Juvitop: Chern-Simons forms and applications

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Abstract

In this talk, we introduce the Chern-Simons forms and explore their relation to the Cheeger-Simons differential characters. We will present a construction of the Chern-Simons invariants of topological 3-manifolds. As an application, we will compute these invariants for Lens spaces and see how they can help us understand the homotopy types of the configuration spaces of Lens spaces.

1 Motivation/Review

Let P be a principal G-bundle over a manifold M and A a G-connection on P with curvature F_A. Recall the Chern-Weil homomorphism

$$CW : \mathrm{Sym}(\mathfrak{g}^*)^{\mathrm{Ad}} \to H^*_{dR}(M;\mathbb{R}),$$

which sends an homogenous invariant polynomial ρ to a real cohomology class $[\rho(F_A)]$.

For instance, the image $\hat{c}_k$ of the k-th chern class c_k in $H^{2k}(M;\mathbb{R}$ is given by $\mathrm{tr}(\bigwedge^k F_A)$. Now suppose that A is a flat connection on a vector bundle over M, then $\hat{c}_k = 0$ and there is a lifting of c_k in the exact sequence

$$\ldots \to H^{2k-1}(M;\mathbb{R}/\mathbb{Z}) \to H^{2k}(M;\mathbb{Z}) \to H^{2k}(M;\mathbb{R}) \to \ldots$$

A canonical way to produce the lifting was first given by the Chern-Simons form in [3]. The Cheeger-Simons differential characters came out as a refinement of the Chern-Simons form.

2 Chern-Simons form

Let G be a compact Lie group and $\pi : P \to M$ a princial G-bundle. Fix a degree k invariant polynomial $\rho \in \mathrm{Sym}(\mathfrak{g}^*)^{\mathrm{Ad}}$. We will write $\rho(A) = \rho(F_A)$ for a connection A.

Recall 2.1. A *principal G-connection A* on P is a $\mathfrak{g}$-valued one form that is compatible with the G-action on P. Explicitly, A is G-equivariant, i.e., $(R_g)^*A = Ad_{g^{-1}}A$, and it is "the identity" on tangent vectors along the fiber, i.e., $A(X_\xi) = \xi$ for $\xi \in \mathfrak{g}$ and X_ξ its fundamental vector field.

Analogous to connections on vector bundles, a G-connection corresponds to a splitting $P = VP \oplus HP$ such that the horizontal bundle HP is G-equivariant in a suitable sense.

The affine space of connections on P can be identified with $\mathscr{A}_P = \Omega^1(M;\mathfrak{g}_P)$, i.e. 1-forms on M with values in the adoint bundle. Given two connections $A_0, A_1 \in \mathscr{A}_P$, the straight-line path $A_t : I \to \mathscr{A}_P$ determines a connection $\bar{A}$ on the G-bundle $P \times [0,1]$ over $M \times [0,1]$.

Definition 2.2. The *Chern-Simons form* associated to $A_0, A_1 \in \mathscr{A}_P$ and ρ is given by

$$CS_\rho(A_1, A_0) = \int_{[0,1]} \rho(\bar{A}) \in \Omega^{2k-1}(M).$$

Here we see again that $dCS_\rho(A_1,A_0) = \rho(A_1) - \rho(A_0)$ by Stokes' theorem, i.e. the de Rham class $CW(\rho) = [\rho(A_1)]$ is independent of the choice of connection. (c.f. Greg's talk.)

One remark: in general, if we choose a different path, the Chern-Simons form will differ by an exact term. This is beyond the scope of our talk. Maybe go talk to a gauge theorist.

Suppose instead we take the G-bundle $\pi^*P \to P$, which has a tautological section and hence a tautological (flat) connection Θ. Then we can define a Chern-Simons form on P (not on M!) for a single connection A, i.e.,

$$CS_\rho(A) = CS_\rho(\pi^*A, \Theta) \in \Omega^{2k-1}(P).$$

This a closed form with differential

$$dCS_\rho(A) = \rho(\pi^*A) = \pi^*\rho(A).$$

Chern and Simons ([3]) showed that when $[\rho(A)]$ is an integral class, then there is a $u \in C^{2k-1}(M, \mathbb{R}/\mathbb{Z})$ such that π^*u is the reduction of $CS_\rho(A) \mod \mathbb{Z}$.

2.1 Relation to Cheegar-Simons differential characters

Now we will briefly explain the relation between Chern-Simons forms and Cheegar-Simons differential cohomology, following [2, Chapter 2].

Recall the definition of the Cheegar-Simons differential cohomology

$$\hat{H}^k(M;\mathbb{Z}) = \{\chi : Z^{sm}_{k-1} \to \mathbb{R}/\mathbb{Z} \big| \exists \alpha \in \Omega^k(M)_\mathbb{Z},\ \chi(\partial c) = \int_c \alpha \mod \mathbb{Z}\}$$

and the differential cohomology diagram. (c.f. Peter's introductory talk.)

