gamma)}$.

Extend $\hat{\chi}$ to all 1-cycles as follows. Let $x \in Z_1$ and choose a closed curve γ and a chain $y \in C_2$ so that $x = \gamma + \partial y$. Set

$$\hat{\chi}(x) = \hat{\chi}(\gamma) + \frac{1}{2\pi}\tilde{\Omega}(y).$$

It is easily seen that $\hat{\chi}$ is well defined and clearly $\hat{\chi} \circ \partial = \frac{1}{2\pi}\tilde{\Omega}$. Thus $\hat{\chi} \in \hat{H}^1(M,R/Z)$. If we let χ denote the integral Euler class then one can check

$$\delta_1(\hat{\chi}) = \frac{1}{2\pi}\Omega, \quad \delta_2(\hat{\chi}) = \chi.$$

$\hat{\chi}$ carries more information than Ω and χ together, since both may vanish when $\hat{\chi}$ does not, e.g. $M = S^1$.

As already mentioned, the differential characters form a graded ring. To define the multiplication we must introduce subdivision. Let $\Delta : C_* \to C_*$ be the standard subdivision map in cubical theory, and let ψ be its chain homotopy to 1 (see [11]). I.e.

$$1 - \Delta = \partial\psi + \psi\partial . \tag{1.6}$$

Since ψ is natural, if σ is a k-simplex then $\psi(\sigma)$ is supported on σ. Thus the (k+1)-dimensional volume of $\psi(\sigma) \in C_{k+1}$ is zero. Consequently, if $\omega \in \Lambda^{k+1}$ then $\omega \circ \psi = 0$. Δ operates on everything, and in particular differential forms (regarded as cochains) are invariant under subdivision. So are differential characters. In fact if $x \in Z_k$ then $\Delta(f)(x) = f(\Delta x) = f(x) - f(\partial\psi x) = f(x) - \tilde{\omega}_f(\psi x) = f(x)$.

Subdivision allows one to connect $\wedge$ product and $\cup$ product. If $\theta, \omega \in \Lambda^*$ we may regard θ, ω, and $\theta \wedge \omega$ as real cochains. We may thus cup θ and ω and get another real cochain $\theta \cup \omega$. In [12] Kervaire has shown

$$\lim_{n\to\infty} \Delta^n(\theta \cup \omega) = \theta \wedge \omega . \tag{1.7}$$

It is because of this formula that we use cubical theory.

Let $\omega_1, \omega_2 \in \Lambda^{\ell_1}, \Lambda^{\ell_2}$ be closed. Using (1.7) we can make $\omega_1 \wedge \omega_2 - \omega_1 \cup \omega_2$ exact in a canonical way. Define $E(\omega_1,\omega_2) \in C^{\ell_1+\ell_2-1}(M,R)$ by

$$E(\omega_1,\omega_2)(x) = -\sum_{i=0}^{\infty} \omega_1 \cup \omega_2(\psi\Delta^i x) \tag{1.8}$$

A straightforward estimate shows that the right hand side of (1.8) is dominated by a geometric series and hence converges. Moreover, it is then obvious that

$$\lim_{n\to\infty} E(\omega_1,\omega_2)(\Delta^n x) = 0 . \tag{1.9}$$

Now

$$\begin{aligned}
\delta E(\omega_1,\omega_2)(x) &= -\sum_{i=0}^{\infty} \omega_1 \cup \omega_2(\psi\Delta^i \partial x) = -\sum_{i=0}^{\infty} \omega_1 \cup \omega_2(\psi\partial\Delta^i x) \\
&= -\sum_{i=0}^{\infty} \omega_1 \cup \omega_2((1-\Delta-\partial\psi)\Delta^i x) = \lim_{n\to\infty} -\sum_{i=0} \omega_1 \cup \omega_2((1-\Delta)\Delta^i x) \\
&= \lim_{n\to\infty} - \omega_1 \cup \omega_2((1-\Delta^{n+1})x) \\
&= (\omega_1 \wedge \omega_2 - \omega_1 \cup \omega_2)(x)
\end{aligned}$$

where we have used (1.6), (1.7) and the fact that $\delta(\omega_1 \cup \omega_2) = 0$, since the ω_i are closed. Hence

$$\delta E(\omega_1,\omega_2) = \omega_1\wedge\omega_2 - \omega_1\cup\omega_2. \qquad (1.10)$$

The main point in the above computation is that $\{\sum_{i=0}^{\infty}\psi\Delta^i\}$ is a sequence of natural chain homotopies between 1 and Δ^{n+1}. In fact, if θ_n is any such sequence with the property that $\lim_{n\to\infty} -\omega_1\cup\omega_2(\theta_n x)$ exists, then we can take $E(\omega_1,\omega_2)(x) = \lim_{n\to\infty} -\omega_1\cup\omega_2(\theta_n x)$. All expressions so obtained will differ universally by exact cochains. One such can be written as a finite sum of terms involving integrals over x of certain expressions in ω_1, ω_2.

Now let $f\in\hat{H}^{k_1}(M,R/\Lambda)$, $g\in\hat{H}^{k_2}(M,R/\Lambda)$ and choose $T_f\in C^{k_1}(M,R)$, $T_g\in C^{k_2}(M,R)$ with $\tilde{T}_f|Z_{k_1} = f$, $\tilde{T}_g|Z_{k_2} = g$.

Definition.

$$f * g = \widetilde{T_f\cup\omega_g} - (-1)^{k_1}\widetilde{\omega_f\cup T_g} - \widetilde{T_f\cup\delta T_g} + \widetilde{E(\omega_f,\omega_g)}|Z_{k_1+k_2+1}.$$

Theorem 1.11. $f*g\in\hat{H}^{k_1+k_2+1}(M,R/\Lambda)$ is well defined independent of the choices of T_f, T_g. Moreover,

1) $(f*g)*h = f*(g*h)$

2) $f*g = (-1)^{(k_1+1)(k_2+1)}g*f$

3) $\omega_{f*g} = \omega_f\wedge\omega_g$ and $u_{f*g} = u_f\cup u_g$ i.e. δ_1 and δ_2 are ring homomorphisms as is $(\delta_1,\delta_2):\hat{H}(M)\to R(M)$.

4) If $\phi: M\to N$ is a C^∞ map, then $\phi^*(f*g) = \phi^*(f)*\phi^*(g)$.

Proof. Let $\delta T_f = \omega_f - c_f$, $\delta T_g - \omega_g - c_g$ with $[c_f] = u_f$, $[c_g] = u_g$. To see that $f*g$ is a different character such that 3) holds, one computes that

$$\delta(T_f\cup\omega_g - (-1)^{k_1}\omega_f\cup T_g - T_g\cup\delta T_g + E(\omega_f,\omega_g))$$
$$= (\omega_f - c_f)\cup\omega_g + \omega_f\cup(\omega_g - c_g) - (\omega_f - c_f)\cup(\omega_g - c_g) + \omega_f\wedge\omega_g - \omega_f\cup\omega_g$$
$$= \omega_f\wedge\omega_g - c_f\cup c_g.$$

3) follows immediately. That the definition is independent of the choices of T_f, T_g is straightforward. 2) can be proved by a simple formal argument and 4) is trivial. To see 1), let $h\in\hat{H}^{k_3}$ and choose T_h as above. A direct computation shows that $(f*g*h - f*(g*h)$ is the mod Λ reduction of

$$E(\omega_f,\omega_g) \cup \omega_h + E(\omega_f\wedge\omega_g,\omega_h) + (-1^{k_1}{}_f \cup E(\omega_g,\omega_h) - E(\omega_f,\omega_g\wedge\omega_h), \qquad (1.12)$$

and that the coboundary of this expression is zero. (1.9) and similar estimates show that the limit of (1.12) under subdivision, is a cocycle with zero periods, and 1) follows. q.e.d.

Note that if Λ is discrete, e.g. $\Lambda = Z$, then by use of (1.7) we have

$$f*g = \lim_{n\to\infty} \Delta^n(\widetilde{T_1 \cup \omega_g} - (-1)^{k_1}\widetilde{\omega_f \cup T_g} - \widetilde{T_f \cup \delta T_g})|Z_{k_1+k_2+1}.$$

Two special cases are important and follow easily from the definition.

$$f*g = (-1)^{k_1+1} u_f \cup g \qquad g \in H^{k_2}(M,R/\Lambda) \qquad (1.14)$$

$$f*g = (-1)^{k_1+1} \omega_f \wedge g \qquad g \in \Lambda^{k_2}/\Lambda_0^{k_2}. \qquad (1.15)$$

Theorem 1.11 may be paraphrased as saying that $\hat{H}^*$ is a functor from manifolds to rings and $(\delta_1,\delta_2):\hat{H}^* \to R^*$ is a natural transformation of functors. The $*$ product is probably characterized by this property. It is also possible to represent differential characters by differential forms with singularities (although not canonically). With respect to this representation, there is a nice formula for the product which generalizes that of Example 1.16 below. (For more details see [7]).

