Chern-Simons Notes

Charlie Reid

November 2, 2019

1 Principal bundles and Chern Weil Theory

Let G be a lie group and let g be its lie algebra. Given a principal G bundle $\pi : P \to M$ with connection $\Theta \in \Omega^1(P, \mathfrak{g})$ we define its curvature to be

$$\tilde{\Omega} = d\Theta + [\Theta \wedge \Theta]$$

The expression $[- \wedge -]$ is defined to be the composition

$$\Omega^1(P, \mathfrak{g}) \otimes \Omega^1(P, \mathfrak{g}) \to \Omega^2(P, \mathfrak{g} \otimes \mathfrak{g}) \to \Omega^2(P, \mathfrak{g})$$

If we have a lie algebra element $X \in \mathfrak{g}$, it induces a vector field ξ_X on P. We can compute that $\mathcal{L}_{\xi_X}\tilde{\Omega} = ad_\xi(\tilde{\Omega})$ and $\iota_{X_\xi}\tilde{\Omega} = 0$. Together these imply that $\tilde{\Omega}$ is the pullback of a form Ω on M valued in the adjoint bundle. We also call Ω the curvature. Now suppose $f : \mathfrak{g}^{\otimes l} \to \mathbb{R}$ is a degree l invariant polynomial. We define the corresponding Chern Weil Form as follows.

$$w_f(\Theta) := f(\Omega \wedge ... \wedge \Omega) \in \Omega^{2l}(M)$$

$w_f(\Theta)$ is closed, thus represents a real cohomology class on M. These closed forms are natural in that if we have a map $\phi : N \to M$, then

$$w_f(\phi^*\Theta) = \phi^* w_f(\Theta)$$

In other words, f gives us a natural transformation

$$w_f : B_\nabla G \to \Omega^{2l}_{cl}$$

Where $B_\nabla G$ denotes the functor sending a manifold to its groupoid of principal G bundles with connection. It turns out we can lift w_f naturally to a Cheeger-Simons differential character.

2 Chern-Simons forms

First we define the Chern-Simons form for a pair of connections. Let $\pi : P \to M$ be a principal bundle. Let Θ_0, Θ_1 be connections on P. Let Θ_t denote the connection on the principal bundle $P \times I \to M \times I$ given by

$$\Theta_t = \Theta_0 + t(\Theta_1 - \Theta_0)$$

where t is the coordinate on I. Let Ω_t be the curvature of Θ_t.

Definition.

$$\alpha_f(\Theta_0, \Theta_1) := \int_{M\times I/M} f(\Omega_t) \in \Omega^{2l-1}(M)$$

By a families version of stokes theorem, we have

$$d\alpha_f(\Theta_0, \Theta_1) = w_f(\Theta_1) - w_f(\Theta_0)$$

Which is the main important property. In some sence we only care about α_f up to exact forms but I'm not sure exactly what sense this is, and anyway it is convenient that it is cannonically defined on the nose. I'm not sure how this is useful, but note that we can easily extend this definition by taking the linear interpolation of $n+1$ connections on M times the n-simplex.

$$\alpha_f^n(\Theta_0, ..., \Theta_n) := \int_{M \times \Delta^n / M} w_f(\Theta_t)$$

These forms satisfy a nice relation

$$d\alpha_f^n(\Theta_0, ..., \Theta_n) = \sum_i (-1)^i \alpha_f^{n-1}(\Theta_0, ..., \widehat{\Theta_i}, ..., \Theta_n)$$

Anyway, if we have a single connection Θ, we can define a natural Chern-Simons form, but it lives on P instead of M. $\pi^* P$ has a tautological section, thus a tautological flat connection $\tilde{\Theta}$

Definition. $\alpha_f(\Theta) = \alpha_f(\tilde{\Theta}, \pi^*\Theta)$

This $2l-1$ form satisfies the relation

$$d\alpha_f(\Theta) = \pi^* w_f(\Theta)$$

3 Cheeger-Simons Differential Character

A Cheeger-Simons differential charactar on M of degree n is a homomorphism

$$\alpha : Z^{n-1}(M) \to \mathbb{R}/\mathbb{Z}$$

from smooth $n-1$ cycles on M to $\mathbb{R}/\mathbb{Z}$ such that there exists a closed n form with integral periods which, when integrated on a smooth n chain c gives $\alpha(\partial c)$. It turns out that this closed n form is unique, and we can also extract an integral degree n cohomology class from α. Cheeger and Simons construct charactaristic differential charactars $S_{f,u}$ of principal bundles with connection given the data of an invariant polynomial f which defines a cohomology class of BG with integral periods, and a lift u of this class to integral cohomology.