<u>Example 1.16</u>. $M = S^1$, $f,g \in \hat{H}^0(S^1,R/Z) = C^\infty(S^1,R/Z)$. f and g may be represented by functions $F,G : R \to R$ so that $F(x+2\pi) = F(x) + n_1$, $G(x+2\pi) = G(x) + n_2$ with $n_1,n_2 \in Z$. Now $f*g \in \hat{H}^1(S^1,R/Z) = H^1(S^1,R/Z)$, and

$$f*g(S^1) = n_1G(0) - \int_0^{2\pi} FG'.$$

§2. <u>A Lift of Weil Homomorphism</u>

Let G be a Lie group with finitely many components, B_G its classifying space and $I^*(G)$ the ring of invariant polynomials on G. Let $\alpha = \{E,M,\theta\}$ be a principle G-bundle with total space E, base space M and connection θ. Let $\varepsilon(G)$ be the category of these objects with morphisms being connection preserving bundle maps. Then we have the functors $\alpha \to I^*(G)$, $H^*(B_G,R)$, $H^*(B_G,\Lambda)$, $H^*(M,\Lambda)$, $H^*(M)$,

$\Lambda^*_{cl}(M)$ (= closed forms). (In the first three cases, to any morphism we assign the identity map). The Weil homomorphism constructs a homomorphism $w : I^k(G) \to H^{2k}(B_G,R)$ and a natural transformation $W : I^k(G) \to \Lambda^{2k}(M)$ such that the following diagram of natural transformations commutes

$$\begin{tikzcd} I^*(G) \arrow[r, "w"] \arrow[d, "W"] & H^*(B_G,R) \arrow[d, "C_R"] & H^*(B_G,\Lambda) \arrow[l, "r"'] \arrow[d, "C_\Lambda"] \\ \Lambda^*_{cl}(M) \arrow[r, "dR"] & H^*(M,R) & H^*(M,\Lambda). \arrow[l, "r"'] \end{tikzcd} \tag{2.1}$$

Here, C_Λ, C_R are provided by the theory of characteristic classes and dR is the de Rham homomorphism. If $P \in I^k(G)$, $u \in H^*(B_G,\Lambda)$ and Ω is the curvature form of $\alpha \in \varepsilon$ then explicitly, $W(P) = P(\overbrace{\Omega,\ldots,\Omega}^{k})$, and $C_\Lambda(u) = u(\alpha)$, the characteristic class. Set

$$K^{2k}(G,\Lambda) = \{(P,u) \in I^k(G) \times H^{2k}(B_G,\Lambda) \mid w(P) = r(u)\}.$$

$K^*(G,\Lambda) = \oplus K^{2k}(G,\Lambda)$ forms a graded ring in an obvious way. Moreover (2.1) induces $K^*(G,\Lambda) \xrightarrow{W\times C_\Lambda} R^*(M,\Lambda)$. Our result may be paraphrased as saying that there exists a unique natural transformation $S : K^*(G,\Lambda) \to \hat{H}^*(M,\Lambda)$ such that the diagram

$$\begin{tikzcd} & \hat{H}^*(M,R/\Lambda) \arrow[d, "(\delta_1,\delta_2)"] \\ K^*(G,\Lambda) \arrow[ur, "S"] \arrow[r, "W\times C_\Lambda"'] & R^*(M,\Lambda) \end{tikzcd}$$

commutes. In more detail:

<u>Theorem 2.2</u>. <u>Let</u> $(P,u) \in K^{2k}(G,\Lambda)$. <u>For each</u> $\alpha \in \varepsilon(G)$ <u>there exists a unique</u> $S_{P,u} \in \hat{H}^{2k-1}(M,R/\Lambda)$ <u>satisfying</u>

1) $\delta_1(S_{P,u}(\alpha)) = P(\Omega)$.

2) $\delta_2(S_{P,u}(\alpha)) = u(\alpha)$.

3) <u>If</u> $\beta \in \varepsilon(G)$ <u>and</u> $\phi : \alpha \to \beta$ <u>is a morphism then</u> $\phi^*(S_{P,u}(\beta)) = S_{P,u}(\alpha)$.

<u>Proof</u>. An object $\beta_N = (E_N,A_N,\mu_N) \in \varepsilon(G)$ is called N-classifying if

any $(E,M,\theta) = \alpha \in \varepsilon(G)$ with $\dim M < N$ admits a morphism to β_N and for any two such morphisms, the corresponding maps $f_1, f_2 \to M$ are smoothly homotopic. By a theorem of Narasimhan-Ramanan [15] such objects exist. It is well known that $H^{odd}(B_G, R) = 0$ and since β_N is topologically N-classifying $H^{2k-1}(A_N, R) = 0$ for N-sufficiently large. Referring to Theorem 1.1, $(\delta_1, \delta_2) : \hat{H}^{2k-1}(A_N) \to R^{2k}(A_N)$ is an isomorphism and the theorem follows trivially in the category of such N-classifying objects by setting $S_{P,u}(\beta_N) = (\delta_1, \delta_2)^{-1}(P(\Omega))$. It will follow in general if we can show that if F_0, F_1 are morphisms of α to N-classifying β_N^0, β_N^1 with $f_i : M \to A_N^i$ the corresponding maps of base spaces, then $f_0^*(S_{P,u}(\beta_N^0)) = f_1^*(S_{P,u}(\beta_N^1))$. There is an N-classifying object $\beta_{N'}$, $N' >> N$, such that β_N^i admit morphisms to $\beta_{N'}$. Let $\phi_i : A_N^i \to A_{N'}$ be the corresponding maps of base spaces. By the above, $\phi_i^*(S_{P,u}(\beta_{N'})) = S_{P,u}(\beta_N^i)$. Therefore, it suffices to check that $(\phi_0 \circ f_0)^*(S_{P,u}(\beta_{N'})) = (\phi_1 \circ f_1)^*(S_{P,u}(\beta_{N'}))$. Let G_t be a homotopy between $\phi_0 \circ f_0$ and $\phi_1 \circ f_1$. Further, choose G_t to be constant near $t = 0$, $t = 1$. Let $z \in Z_{2k-1}(M)$ and $\bar{\Omega}$, Ω be the curvature forms of $\beta_{N'}$, and $G_t^*(\beta_{N'})$, (the latter being a bundle over $M \times I$). Since

$$(\phi_1 \circ f_1)^*(S_{P,u}(\beta_{N'})) - (\phi_0 \circ f_0)^*(S_{p,u}(\beta_{N'}))(z) = S_{P,u}(\partial G_t(z \times I)) = \int_{G_t(z\times I)} P(\bar{\Omega})$$

we must show that $\int_{G_t(z\times I)} P(\bar{\Omega}) \in \Lambda$. Since G_t is constant near $t = 0$, $t = 1$, the induced connection on $G_t^*(E_{N'})$ is independent of t near these points. By identifying $G_t^*(E_{N'})|M \times 0$ with $G_t^*(E_{N'})|M \times 1$, we obtain a bundle with smooth connection over $M \times S^1$. Let $P(\hat{\Omega})$ denote the characteristic form for this bundle. Clearly

$$\int_{G_t(z\times I)} P(\bar{\Omega}) = \int_{z\times I} P(\Omega) = \int_{z\times S^1} P(\hat{\Omega}).$$

Since, $z \times S^1$ is a cycle and $P(\hat{\Omega}) \in \Lambda_0$, the theorem follows.

Corollary 2.3. The map $S : K^*(G,\Lambda) \to \hat{H}^*(M, R/\Lambda)$ is a ring homomorphism. i.e.

$$S_{PQ,u\cup v}(\alpha) = S_{P,u}(\alpha) * S_{Q,v}(\alpha).$$

This follows immediately from the properties of $*$ product and the uniqueness statement in the theorem. From Theorem 1.1 and Corollary 1.2 we see

<u>Corollary 2.4</u>. <u>Suppose</u> $P(\Omega) = 0$. <u>Then</u>

1) $S_{p,u}(\alpha) \in H^{2k-1}(M,R/\Lambda)$

2) $B(S_{P,u}(\alpha)) = -u(\alpha)$.

<u>Example 2.5</u>. Suppose $\alpha = \{E,M^{2k-1},\theta\}$ where M^{2k-1} is compact and oriented. If $(P,u) \in K^{2k}(G,\Lambda)$ then $P(\Omega)$ vanishes for dimension reasons and $S_{P,u}(\alpha) \in H^{2k-1}(M,R/\Lambda)$. Evaluating on the fundamental cycle we get the <u>characteristic number</u>

$$S_{P,u}(\alpha)(M^{2k-1}) \in R/\Lambda.$$

Now suppose $M^{2k-1} = \partial\overline{M}$ and that E extends to $\overline{E}$, a principal G-bundle over $\overline{M}$. Let $\overline{\theta}$ be any extension of θ to a connection in $\overline{E}$. Setting $\overline{\alpha} = \{\overline{M},\overline{E},\overline{\theta}\}$ we have the morphism $\alpha \hookrightarrow \overline{\alpha}$. Thus $S_{P,u}(\overline{\alpha}) \mid M^{2k-1} = S_{P,u}(\alpha)$. Since $\delta_1(S_{P,u}(\overline{\alpha})) = P(\overline{\Omega})$,

$$S_{P,u}(\alpha)(M^{2k-1}) = \widetilde{\int_{\overline{M}} P(\overline{\Omega})}. \tag{2.6}$$

It might appear from this formula that these numbers depend only on P, but this is false since in general E only extends over a manifold whose boundary is a finite union of copies of M^{2k-1}, and the choice of u removes a rational ambiguity.

In [9] the authors considered the forms $TP(\theta)$ defined in E by

$$TP(\theta) = k\int_0^1 P(\theta \wedge \phi_t^{k-1})dt$$

where $\phi_t = t\Omega + \frac{1}{2}(t^2-t)[\theta,\theta]$, and showed

$$dTP(\theta) = P(\Omega) \quad \text{in } E. \tag{2.7}$$

These forms, reduced mod Λ, are the lifts of the $S_{P,u}(\alpha)$. Letting $\pi : E \to M$, one may show

<u>Proposition 2.8</u>. $\pi^*(S_{P,u}(\alpha)) = \widetilde{TP(\theta)} \mid Z_{2k-1}(E)$.

This makes the characters representable by specific differential forms when E has a global cross-section.

If θ_0,θ_1 are connections on E set $\alpha_i = \{E,M,\theta_i\}$. Then $\delta_2(S_{P,u}(\alpha_1) - S_{P,u}(\alpha_0)) = u(\alpha_1) - u(\alpha_0) = 0$. Thus by (1.1) the difference of the characters must be the reduction of a form. Let θ_t be a smooth curve of connections joining θ_0 to θ_1, let Ω_t be the curvature at time t, and set $\theta_t' = d/dt(\theta_t)$. As in [9] we have

Proposition 2.9. $S_{P,u}(\alpha_1) - S_{P,u}(\alpha_0) = k \widehat{\int_0^1 P(\theta_t' \wedge \Omega_t^{k-1})dt} \mid Z_{2k-1}(M)$.

This makes sense since θ_t' vanishes on vertical vectors, and the integrand is the lift of a form on M.

A bundle is called flat if $\Omega = 0$. The holonomy theorem [1] shows that in this case the holonomy group $H \subseteq G$ is arcwise totally disconnected. It is called globally flat if it is trivially a product. The G-bundle $\{E,M\}$ is always reducible to an H-bundle $\{E_H,M\}$, and this is induced by a map $\rho : M \to B_H$. The inclusion $H \subseteq G$ induces $\phi : B_H \to B_G$, and for $u \in H^{2k}(B_G,\Lambda)$, we get $\phi^*(u) \in H^{2k}(B_H,\Lambda)$. These are sometimes called the characteristic classes of the representation, see [2]. We recall that if H is finite its integral cohomology is all torsion, and $H^{2k-1}(B_H,R/Z) \cong H^{2k-1}(B_H,Q/Z) \xrightarrow[B]{\cong} H^{2k}(B_H,Z)$.

Proposition 2.10. *If α is flat then all $S_{P,u}(\alpha) \in H^{2k-1}(M,R/\Lambda)$. If α is globally flat then all $S_{P,u}(\alpha) = 0$. If α has finite holonomy (and is consequently flat) and $\Lambda = Z$ then*

$$S_{P,u}(\alpha) = \rho^*(B^{-1}(\phi^*(u)) \in H^{2k-1}(M,Q/Z).$$

This formula was pointed out to us by John Millson; its proof, which is straightforward appears in his dissertation [13].

§3. The Euler Character

It is possible to give a more intrinsic construction of the Euler character $\hat{\chi}$. Let $V^{2n} = \{V,M,\nabla\}$ be a real 2n-dim Riemannian vector bundle over M, with ∇ denoting covariant differentiation. Let $SV \xrightarrow{\pi} M$ be the associated sphere bundle. We have the homology sequence

$$H_{2n-1}(S^{2n-1}) \to H_{2n-1}(SV) \xrightarrow{\pi_*} H_{2n-1}(M) \to 0. \tag{3.1}$$

Let $\chi \in H^{2n}(B_{SO(2n)}Z)$ be the integral Euler class and let $P_\chi \in I^n(SO(2n))$ be the unique polynomial with $w(P_\chi) = \chi$. (P_χ is unique since G is compact). Let $F(V) \in \varepsilon(SO(2n))$ be the orthonormal frame bundle of V, with connection θ and curvature Ω. The Euler form $P_\chi(\Omega)$ becomes exact in SV, and in fact there is a canonical $(2n-1)$ form Q (see [8]) on SV which is natural in the category and which satisfies

$$\pi^*(P_\chi(\Omega)) = dQ \quad \text{and} \quad \int_{S^{2n-1}} Q = 1. \tag{3.2}$$

Let $z \in Z_{2n-1}(M)$. By (3.1) we can find $y \in Z_{2n-1}(SV)$ and $w \in C_{2n}(M)$ with

$$z = \pi_*(y) + \partial w.$$

We define the <u>Euler character</u>, $\hat{\chi}(V) \in \hat{H}^{2n-1}(M,R/Z)$,

$$\hat{\chi}(V)(z) = \tilde{Q}(y) + \widetilde{P_\chi(\Omega)}(w). \tag{3.3}$$

An analysis of (3.1) easily shows $\hat{\chi}(V)$ to be well defined. It is immediate from (3.3) that $\delta_1(\hat{\chi}(V)) = P_\chi(\Omega)$, and it is not difficult to show $\delta_2(\hat{\chi}(V)) = \chi(V)$. Since Q is natural $\hat{\chi}$ is natural, and so by Theorem 2.2

$$\hat{\chi}(V) = S_{P_\chi,\chi}(F(V)). \tag{3.4}$$

In the special case that $\dim M = 2n-1$, (3.3) simplifies. We may then choose a global cross-section $\phi : M \to SV$, and if M is compact and oriented

$$\hat{\chi}(V)(M) = \phi^*(Q)(M). \tag{3.5}$$

Let $V = \{V,M,\nabla\}$ and $W = \{W,M,\nabla'\}$ be two Riemannian vector bundles over M. The inner product on V and W naturally induces one on $V \oplus W$, and letting $\nabla \oplus \nabla'$ denote the natural connection we get the new Riemannian vector bundle $V \oplus W = \{V \oplus W, M, \nabla \oplus \nabla'\}$.

<u>Theorem 3.6</u>. $\hat{\chi}(V \oplus W) = \hat{\chi}(V) * \hat{\chi}(W)$.

<u>Proof</u>. Since $V \oplus W$ may be induced from a bundle over a product of classifying spaces, by naturality it suffices to check the theorem there. Such a product again has vanishing real cohomology in odd dimensions and the theorem follows from Theorem 1.1 and Theorem 2.2.

§4. <u>Chern Characters</u>.

Let $G = Gl(n,C)$. The Weil map W is onto, but has a large kernel. Let A be an $n \times n$ complex matrix and define the k^{th} <u>Chern polynomial</u>, $C_k \in I^k(Gl(n,C))$ by

$$\det(\lambda I - \frac{1}{2\pi i} A) = \sum_{k=0}^{n} [C_k(A) + iD_k(A)]\lambda^{n-k}. \tag{4.1}$$

Letting c_k denote the k^{th} integral <u>Chern class</u>, $w(C_k) = c_k$, and $(C_k, c_k) \in K^{2k}(Gl(n,C), Z)$.

Let E_{n-k+1} be the Stiefel manifold of $n-k+1$ bases in C^n. We do not require these to be orthonormal. $H_i(E_{n-k+1}) = 0$ for $i < 2k-1$. $H_{2k-1}(E_{n-k+1}) \cong Z$ and the image of $S^{2k-1} = U(k)/U(k-1) \subset U(n)/U(k-1) \subset E_{n-k+1}$ gives a generator, h_{2k-1}, of this group. Let $V = \{V^n, M, \nabla\}$ be a complex vector bundle with connection. Let

$$E_{n-k+1} \to V^n_{n-k+1} \xrightarrow{\pi} M$$

be the <u>Stiefel bundle</u> of $n-k+1$ dim bases of V^n. Analogous to (3.1) we have

$$Z \cong H_{2k-1}(E_{n-k+1}) \to H_{2k-1}(V^n_{n-k+1}) \xrightarrow{\pi_*} H_{2k-1}(M) \to 0 \tag{4.2}$$

an exact sequence. Letting $E(V)$ be the $GL(n,C)$ <u>basis bundle</u> of V, with connection θ and curvature Ω, $\pi^*(C_i(\Omega))$ is exact in V^n_{n-k+1}. In fact, there is a family of canonical $(2k-1)$ forms, Q_{2k-1}, in V^n_{n-k+1}, <u>defined modulo exact forms</u> which is natural in the category and which satisfies

$$dQ_{2k-1} = \pi^*(C_k(\Omega)), \quad \int_{h_{2k-1}} Q_{2k-1} = 1. \tag{4.3}$$

Let $z \in Z_{2k-1}(M)$. By (4.2) we can find $y \in Z_{2k-1}(V^n_{n-k+1})$ and $w \in C_{2k}(M)$ with

$$z = \pi_*(y) + \partial w.$$

We define the k^{th} <u>Chern character</u>, $\hat{c}_k(V) \in \hat{H}_{2k-1}(M, R/Z)$ by

$$\hat{c}_k(V)(z) = \tilde{Q}_{2k-1}(y) + \widetilde{C_k(\Omega)}(w). \tag{4.4}$$

As with $\hat{\chi}$ it is easily shown that $\hat{c}_k$ is well defined and that $\delta_1(\hat{c}_k(V)) = C_k(\Omega)$, and $\delta_2(\hat{c}_k(V)) = c_k(V)$. Since the family Q_{2k-1} is natural, Theorem 2.2 shows

$$\hat{c}_k(V) = S_{C_k, c_k}(E(V)). \tag{4.5}$$

We set $\hat{c}(V) = 1 + \hat{c}_1(V) + \ldots + \hat{c}_n(V) \in \hat{H}^{odd}(M, R/Z)$.