3.1 Trivializable Bundles

For simplicity, assume $P \to M$ has sections. Choose a section s. $s^*\alpha_f(\Theta)$ is a $2l-1$ form, thus gives us a $2l-1$ co-chain h on M via integration. Suppose we have a $2l$ dimensional smooth chain c in M. Then by stokes theorem,

$$\int_c w_f = \int_{\partial c} s^*\alpha_f = h(\partial c)$$

Thus we have a Cheeger-Simons differential charactar of degree $2l$ on M. We should check if it depended on the section s. If we have another section s', than there is a gauge transformation $\phi : P \to P$ taking s to s'. We can put the connection Θ_t on $P \times I$ which linearly interpolates from Θ to $\phi^*\Theta$. Let z be a cycle in M. By stokes theorem:

$$\int_z \alpha_f(\phi^*\Theta) - \int_z \alpha_f(\Theta) = \int_{z \times I} w_f(\Theta_t)$$

We can form a principal bundle $\tilde{P}$ on $M \times S^1$ using ϕ as a clutching function. This shows the difference is an integer, because w_f represents an integral cohomology class. Thus h is a well defined homomorphism from $2l-1$ cycles to $\mathbb{R}/\mathbb{Z}$.

In slightly more generality, we could just assume P restricted to any $2l-1$ cycle has sections. This covers the important case where G is simply connected and $l=2$, which covers the first Pontrjagin Class which is the thing we often care about in QFT.

3.2 General Case

In the general case, Cheeger and Simons use n-classifying spaces of Narashiman and Ramanan. Probably this can, and maybe this has been translated into the language of simplicial sheaves by someone. A manifold $B_{\nabla,n}G$ with principal G bundle with connection (P,Θ) is n-classifying if any principal G bundle with connection on any manifold M of dimension less than or equal to n is a pullback of (P,Θ), and furthermore, any two maps $M \to B_{\nabla,n}G$ inducing the same G bundle with connection are smoothly homotopic.

To define a cheeger simons differential charactar, we need to choose a lift $u \in H^{2l}(BG,\mathbb{Z})$ of $[f(\Omega)]$. For n big enough, any n-classifying space $B_{\nabla,n}G$ will have vanishing cohomology in degree $2l-1$. By one of the important short exact sequences, that means that the differential cohomology of degree $2l$ is just built out of closed forms and integral cohomology.

$$0 \longrightarrow \frac{H^{2l-1}(M,\mathbb{R})}{H^{2l-1}(M,\mathbb{Z})} \longrightarrow \hat{H}^{2l}(M) \longrightarrow \Omega^{2l}_{cl}(M) \times_{H^{2l}(M,\mathbb{R})} H^{2l}(M,\mathbb{Z}) \longrightarrow 0$$

So the pullback of u to $B_{\nabla,n}G$ along with f, defines unique differential cohomology classes on all n-classifying spaces for n sufficiently large. These together sort of form the universal Cheeger Simons charactaristic class.

Now it is necessary to show that given two classifying maps $f : M \to B_{\nabla,n}G$ and $f : M \to B_{\nabla,n}G'$, the pullbacks of the universal differential charactars are the same. $B_{\nabla,n}G$ and $f : M \to B_{\nabla,n}G'$ both have classifying maps to some huge $B_{\nabla,N}G$ so we assume our two classifying maps are to this bigger classifying space. Now, there exists a smooth homotopy between the two maps. Pulling back along this homotopy we get a principal bundle with connection on $M \times I$. Given some $2l-1$ cycle Z in M, the difference in the values of the two differential charactars we get can be computed by integrating $f(\Omega)$ on $Z \times I$. However, the principal bundle with connection descends to a connection on $Z \times S^1$, thus this integral must be an integer.

4 Examples

Now we get lots of interesting characteristic classes to think about. For the rest of the paper Cheeger and Simons specialize to vector bundles. The differential charactaristic classes tell us things about vector bundles with connection that are sort of reminiscent of what the old charactaristic classes told us about just vector bundles.

1. Differential Euler class: If we have a real vector bundle with connection of even rank we get $\hat{\chi}(V)$ defined by the phaffian.
2. Differential Chern classes: These are defined for complex vector bundles with connection and are denoted $\hat{c}_i$. They are defined by symmetric polynomials.
3. Differential Pontrjagin Classes classes: These are defined for real vector bundles and are denoted $\hat{p}_i$. They can be defined by complexifying, taking Chern classes, and multiplying by some signs.