If V, W are two complex vector bundles with connections we form their Whitney sum as in §3 and using (4.5) and Theorem 2.2 show as in Theorem 3.6

<u>Theorem 4.6</u>. $\hat{c}(V \oplus W) = \hat{c}(V) * \hat{c}(W)$.

The Whitney sum connection on $V \oplus W$ is not always the most useful. Let $\bar{\nabla}$ be another connection on $V \oplus W$, and for $x, y \in T(M)_m$ let $\bar{R}_{x,y} : V_m \oplus W_m \to V_m \oplus W_m$ be its curvature transformation. By direct sum projection $\bar{\nabla}$ induces connections $\bar{\nabla}^V$ and $\bar{\nabla}^W$ on V and W. We call $\bar{\nabla}$ <u>compatible</u> with $\nabla \oplus \nabla'$ if

1) $\bar{\nabla}^V = \nabla$, $\bar{\nabla}^W = \nabla'$.

2) $\bar{R}_{x,y}(V_m) \subseteq V_m$, $\bar{R}_{x,y}(W_m) \subseteq W_m$.

Using the variation formula (2.9) one shows

<u>Theorem 4.7</u>. <u>Let</u> $\bar{\nabla}$ <u>be compatible on</u> $V \oplus W$.

$$\hat{c}(\{V \oplus W, M, \bar{\nabla}\}) = \hat{c}(V) * c(W).$$

The most important instance of Theorem 4.7 occurs in the following situation. W is called an <u>inverse</u> of V if there exists a <u>globally flat compatible</u> connection on $V \oplus W$. e.g. the complexified tangent and normal bundles of a manifold immersed in R^n are inverses of each other. In general the inverse bundle is not unique.

<u>Corollary 4.8</u>. <u>If</u> V^{-1} <u>is inverse to</u> V

$$\hat{c}(V) * \hat{c}(V^{-1}) = 1.$$

If V has a covariant constant hermitian inner product on its fibers then the underlying real bundle, V^R, is a $2n$-dim Riemannian vector bundle. The expected relation holds

$$\hat{c}_n(V) = -\hat{\chi}(V^R) \tag{4.9}$$

It is sometimes useful to work modulo elements of finite order, i.e. in R/Q. Let $ch \in H^{even}(B_{Gl(n,c)}, Q)$ be the <u>topological Chern character</u>, and let $P_{ch} = P_{ch}(C_1, \ldots, C_n) \in I(Gl(n,c))$ be the corresponding polynomial. So $w(P_{ch}) = ch$, and we set

$$\hat{ch}(V) = S_{P_{ch}, ch}(E(V)) \in \hat{H}^{odd}(M, R/Q)$$

$\hat{ch}$ is just the usual formula for ch, with $*$ replacing $\cup$. We work in R/Q because of denominators. e.g. if V <u>is a line bundle</u>.

$$\hat{c}h(\mathcal{V}) = 1 + \hat{c}_1 + \frac{\hat{c}_1 * \hat{c}_1}{2} + \ldots + \frac{\overbrace{\hat{c}_1 * \ldots * \hat{c}_1}^{n}}{n!} + \ldots \quad . \tag{4.10}$$

Given $\mathcal{V}$, $\mathcal{W}$ there is a standard connection on $V \oplus W$, and one shows as in Theorem 3.6.

Theorem 4.11. $\hat{c}h(\mathcal{V} \oplus \mathcal{W}) = \hat{c}h(\mathcal{V}) + \hat{c}h(\mathcal{W})$

$\hat{c}h(\mathcal{V} \otimes \mathcal{W}) = \hat{c}h(\mathcal{V}) * \hat{c}h(\mathcal{W})$.

§ 5. Pontrjagin Characters

Let $\mathcal{V} = \{V^n, M, \nabla\}$ be a real vector bundle with connection, and let $\mathcal{V}^c$ be its complexification. We set

$$\hat{p}_k(\mathcal{V}) = (-1)^k \hat{c}_{2k}(\mathcal{V}^c). \tag{5.1}$$

In $B_{Gl(n,R)}$ we have the Pontrjagin class p and the polynomial $P_k \in I^{2k}(Gl(n,R))$ with $w(P_k) = p_k$. Letting $E(\mathcal{V}) \in \varepsilon(Gl(n,R))$ denote the basis bundle of $\mathcal{V}$ one easily shows

$$\hat{p}_k(\mathcal{V}) = S_{P_k, p_k}(E(\mathcal{V})). \tag{5.2}$$

We define $\oplus$ and compatible connection as in the complex case, and obtain

$$\hat{p}(\mathcal{V} \oplus \mathcal{W}) = \hat{p}(\mathcal{V}) * \hat{p}(\mathcal{W}) + \text{order 2 elements in } H^{odd}(M, R/Z) \tag{5.3}$$

$$\hat{p}(V \oplus W, M, \overline{\nabla}) = \hat{p}(\mathcal{V} \oplus \mathcal{W}) \quad \overline{\nabla} \text{ compatible where} \tag{5.4}$$

$\hat{p} = 1 + \sum_{k=1}^{[n/2]} \hat{p}_k$.

In order to get a proper inverse formula we inductively define

$$\hat{c}_k^{\perp} = -\hat{c}_k - \hat{c}_{k-1} * \hat{c}_1^{\perp} - \ldots - \hat{c}_1 * \hat{c}_{k-1}^{\perp}$$

$$\hat{p}_k^{\perp} = \hat{c}_k^{\perp}(\mathcal{V}^c).$$

Defining inverse as in the complex case we see

$$\hat{p}_k(\mathcal{V}^{-1}) = \hat{p}_k^{\perp}(\mathcal{V}). \tag{5.5}$$

§6. Applications to Riemannian Geometry

Let M be a Riemannian manifold, with metric, g, and Riemannian connection, ∇, in the tangent bundle $T(M)$. Set $\mathcal{T}(M) = \{T(M),M,\nabla\}$, and set $\hat{p}_k(M) = \hat{p}_k(M,g) = \hat{p}_k(\mathcal{T}(M))$.

Theorem 6.1. Let g, $\bar{g}$ be conformally equivalent metrics on M. Then

$$\hat{p}_k(M,g) = \hat{p}_k(M,\bar{g}).$$

Proof. The difference formula, (2.9), is the same as that in [9] for the difference of the $TP(\theta)$ forms. It is proved in [9] that these forms are conformally invariant, and thus so are the $\hat{p}_k$ characters.

Theorem 6.2. A necessary condition that M^n admit a conformal immersion in R^{n+k} is that $\hat{p}_i^{\perp}(M^n) = 0$ for $i > [\frac{k}{2}]$.

Proof. By the previous theorem we can assume the immersion is isometric. Let $N(M^n)$ be the normal bundle; and let $\bar{\nabla}$ be the globally flat Euclidean connection on $T(M) \oplus N(M)$. $\bar{\nabla}$ induces the Riemannian connection, ∇, on $T(M)$ and also induces a connection ∇' on $N(M)$. Setting $\mathcal{N}(M) = \{N(M),M,\nabla'\}$ we see that $\mathcal{N}(M)$ is an inverse of $\mathcal{T}(M)$. Thus by (5.5) $\hat{p}_i^{\perp}(M^n) = \hat{p}_i(\mathcal{N}(M^n)) = 0$ for $i > [\frac{k}{2}]$.

This theorem, together with Proposition 2.8 show that conformal immersion in R^{n+k} implies that the forms $TP_i^{\perp}(\theta)$ in the frame bundle are closed and represent integral cohomology. This is a main result of [9], where in fact it is shown that $\frac{1}{2}\, TP_i^{\perp}(\theta)$ is integral.

The next two theorems are due to John Millson, and are part of his doctoral dissertation.

Theorem 6.3 (Millson). Let M be a compact nonnegative space form. Then all $\hat{p}_i(M) \in H^{4i-1}(M,Q/Z)$.

Proof. In the flat case the tangent bundle is flat with finite holonomy and the theorem follows from Proposition 2.10. In the positive case $M = S^n/\Gamma$, where $\Gamma \subseteq O(n+1)$ is finite and acts freely. Let $\mathcal{F} = \{V^{n+1},M,\bar{\nabla}\}$ be the flat bundle over M associated to the inclusion representation of its fundamental group Γ. By topology $T(M) \oplus L' \cong V^{n+1}$, where L is a trivial line bundle, and one easily sees that the connection on $T(M)$ induced by $\bar{\nabla}$ is the Riemannian connection, ∇. Since $\mathcal{F}$ is flat $\bar{\nabla}$ is compatible with $\nabla \oplus \nabla'$, where ∇' is the trivial connection on L'. Thus by (5.3) and (5.4) $\hat{p}_i(M) = \hat{p}_i(\mathcal{F})$, and we again may use Proposition 2.10.

Example 6.4 (Millson). Let $M^{4k-1} = L_{p;q_1,\ldots,q_{2k}}$ be the lens space obtained by dividing S^{4k-1} by the cyclic group of order p generated by $(e^{\frac{2\pi i q_1}{p}},\ldots,e^{\frac{2\pi i q_{2k}}{p}})$, where the q_i and p are pairwise relatively prime. As in Ex. 2.5 the top characters give numbers,

$$\hat{p}_k(M^{4k-1}) \equiv \frac{q_1'\cdots q_{2k}'\sigma_k(q_1^2,\ldots,q_{2k}^2)}{p} \bmod Z.$$

where σ_k is the k^{th} elementary symmetric functions and $q_1'q_1 \equiv 1 \bmod p$. e.g. in the standard notation, the 3-dim lens space $L_{p,q} = L_{p;1,q}$, and

$$\hat{p}_1(L_{p,q}) \equiv \frac{q'(1+q^2)}{p} \equiv \frac{q+q'}{p} \bmod Z.$$

Coupling these calculations with the non-immersion theorem shows

Theorem 6.4 (Millson). *For each k there are infinitely many $(4k-1)$ dim lens spaces smoothly immersible in R^{4k} but not conformally immersible in R^{6k-1}.*

The nonnegative space forms themselves may be used as target manifolds.

Theorem 6.5. *A necessary condition that M^n be conformally immersible in a nonnegative space form $\overline{M}^{n+k}$ is that $\hat{p}_i^{\perp}(M^n) \in H^{4i-1}(M^n,Q/Z)$ for $i > [\frac{k}{2}]$.*

Proof. Reduce mod Q, and regard all $\hat{p}_i \in \hat{H}^{4i-1}(M,R/Q)$. Let T, $\overline{T}$, N be the Riemannian tangent bundles of M and $\overline{M}$, and the normal bundle of M together with the connection induced by $\overline{M}$. By restriction we regard $\overline{T}$ as Riemannian vector bundle over M. Because the curvature tensor of $\overline{T}$ is constant, its connection is compatible with that on $T \oplus N$, and by (5.3) and (5.4)

$$\hat{p}(\overline{T}) = \hat{p}(T) * \hat{p}(N) \quad \bmod Q.$$

But Theorem 6.3 shows $\hat{p}(\overline{T}) = 1 \bmod Q$, and so

$$\hat{p}_i^{\perp}(\overline{T}) = \hat{p}_i(N) = 0 \quad i > k.$$

To vanish as an R/Q character is equivalent to being a Q/Z cohomology class.

The case of constant negative curvature is considerably deeper. Since the characteristic forms all vanish, the $\hat{p}_i$ are R/Z classes, but it seems highly unlikely that they are all rational. These manifolds are all of the form $M^n = H^n/\Gamma$, where Γ acts freely and property discontinuously as isometries on the hyperbolic space H^n. Letting $<\ ,\ >_{n,1}$ be the Lorenz metric in $n+1$ space, we may identify $H^n = \{x \mid \|x\|_{n,1} = -1\}$, with the induced metric from $<\ ,\ >_{n,1}$, which is positive definite on this hypersurface. Now Γ is the fundamental group of M^n, and $\Gamma \subset 0(n,1)$. This gives a flat $0(n,1)$ vector bundle, F, over M. As in the proof of Theorem 6.3 one shows $\hat{p}_i(M^n) = \hat{p}_i(F)$. However, we get no rationality conclusion because the holonomy group, Γ, is not finite.

7. Foliations

Let $G = Gl(n,R)$, and set $I_o(G) = \ker w$. Then $I_o = \Sigma I_o^k$ is the ideal generated by the polynomials trA^{2k-1}. Taking $\Lambda = \{0\}$ we see that $Q \leftrightarrow (Q,0)$ is an isomorphism between I_o^k and $K^{2k}(G,\{0\})$. If $\alpha \in \varepsilon(Gl(n,R))$ and $Q \in I_o^k$ set

$$\hat{Q}(\alpha) = S_{Q,0}(\alpha) \in \hat{H}^{2k-1}(M,R).$$

Let F be a foliation of co-dim n in a manifold M, and let $N(F)$ be its normal bundle. In [4] Bott has defined a family of connections in $N(F)$, all of which have the property that their curvature transformations, $R_{x,y}$ vanish if $x, y \in F_m$. This guarantees $\overbrace{\Omega \wedge \ldots \wedge \Omega}^{k} = 0$ if $k > n$, and thus $P(\Omega) = 0$ if $P \in I^k(G)$. This shows that certain Pontrjagin classes vanish, and it also leads one to construct secondary cohomology invariants. Let $N(F) = \{N(F),M,\nabla\}$ where ∇ is such a Bott connection and set $\hat{Q}(F) = \hat{Q}(N(F))$, $\hat{p}_i(F) = \hat{p}_i(N(F))$. Bott's curvature vanishing theorem shows:

$$\hat{Q}(F) \in H^{2k-1}(M,R) \qquad Q \in I_o^k,\ \ k > n \tag{7.1}$$

$$\hat{p}_i(F) \in H^{4i-1}(M,R/Z) \qquad 2i > n. \tag{7.2}$$

A simple application of Proposition 2.9 yields

<u>Proposition 7.3</u>. <u>The classes $\hat{Q}(F)$ and $\hat{p}_i(F)$ are defined independently of choice of Bott connection, and are thus invariants of F.</u>

It is straightforward to show that the classes $\hat{Q}(F)$ and $\hat{p}_i(F)$ are <u>natural</u> under smooth maps transverse to F, and that they are cobordism invariants. Thus, letting $B\Gamma_n$ denote Haefliger classifying space for foliations, see [5], we get

$$\hat{Q} \in H^{2k-1}(B\Gamma_n, R) \qquad k > n$$

$$\hat{p}_i \in H^{4i-1}(B\Gamma_n, R/Z) \qquad 2i > n.$$

The classes $\hat{Q}$ have been defined independently by others. For example $(\hat{tr}A)^2 \in H^3(B\Gamma_1, R)$ is the Godbillon-Vey invariant. An extensive treatment of the $\hat{Q}$ classes may be found in [6]. The $\hat{p}_i$ classes are non-vanishing:

Theorem 7.4. *Let* $\psi : B\Gamma_n \to B_{Gl(n,R)}$ *be the natural map. Then letting* B *denote Bockstein*

$$B(\hat{p}_i) = -\psi^*(p_i).$$

Corollary 7.5. $\hat{p}_i \neq 0$.

The proof of the theorem is immediate from Corollary 2.4. The corollary then follows from examples of Bott-Heitsch [6] of foliations of co-dim n, the i^{th} integral Pontrjagin classes of whose normal bundles do not vanish for $i > n$. This shows $\psi^*(p_i) \neq 0$ and thus $\hat{p}_i \neq 0$.

§8. Flat Bundles

Let G be a Lie group with finitely many components, and let $\rho : \pi_1(M) \to G$ be a representation. Associated to ρ we get a flat G-bundle E_ρ. For $(P,u) \in K^{2k}(G,\Lambda)$ we set $u(\rho) = u(E_\rho)$ and $S_{P,u}(\rho) = S_{P,u}(E_\rho)$. Corollary 2.4 shows

$$S_{P,u}(\rho) \in H^{2k-1}(M, R/\Lambda) \tag{8.1}$$

$$B(S_{P,u}(\rho)) = -u(\rho). \tag{8.2}$$

Let $\phi : N \to M$ be smooth. Then $\rho \circ \phi : \pi_1(N) \to G$, and Theorem 2.2 shows

$$S_{P,u}(\rho \circ \phi) = \phi^*(S_{P,u}(\rho)). \tag{8.3}$$

Let G_o denote G equipped with the *discrete* topology, and let B_{G_o} denote its classifying space. The identity map $G_o \xrightarrow{j} G$ is continuous and induces $B_{G_o} \xrightarrow{j} B_G$. For $u \in H^{2k}(B_G,\Lambda)$, we get $u = j^*(u) \in H^{2k}(B_{G_o},\Lambda)$. From (8.1) and (8.3) one shows

$$S_{P,u} \in H^{2k-1}(B_{G_o},R/\Lambda) \tag{8.4}$$

$$B(S_{P,u}) = -u. \tag{8.5}$$

Any representation $\rho : \pi_1(M) \to G$ can be factored as $\pi_1(M) \xrightarrow{\rho_o} G_o \xrightarrow{i} G$. Since ρ_o is continuous it induces $\rho_o : M \to B_{G_o}$, and

$$\rho_o^*(S_{P,u}) = S_{P,u}(\rho). \tag{8.6}$$

<u>Proposition 8.7</u>. <u>Let</u> $(P,u) \in K^{2k_1}(G,\Lambda), (Q,v) \in K^{2k_2}(G,\Lambda)$, <u>and let</u> $(R/\Lambda)_{tor} \subseteq R/\Lambda$ <u>denote the torsion subgroup</u>.

$$S_{PQ,u\cup v} = u \cup S_{Q,v} \in H^*(B_{G_o},(R/\Lambda)_{tor}).$$

<u>Proof</u>. For any $\rho : \pi_1(M) \to G$, Corollary 2.3 and (1.14) show $S_{PQ,u\cup v}(\rho) = S_{P,u}(\rho) * S_{Q,v}(\rho) = u(\rho) \cup S_{Q,v}(\rho)$. Moreover, since E_ρ is flat, $u(\rho) = u(E_\rho) \in H^{2k}_{tor}(M,\Lambda)$ and $H^*_{tor}(M,\Lambda) \cup H^*(M,R/\Lambda) \subseteq H^*(M,R/\Lambda)_{tor})$.