5 Conformal Invariance

Let M be a Riemannian 3 manifold. Let Θ_{LC} be the Levi-Civita connection. We can use the variation formula to show that the Pontrjagin Classes classes $\hat{p}_i(\Theta_{LC})$ are invariant under conformal changes in metric. It is sufficient to check invariance along a path of conformally related metrics $e^{tf}g$ where $f \in C^\infty(M)$, t is a parameter, and g is a metric. The proof is a little hard for people like me who aren't super up on their Riemannian geometry, but an important part is the variation formula.

5.1 Variation Formula

Suppose Θ_t is a path of connections on a G bundle $P \to M$. Let f be an invariant polynomial. Let $CS_f(\Theta_t)$ denote the resulting path of Chern Simons forms. Than

$$\frac{d}{dt}CS_f(\Theta_t) = lf(\Theta_t' \wedge \Omega_t^{l-1}) + exact$$

Note that Θ_t' is a lie algebra valued one form which descends to M, so the variation in this Chern Simons form is by the pullback of a $2l-1$ form on M.

Now, let u be a lift of $[f(\Omega)]$ to integral cohomology of BG, so we have the differential charactar $S_{f,u}$. $S_{f,u}(\Theta_t)$ will also be changing by this $2l-1$ form on M. As a consequence, we get back the fact of Chern Weil theory that the characteristic class of $S_{f,u}(\Theta_t)$ in $H^{2l}(M,\mathbb{Z})$ remains constant if we change the connection. In the case that the $2l-1$ form is exact, our differential character isn't changing at all. This is what happens in the case of our path of conformally related Levi Civita connections.

6 Obstructions to Conformal Immersions

6.1 Differential Chern Classes

Define the Chern polynomials $C_k \in I^k(GL_n(\mathbb{C}))$ by

$$det(\lambda I - \frac{1}{2\pi i}A) = \sum_{k=0}^{n}(C_k(A) + iD_k(A))\lambda^{n-k}$$

We don't care about the D's. Since $BGL_n(\mathbb{C})$ has no torsion in it's cohomology, these polynomials are enough to define differential charactaristic classes $\hat{c}_k(V)$ of a complex vector bundle with connection of rank n, which refine the ordinary Chern classes $c_k(V)$. We also define the total chern class

$$\hat{c}(V) = 1 + \hat{c}_1(V) + ... + \hat{c}_n(V) \in \hat{H}^{even}(M)$$

We also define inverse Chern polynomials by $(1 + C_1 + ... + C_n)(1 + C_1^{\perp} + C_2^{\perp} ...) = 1$. We can solve for them and write explicit inductive formulas:

$$C_1^{\perp} = -C_1$$
$$C_2^{\perp} = -C_2 - C_1C_1^{\perp}$$
$$C_3^{\perp} = -C_3 - C_2C_1^{\perp} - C_1C_2^{\perp}$$
$$...$$

We then define differential characteristic classes $\hat{c}_k^{\perp}$ using these polynomials. I haven't mentioned it yet, but the Weil map and it's differential refinement are actually ring homomorphisms, so for example $\hat{c}_2^{\perp} = -\hat{c}_2 - \hat{c}_1\hat{c}_1^{\perp}$.

6.2 Differential Pontrjagin Classes

For a real vector bundle V with connection, let $V^{\mathbb{C}}$ be its comlexification, and define

$$\hat{p}_k(V) = (-1)^k\hat{c}_{2k}(V^{\mathbb{C}})$$

(I'm not sure whats up with the sign.) Note that $\hat{p}_k$ is a class of degree $4k$. Also define

$$\hat{p}_k^{\perp}(V) = \hat{c}_k^{\perp}(V^{\mathbb{C}})$$

This is what they wrote, but it looks weird to me...

6.3 Direct Sums

As with ordinary Chern classes, we have the following formula.

$$\hat{c}(V \oplus W) = \hat{c}(V) * \hat{c}(W)$$

Where $V \otimes W$ has the Whitney sum connection. $V \oplus W$ can be induced from a product of classifying spaces which we may assume to have vanishing odd cohomology, so it suffices to check the formula there. There it follows basically because determinants multiply under direct sums. In the case we are going to apply this formula to however, $V \otimes W$ will not have the Whitney sum connection. Instead $V \otimes W$ will have a connection which is compatable with the connections on V and W in the following sense. It compresses to the connections on V and W and has curvature tensor $R \in \Omega(M, End(V \oplus W))$ which respects the direct sum decomposition. Cheeger and Simons claim that you can use the variation formula to show that differential Chern classes for a compatable connection on $V \oplus W$ are the same as those for the whitney sum connection.

We have the same formula for Pontrjagin classes of real vector bundles:

$$\hat{p}(V \oplus W) = \hat{p}(V) * \hat{p}(W)$$

6.4 Conformal Immersions

Theorem. *Suppose $\phi : M^n \to \mathbb{R}^{n+k}$ is a conformal immersion of riemannian manifold into euclidean space. Then*

$$\hat{p}_i^{\perp}(M^n) = 0$$

for all $i > k/2$.

Proof. WLOG, by conformal invariance, we can assume ϕ is isometric. Let NM denote the orthogonal normal bundle.

$$TM \otimes NM = \underline{\mathbb{R}}^{n+k}$$

The Levi-Civita connection on $\mathbb{R}^{n+k}$ compresses to the Levi-Civita connection on TM, and some connection we don't really care about on NM. It also has zero curvature so it is compatable with these two connections. We thus have

$$\hat{p}(TM) * \hat{p}(NM) = 1$$

This implies

$$\hat{p}^{\perp}(TM) = \hat{p}(NM)$$

Since NM has rank k, $\hat{p}_i(NM)$ vanishes for $i > k/2$. □

6.5 $\mathbb{R}P^3$

In the Chern-Simons paper, they prove that $\mathbb{R}P^3$ cannot conformally immerse in $\mathbb{R}^4$. One would hope to show this by showing $\hat{p}_1^{\perp}(T\mathbb{R}P^3)$ is non vanishing. Note that $\hat{p}_1^{\perp} = -\hat{p}_1$. Also note that we are on a 3 manifold so the characteristic class and curvature of $\hat{p}_1$ both vanish, but there is a little more information in the differential charactar. One can compute this it by choosing a section of the orthogonal frame bundle of $\mathbb{R}P^3$, and integrating the pullback of the Chern Simons form by this section. Unfortunately this integral gives 1, and we are supposed to consider it as an element of $\mathbb{R}/\mathbb{Z}$. In the paper they go further and show that it is actually well defined mod $2\mathbb{Z}$. At least thats what I understood.

7 Differential Euler Class

Suppose we have a rank $2n$ real vector bundle V with connection on a manifold M. The differential Euler class $\hat{\chi}(V)$ has a nice interpretation as an obstruction to flat sections of the unit sphere bundle $\pi : S \to M$ of V.

In an early paper of Chern, he defined a $2n-1$ form Q on S which restricts to the unit volume form on each fiber, and such that $dQ = \pi^* Pf(\Omega)$. Here Pf is the Pfaffian and Ω is the curvature. He used this to give a proof of the higher dimensional gauss bonnet theorem.

It turns out that Q is sort of an avatar of the differential Euler class. If we have any $2n-1$ cycle $Z \subset M$, we can evaluate $\hat{\chi}(Z)$ by lifting it to S and integrating Q along it. Actually such a lift might not exist but Z is always homologous to a cycle Z' which does have a lift $\tilde{Z}'$. This means there is a $2n$ chain C with $\partial C = Z - Z'$. Then we have

$$\hat{\chi}(Z) = \int_{\tilde{Z}'} Q + \int_C Pf(\Omega)$$

There are similar descriptions of differential Chern classes in the Cheeger Simons paper with sphere bundles replaced by Stiefel bundles. The recipie for making Q and its generalizations is similar to how we made Chern Simons forms. Dan Freed has a nice description of it. Let me know if you're interested.

7.1 Flat Connections

Now assume the connection on V is flat. Suppose M is oriented and $2n-1$ dimensional. Choose a triangulation of M by smooth simplices $\sigma_1, ..., \sigma_r$ with orientations agreeing with M so that they add up to the fudamental class.

$$[M] = \sum_i \sigma_i$$

Let $v_1, ..., v_n$ be the vertices of the triangulation. Choose an element $s_i \in \pi^{-1}(v_i)$ of the fiber of S over each vertex v_i. Also choose an interior point of each simplex $b_i \in \sigma_i$. Inside each fiber $\pi^{-1}(b_i)$ we consider the geodesic simplex Σ_i who's vertices are the paralell transports of the s_i over the vertices of σ_i. According to Cheeger and Simons the sum of the volumes of these simplices is the differential euler class evaluated on $[M]$.

$$\hat{\chi}(V)([M]) = \sum_i Vol(\Sigma_i)$$