In particular we see

$$S_{PQ,u\cup v} \in H^*(B_{G_o},Q/Z) \qquad \Lambda = Z \tag{8.8}$$

$$S_{PQ,u\cup v} = 0 \qquad \Lambda = Q.$$

If $\rho^t : \pi_1(M) \to G$ is a family of representations we call it <u>smooth</u> if, for each $h \in \pi_1(M)$, $\rho^t(h)$ is a smooth curve in G. Using Proposition 2.9 one easily shows

<u>Proposition 8.10</u>. <u>If</u> $\rho^t : \pi_1(M) \to G$ <u>is smooth, and</u> $k \geq 2$, <u>then</u> $S_{P,u}(\rho^0) = S_{P,u}(\rho^1)$.

As we will see below in the case of $\hat{\chi} \in H^1(B_{SO(2)_o},R/Z)$; the condition $k \geq 2$ is necessary in this theorem.

A dominating question will be the <u>values</u> taken by the $S_{P,u}$ classes when they are regarded as characters on $H_{2k-1}(B_{G_o})$. Propositions 8.7 and 8.10 show that for $k \geq 2$ elementary constructions will not produce values outside of $(R/\Lambda)_{tor}$. Moreover, Proposition

8.10 seems to indicate that the range of values is countable.

The Euler Character. Let $\rho : \pi_1(M) \to SO(n)$, and let V_ρ be the corresponding flat vector bundle. Let $SV \to M$ be the associated sphere bundle, and let $\omega \in \Lambda^{n-1}SV$ be the volume form on S^{n-1}, normalized so $\int_{S^{n-1}} \omega = 1$, and extended via the connection to a form, ω, on SV. Since SV is flat, $d\omega = o$ and defines $[\omega] \in H^{n-1}(SV,R)$.

Let $z \in H_{n-1}(M)$, and choose $y \in H_{n-1}(SV)$ so that $\pi_*(y) = z$. Sequence (3.1) shows that such a y exists and is unique up to a homology class in the fibre. We then define $\hat{\chi}(\rho) \in Hom(H_{n-1}(M),R/Z) = H^{n-1}(M,R/Z)$ by

$$\hat{\chi}(\rho)(z) = [\omega](y). \tag{8.11}$$

This agrees with the original definition of $\hat{\chi}$ given in §3 for arbitrary SO(2n) bundles, and extends it, in the flat case, to all SO(n).

Proposition 8.12.

1) $\hat{\chi}(\rho)$ has order 2 $\qquad$ n odd

2) $\hat{\chi}(\rho) \in H^{n-1}(M,Q/Z)$ $\qquad$ $\rho(\pi_1(M))$ finite

3) $\hat{\chi}(\rho_1 \oplus \rho_2) = \chi(\rho_1) \cup \hat{\chi}(\rho_2) \in H^*(M,Q/Z)$.

Proof. Let $A : SV \to SV$ be the antipodal map. Since $\pi \circ A = \pi$, we could use $A_*(y)$ instead of y in (8.11). For n odd, $A^*(\omega) = -\omega$, and this shows 1). 2) follows from Proposition 2.10, and 3) follows from Theorem 3.6 and (1.14).

In the case $n = 2$, we are dealing with a flat circle bundle, and $\hat{\chi}(\rho)$ assigns to each closed curve in M its associated angle of holonomy (see Ex. 1.5). In particular if $M = S^1$ and $\rho : \pi_1(S)^1 \to SO(2)$, then $\rho(1) = e^{2\pi i\alpha}$, and $\hat{\chi}(\rho)(S^1) = \alpha$. Since ρ may be smoothly perturbed so that $\rho(1)$ takes any value in SO(2), we see that $k \geq 2$ is necessary in Proposition 8.10. Now $H_1(B_{SO(2)_o}) \cong SO(2)_o/[SO(2)_o,SO(2)_o] \cong SO(2)_o \cong R/Z$. One easily shows

$$\hat{\chi} : H_1(B_{SO(2)_o}) \xrightarrow{\cong} R/Z. \tag{8.13}$$

The higher dimensional cases are more interesting. Let M^{2n-1} be compact and oriented, let $\rho : \pi_1(M^{2n-1}) \to SO(2n)$ be a representation, and let SV be the associated flat (2n-1) dim sphere bundle.

Let $m_1,\ldots,m_N$ be the vertices of a simplicial subdivision of M^{2n-1}. For each vertex choose $v_j \in SV_{m_j}$. If $\sigma_i = (m_{i_0},\ldots,m_{i_{2n-1}})$ is a top dimensional simplex let b_1 denote its barycenter, and let $w_{i_0},\ldots,w_{i_{2n-1}} \in SV_{b_i}$ be the vectors obtained by parallel translating $v_{i_0},\ldots,v_{i_{2n-1}}$ along curves in σ_i. Note that since SV is flat the $\{w_{i_j}\}$ do not depend on the choices of these curves. We call $v_1,\ldots,v_N$ generic if for each σ_i the vectors $w_{i_0},\ldots,w_{i_{2n-1}}$ are linearly independent. It is easily seen that the set of generic N-tuples $v_1,\ldots,v_N$ form an open dense subset of $SV_{m_1}\times\ldots\times SV_{m_N}$. In the generic case let $\Sigma_i \subsetneq S^{2n-1}$ denote the unique convex oriented geodesic simplex spanned by $w_{i_0},\ldots,w_{i_{2n-1}}$, and let $\mathrm{Vol}(\Sigma_i)$ denote its oriented volume.

Theorem 8.14. *Let $v_1,\ldots,v_N$ be generic and let $\sigma_1+\ldots+\sigma_k$ be a fundamental cycle of M^{2n-1}. Let S^{2n-1} be normalized to have unit volume. Then*

$$\hat{\chi}(\rho)(M^{2n-1}) = \Sigma\ \widetilde{\mathrm{Vol}(\Sigma_i)}.$$

This theorem suggests a direct definition of $\hat{\chi}$ as a cocycle in the bar resolution of $SO(2n)_o$. We recall that k-simplices are $(k+1)$ tuples of group elements $(g_o,\ldots,g_k)$, under the equivalence $(g_o,\ldots,g_k) \sim (hg_o,\ldots,hg_k)$, and $\partial(g_o,\ldots,g_k) = \Sigma_{i=0}^{k}(-1)^i(g_o,\ldots,\hat{g}_i,\ldots,g_k)$. The homology and cohomology of this complex are isomorphic to that of B_{G_o}.

Let us fix $e \in S^{2n-1}$, and call $\sigma = (g_o,\ldots,g_{2n-1})$ generic if $g_o(e),\ldots,g_{2n-1}(e)$ are linearly independent in R^{2n}. The generic simplices form an open dense subset of $\overbrace{SO(2n \times\ldots\times SO(2n)}^{2n}$. For σ generic let $\Sigma(\sigma) \subsetneq S^{2n-1}$ be the convex, oriented, geodesic simplex spanned by $g_o(e),\ldots,g_{2n-1}(e)$. Let S^{2n-1} be normalized to have unit volume, and set

$$\widetilde{\mathrm{Vol}}(\sigma) = \widetilde{\mathrm{Vol}(\Sigma(\sigma))} \in R/Z.$$

Since $\widetilde{\mathrm{Vol}}((hg_o,\ldots,hg_{2n-1})) = \widetilde{\mathrm{Vol}}((g_o,\ldots,g_{2n-1}))$, $\widetilde{\mathrm{Vol}}$ defines an $(2n-1)$ cochain on the generic simplexes.

Let $\gamma = (g_o,\ldots,g_{2n})$ be a $2n$ simplex, all of whose $(2n-1)$ faces are generic, and let γ_i denote its i^{th} face. Then

$$(\delta\ \widetilde{\mathrm{Vol}})(\gamma) = \widetilde{\mathrm{Vol}}(\sigma\gamma) = \widetilde{\mathrm{Vol}(\sum_{i=0}^{2n}(-1)^i\ \Sigma(\gamma_i)0} = 0$$

since $\Sigma_{i=0}^{2n}(-1)^i\Sigma(\gamma_i)$ is a $(2n-1)$ dim singular cycle on S^{2n-1}, and so by our normalization has integral volume. Thus $\widetilde{\mathrm{Vol}}$ is a co-cycle on its domains of definition and clearly defines, almost everywhere a Borel map (in the sense of [14] from $SO(2n)\times\ldots\times SO(2n)\to R/Z$. It is shown in [14] that such a cochain can be extended to a cocycle on all chains, and all such extensions are cohomologous. Thus $[\widetilde{\mathrm{Vol}}]\in H^{2n-1}(B_{SO(2n)_o},R/Z)$ is well defined. It is easily shown that $[\widetilde{\mathrm{Vol}}]$ is independent of the choice of e. By using the previous theorem one shows

Theorem 8.15. $\hat{\chi}=[\widetilde{\mathrm{Vol}}]$

and thus we obtain

Corollary 8.16. $\hat{\chi}\in H^{2n-1}(B_{SO(2n)_o},R/Z)$ *is a Borel cohomology class.*

Let Range $\hat{\chi}$ be the image of the map $\hat{\chi}:H_{2n-1}(B_{SO(2n)_o})\to R/Z$. By (8.13) this is of interest only for $n\geq 2$.

Proposition 8.17. Range $\hat{\chi}\supseteq Q/Z$.

Proof. Let $\Gamma\subseteq SO(2n)$ be a finite subgroup acting freely on S^{2n-1}. Let $E\to S^{2n-1}/\Gamma$ be the associated flat $SO(2n)$ bundle. The associa- sphere bundle has a canonical cross-section defined by the normal field to S^{2n-1} in R^{2n}. Using it in (8.11) shows $\hat{\chi}(E)(S^{2n-1}/\Gamma)\equiv 1/\mathrm{ord}(\Gamma)$ mod Z. For any k we may choose $\Gamma\cong Z_k$.

Let $\Sigma\subseteq S^n$ be an n-dim goedesic simplex. Let $\pi x_{i,j}$ denote the dihedral angle between the i^{th} and j^{th} face. The set $(x_{1,2},\ldots,x_{n,n+1})$ determines Σ up to congruence, and we call Σ rational if all $x_{i,j}\in Q$. Normalize S^n to have volume 1. Then for $n=2$, the Gauss-Bonnet theorem gives

$$\mathrm{Vol}(\Sigma) = \frac{1}{4}(x_{1,2}+x_{1,3}+x_{2,3}-2),$$

and this shows that rational simplexes in S^2 have rational volume. For $n\geq 3$ this is probably false. In particular

$$\mathrm{Vol}(\Sigma) = f(x_{1,2},\ldots,x_{n,n+1})$$

is a non-elementary transcendental function (see [10], [16]). It seems highly unlikely that f takes rational values at all rational points, but we do not know a counterexample. The following theorem is due to W. Thurston.

Theorem 8.18 (Thurston). For all but a finite number of rational 3-simplexes $\Sigma \subseteq S^3$, there exists an integer $m \neq 0$ and $r \in Q$ so that

$$\widetilde{m\ \mathrm{Vol}(\Sigma)} + r \in \mathrm{Range}\ \hat{\chi}.$$

Thus $\hat{\chi}$ takes irrational values on $H_3(B_{SO(3)_o})$ unless all but a finite number of rational geodesic 3-simplices have rational volume.

The proof of this theorem depends on Theorem 8.14 and a recent and unpublished construction of Thurston. This construction consists of associating to a given rational simplex Σ, the denominators of whose dihedral angles are sufficiently large, a compact manifold M^3 of constant negative curvature and a representation $\rho : \pi_1(M^3) \to SO(4)$. He then shows that $\hat{\chi}(\rho) \equiv m\ \mathrm{Vol}(\Sigma) \bmod Q$.

We should emphasize that information on the values of $\hat{\chi}$ gives a lower bound for the homology group $H_{2n-1}(SO(2n))$. For example, Proposition 8.17 shows that this group has a nontrivial homomorphism onto some group, $H \supseteq Q/Z$ and hence is not finitely generated. Similarly Theorem 8.18 implies

Corollary 8.19. Let V be the vector space over the rationals generated by the volumes of the (all but finitely many) rational geodesic simplices of Theorem 8.18. Then $\mathrm{Rank}\ H_3(B_{S)(4)_o}) \geq \dim V - 1$.

The Chern Characters. Let $\rho : \pi_1(M) \to U(n)$ and let V_ρ be the associated, flat, hermitian bundle. Let $V_{n-k+1} \to M$ be the flat Stiefel bundle with fiber the Stiefel manifold E_{n-k+1}. We recall from (4.2) that $H_{2k-1}(E_{n-k+1}) \cong Z$. Let ω_{2k-1} be the unique harmonic (2k-1) form on E_{n-k+1} whose value on the generator $U(k)/U(k-1)$ is 1. Since V_{n-k+1} is flat, ω_{2k-1} defines a closed form on V_{n-k+1}, and we denote its cohomology class by $[\omega_{2k-1}]\ H^{2k-1}(V_{n-k+1},R)$.

Let $z \in H_{2k-1}(M)$, and choose $y \in H_{2k-1}(V_{n-k+1})$ with $\pi_*(z) = y$. Sequence (4.2) shows that such a z exists and is unique up to a multiple of the generator of $H_{2k-1}(E_{n-k+1})$. We define $\hat{c}_k(\rho) \in H^{2k-1}(M,R/Z) \cong \mathrm{Hom}\ (H_{2k-1}(M),R/Z)$ by

$$\hat{c}_k(\rho)(z) = [\tilde{\omega}_{2k-1}](y). \tag{8.19}$$

This definition agrees, for flat bundles, with the general definition given in (4.4).

Since $U(n)_o \subseteq SO(2n)_o$, we may also consider $\hat{\chi}(\rho)$, and (4.9) shows

$$\hat{c}_n(\rho) = -\hat{\chi}(\rho). \tag{8.20}$$

For any space X with $\alpha \in H^{\ell}(X,R/Z)$, $\beta \in H^k(X,R/Z)$, set $\alpha * \beta = -B(\alpha) \cup \beta \in H^{k+\ell+1}(X,R/Z)$. Note

$$B(\alpha * \beta) = B(\alpha) \cup B(\beta). \tag{8.21}$$

It is easily seen that for any Lie group, G, $H^*_{\text{Borel}}(B_{G_o},R/Z)$ forms a ring under $*$ product.

<u>Theorem 8.21</u>. $\hat{c}_1,\ldots,\hat{c}_n \in H_{\text{Borel}}(B_{U(n)_o},R/Z)$. <u>Moreover, under $*$ product, they are a complete set of generators of the ring</u>.

<u>Proof</u>. That $\hat{c}_n$ is Borel follows from (8.20) and Corollary (8.16). However, to prove the lower $\hat{c}_k$ are Borel one needs a formula for $\hat{c}_k$ in the bar resolution of $U(n)_o$, similar to that for $\hat{\chi}$. This may be derived using (8.19), however, it does not have the canonical flavor of $[\widetilde{\text{Vol}}]$, and we omit the details.* The simple exception is $\hat{c}_1$, and

$$\hat{c}_1(g_o,g_1) = \widetilde{\frac{1}{2\pi i}\log\det(g_o^{-1}g_1)}.$$

Let $j : B_{U(n)_o} \to B_{U(n)}$ be the natural map, and let $j : H^*(B_{U(n)},Z) \to H^*(B_{U(n)_o},Z)$ be the associated map. In his thesis, [18] Wigner shows

$$B : H^*_{\text{Borel}}(B_{U(n)_o},R/Z) \xrightarrow{\cong} \operatorname{Im} j^*.$$

But $\operatorname{Im} j^*$ is the ring generated by $j^*(c_1),\ldots,j^*(c_n)$ and by (8.5)

$$B(\hat{c}_k) = -j^*(c_k).$$

Since the $\hat{c}_k$ are Borel, and B maps $*$ products into cup products we are done.

We need not be restricted to $U(n)$. If $\rho : \pi_1(M) \to Gl(n,c)$, following (4.5), (4.10), and (8.1) we define

$$\hat{c}_k(\rho) \quad H^{2k-1}(M,R/Z)$$

$$\hat{ch}(\rho) \quad H^{odd}(M,R/Q).$$

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* We wish to thank John Millson for acquainting us with Wigner's thesis and for suggesting that the $\hat{\chi}$, $\hat{c}_k$ might be Borel.

Theorem 8.22.

1) $\hat{c}_k(\rho_1 \oplus \rho_2) = \hat{c}_k(\rho_1) + \hat{c}_k(\rho_2) + \sum_{i=1}^{k-1} c_i(\rho_1) \cup \hat{c}_{k-i}(\rho_2)$

2) $\hat{ch}(\rho) = n + \sum_{i=1}^{n} \frac{(-1)^{i-1}}{(i-1)!} \hat{c}_i(\rho) \mod Q$

3) $\hat{ch}(\rho_1 \oplus \rho_2) = \hat{ch}(\rho_1) + \hat{ch}(\rho_2)$

4) $\hat{ch}(\rho_1 \otimes \rho_2) = n_1\hat{ch}(\rho_2) + n_2\hat{ch}(\rho_1) - n_1 n_2.$

Proof. 1) follows immediately from Theorem 4.6 and (1.14). $\hat{ch}$ is only defined mod Q, and $\hat{c}_i(\rho) * \hat{c}_j(\rho) = c_i(\rho) \cup \hat{c}_j(\rho) = 0 \mod Q$ since $c_i(\rho)$ is torsion. Thus in the general formula for $\hat{ch}$ (e.g. see 4.10) all product terms vanish, and 2) is what remains. 3) is immediate from Theorem 4.11, and so is 4) by virtue of 2).

Let $R(\pi_1(M))$ denote the rational unitary representation ring of $\pi_1(M)$, and $I(\pi_1(M))$ the augmentation ideal of virtual representations of dim 0. Clearly $\hat{ch}$ extends to $R(\pi_1(M))$ and defines a homomorphism $\hat{ch} : R(\pi_1(M)) \to H^{odd}(M,R/Q)$ as Q-modules. 4) of the above theorem shows $\hat{ch}(\rho_1 \otimes \rho_2) = 0$ if $\rho_1, \rho_2 \in I(\pi_1(M))$. This

$$\hat{ch} : I(\pi_1(M))/I^2(\pi_1(M)) \to H^{odd}(M,R/Q)$$

is a well defined Q-module homomorphism. If we suppose L to be a finitely generated group whose classifying space, B_L, is a finite dimensional manifold we get

$$\hat{ch} : I(L)/I^2(L) \to H^{odd}(B_L,R/Q).$$

At this point we have no information as to the kernel and range of this map.

We remark that by constructions completely analogous to those given in this section, it is possible to give explicit cocycles representing the real continuous cohomology of noncompact Lie groups. This cohomology, which corresponds to invariant polynomials on g/k (k-maximal compact) does not in general come as a special case of the $\hat{H}^k$.

9. The Geometric Index Theorem of Atiyah-Patodi-Singer

Let $L_k = L_k(p_1,\ldots,p_k) \in H^{4k}(B_{Gl(n,R)},Q)$ denote the k^{th} universal L-class and let $P_{L_k} = L_k(P_1,\ldots,P_k)$ denote the corresponding invariant polynomial. If $V = \{V^n,M,\nabla\}$ is a real vector bundle with connection we let

$$L(V) = 1 + L_1(V^n) + \ldots + L_{[k/2]}(V) \in H^*(M,Q)$$

$$P_L(V) = 1 + P_{L_1}(\Omega) + \ldots + P_{L_{[k/2]}}(\Omega)$$

$$\hat{L}(V) = 1 + \hat{L}_1(V) + \ldots + \hat{L}_{[k/2]}(V) \in \hat{H}^*(M,R/Q)$$

denote the corresponding rational class, characteristic form, and differential character. The $\hat{L}_i$ can of course be written in terms of the $\hat{p}_i$ and $*$ product, e.g.

$$\hat{L}_1 = \frac{1}{3}\hat{p}_1 \qquad \hat{L}_2 = \frac{7\hat{p}_2 - \hat{p}_1 * \hat{p}_1}{45}$$

If $\{M,g\}$ is a Riemannian manifold we let $L(M)$, $P_L(M,g)$, $\hat{L}(M,g)$ be the class, form, and character corresponding to the Riemannian tangent bundle. In spite of the fact that $L(M)$ is an <u>integral</u> class, it is impossible to refine $\hat{L}(M,g)$ to get an R/Z character which maps naturally under isometries. The R/Q character, $\hat{L}(M,g)$ is of course natural.

Let $\{M,g\}$ be compact, oriented, and odd dimensional, and let $V = \{V,M,\nabla\}$ be a complex Hermitian vector bundle. Let $\Lambda^k(M,V)$ denote V-valued k-forms. The connection on V allows one to define $d : \Lambda^k(M,V) \to \Lambda^{k+1}(M,V)$, and the metric on M defines $* : \Lambda^k(M,V) \to \Lambda^{n-k}(M,V)$. Define

$$T : \Sigma \oplus \Lambda^{2p}(M,V) \to \Sigma \oplus \Lambda^{2p}(M,V)$$

by

$$T = *d + (-1)^p d* \qquad \dim M = 4k-1$$

$$T = i(*d + (-1)^p d_*) \qquad \dim M = 4k+1.$$

In [3], Atiyah-Patodi-Singer study this <u>symmetric elliptic operator</u>. It has discrete spectrum with infinite positive and negative range. Letting $\{\lambda_i\}$, $\{\gamma_i\}$ denote its strictly positive and strictly negative spectrum they form the function of a complex variable s,

$$N_V(s) = \sum_{i=1}^{\infty} \lambda_i^{-s} - \sum_{i=1}^{\infty} (-\gamma_i)^{-s}$$

and show this to be continuable to a meromorphic function in the entire plane. They also show that $N(0)$ <u>is real and finite</u>. Set

$$N(V) = N_V(0).$$

Now suppose that $M = \partial\bar{M}$ and that $\mathcal{V}$ extends to $\bar{\mathcal{V}} = \{\bar{V},\bar{M},\bar{\nabla}\}$. Let $\bar{g}$ be any metric on $\bar{M}$ which induces g on M, and which is product metric in a collar neighborhood of M.

Theorem 9.1 (Atiyah-Patodi-Singer)

$$(-1)^{k+1}\eta(\mathcal{V}) = \int_{\bar{M}} P_{ch}(\bar{\mathcal{V}}) \wedge P_L(\bar{M},\bar{g}) + N(\bar{M},M,V)$$

where $N(\bar{M},M,V)$ *is the index of a certain boundary value problem associated to the data and is therefore an integer.*

The left side of this equation is clearly an intrinsic function of the odd dimensional Riemannian manifold, M, and the Hermitian vector bundle $\{V,M,\nabla\}$. Therefore of course, so is the right side. It, however, is, defined only when M is a boundary and when V extends over the interior. One can avoid this restriction and get an intrinsic right hand side which is always defined by working mod A. Some topology is lost, but one gains naturality and some computational facility.

Theorem 9.2. *For all complex, Hermitian, Riemannian vector bundles* $\mathcal{V}$ *over* $\{M,g\}$

$$(-1)^{k+1}\eta(\mathcal{V}) \equiv (\hat{ch}(\mathcal{V}) * \hat{L}(M,g))(M) \mod Q.$$

Proof. It is always the case that one can find an integer ℓ, and a compact manifold, $\bar{M}$, so that $\partial\bar{M} = \ell M$ and so that ℓV extends to $\bar{V}$ over $\bar{M}$, where ℓV is V on each component of ℓM. Extending the connection to $\bar{\nabla}$ on $\bar{V}$, and choosing a metric $\bar{g}$ on $\bar{M}$, product near the boundary, we get

$$(-1^{k+1}\eta(\ell\mathcal{V}) = \int_{\bar{M}} P_{ch}(\bar{\mathcal{V}}) \wedge P_L(\bar{M},\bar{g}) + \text{integer}.$$

Clearly $\eta(\ell\mathcal{V}) = \ell\eta(\mathcal{V})$, and working mod Q,

$$(-1)^{k+1}\eta(\mathcal{V}) = \frac{1}{\ell}\int_{\bar{M}} P_{ch}(\bar{\mathcal{V}}) \wedge P_L(\bar{M},\bar{g})$$

$$= \frac{1}{\ell}\,[\delta_1(\hat{ch}(\bar{\mathcal{V}})) \wedge \delta_1(\hat{L}(\bar{M},\bar{g}))](\bar{M})$$

$$= \frac{1}{\ell}\;\hat{\delta}_1(\hat{ch}(\bar{\mathcal{V}}) * \hat{L}(\bar{M},\bar{g}))(\bar{M})$$

$$= \frac{1}{\ell}\,[\hat{ch}(\ell\mathcal{V}) * \hat{L}(\mathcal{T}(\bar{M}) \mid \ell M)](\ell M).$$

The assumption of product metric means that $\mathcal{T}(M) \mid \ell M = \mathcal{T}(\ell M) \oplus L$, where L is a trivial Riemannian line bundle. Thus

$$(-1^{k+1}\eta(V) = \frac{1}{\ell}[\hat{c}h(\ell V) * \hat{L}(\ell M,g)](\ell M) = [\hat{c}h(V) * \hat{L}(M,g)](M).$$

This formula seems of interest for flat bundles. Let $\rho : \pi_1(M) \to U(n)$, and set

$$\eta(\rho) = \eta(V_\rho).$$

Using Theorem 8.22 and (1.14) we see

<u>Corollary 9.3</u>. If $\dim M = 4k+1$ then

$$(-1)^{k+1}\eta(\rho) \equiv \sum_{i=0}^{k} \frac{1}{(2k-2i)!} L_i(M) \cup \hat{c}_{2(k-1)+1}(p)\ (M) \quad \text{mod } Q$$

$$\eta(\rho_1 \otimes \rho_2) \equiv n_2\eta(\rho_1) + n_1\eta(\rho_2) \quad \text{mod } Q.$$

If $\dim M = 4k-1$ then

$$(-1^{k+1}\eta(\rho) \equiv n\hat{L}_k(M-g) - \sum_{i=0}^{k-1} \frac{1}{(2k-2i-1)!} L_i(M) \cup \hat{c}_{2(k-1)}(\ell)(M) \quad \text{mod } Q$$

$$\eta(\rho_1 \otimes \rho_2) \equiv n_2\eta(\rho_1) + n_1\eta(\rho_2) - n_1 n_2 \hat{L}_k(M,g) \quad \text{mod } Q.$$

Let L^n be the trivial complex bundle of dimension $n = \dim \rho$. That $\eta(\rho) - \eta(L^n)$ depends only on ρ was proved in [3] by showing that the derivative under change of metric is zero. The above formulae clarify this independence and calculate $\eta(\rho) - \eta(L^n)$ mod Q, explicitly in terms of characteristic classes of ρ and M. (Note that in case $\dim M = 4k+1$, $\eta(L^n) = 0$).

<u>Example 9.4</u>. Let M be a compact 3-manifold with an orientation reversing isometry. Then $\hat{L}_1(M,g) = 0$ and

$$\eta(\rho) \equiv -\hat{c}(\rho)(M) = \hat{\chi}(\rho)(M) \quad \text{mod } Q.$$

By (8.20) and Theorem 8.14 we see that the series $\eta(\rho)$ may be evaluated, up to a rational, as a sum of simplex volumes on S^3.

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