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H^{*-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "-\beta"] \arrow[dr] & & H^*(M;\mathbb{Z}) \arrow[ur] \arrow[dr] & \\
H^{*-1}_{dR}(M) \arrow[ur] \arrow[dr] & & \hat{H}^*(M;\mathbb{Z}) \arrow[ur, "\mathrm{ch}"] \arrow[dr, "\mathrm{curv}"] & & H^*_{dR}(M) \\
& \frac{\Omega^{*-1}(M)}{\Omega^{*-1}_{cl}(M)_\mathbb{Z}} \arrow[ur, "\iota"] \arrow[rr, "d"'] & & \Omega^*_{cl}(M)_\mathbb{Z} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

The curvature map $\mathrm{curv} : \hat{H}^k(M;\mathbb{Z}) \to \Omega^k(M)$ sends χ to α. The characteristic class map $\mathrm{ch} : \hat{H}^k(M;\mathbb{Z}) \to H^k(M;\mathbb{Z})$ is obtained by lifting χ to $\tilde{\chi} : Z^{sm}_{k-1} \to \mathbb{R}$ and sending χ to the integral class defined by

$$c \mapsto -\tilde{\chi}(\partial c) + \int_c \alpha,\ c \in C^{sm}_k(M;\mathbb{Z}).$$

The map ι is the descent of the map $\Omega^{k-1}(M) \to \hat{H}^k(M,\mathbb{Z})$ defined by

$$\iota(\omega)(z) = \exp(2\pi i \int_z \omega)$$

for $z \in Z^{sm}_{k-1}$.

Here we describe a way to lift the Chern-Weil homomorphism $CW = CW_\theta$ to $\hat{H}^{2k-1}(M;\mathbb{Z})$. Set

$$K^{2k}(G;\mathbb{Z}) = \{(\rho, u) \in \mathrm{Sym}^k(\mathfrak{g}^*)^{\mathrm{Ad}} \times H^{2k}(BG;\mathbb{Z}) \big| \ CW(\rho) = u_\mathbb{R}\},$$

$$R^{2k}(M;\mathbb{Z}) = \{(\omega, v) \in \Omega^{2k}(M)_\mathbb{Z} \times H^{2k}(M,\mathbb{Z}) \big| [\omega]_{dR} = v_\mathbb{R}\}.$$

Here $u_\mathbb{R}$ denotes the image of u in real cohomology. Then there is a unique natural map $\widehat{CW}_\theta$ that makes the diagram commutes.

$$\begin{tikzcd}
& \hat{H}^{2k}(M;\mathbb{Z}) \arrow[d, "(\mathrm{curv},\mathrm{ch})"] \\
K^{2k}(G;\mathbb{Z}) \arrow[ur, dashed, "\widehat{CW}_\theta"] \arrow[r, "(CW_\theta, f^*)"'] & R^{2k}(M;\mathbb{Z})
\end{tikzcd}$$

The universal bundle $\pi_G : EG \to BG$ comes with a universal connection Θ. Let $f : M \to BG$ be the classifying map with lift $F : (P,A) = (f^*(EG), f^*(\Theta)) \to (EG,\Theta)$. Recall that $\mathrm{Sym}(\mathfrak{g}^*)^{\mathrm{Ad}} \to H^*_{dR}(BG;\mathbb{R})$ is an isomorphism. Fix a pair $(\lambda,u) \in K^{2k}(G;\mathbb{Z})$. Then

$$dCS_\rho(\Theta) = \pi_G^*\rho(\Theta) = \mathrm{curv}(\pi_G^*\widehat{CW}_\Theta(\rho,u)).$$

Since EG is contractible, we conclude that

$$\iota(CS_\rho(\Theta)) = \pi_G^*\widehat{CW}_\Theta(\rho,u).$$

Pulling back to the bundle P and the connection θ, we obtain the desired relation between Chern-Simons forms and Cheegar-Simons differential characters:

$$\iota(CS_\rho(A)) = F^*\iota(CS_\rho(\Theta)) = F^*\pi_G^*\widehat{CW}_\Theta(\rho,u) = \pi^* f^*\widehat{CW}_\Theta(\rho,u) = \pi^*\widehat{CW}_A(\rho,u).$$

Now suppose that $\pi : P \to M$ admits a section $\sigma : M \to P$. Then we further deduce that

$$\widehat{CW}_A(\rho,u) = \sigma^*(\pi^*\widehat{CW}_A(\rho,u)) = \iota(\sigma^*CS_A(\rho)).$$

Then (the generalization of) the *Chern-Simons functional* is the evaluation at the fundamental class

$$\widehat{CW}_A(\rho,u)([M]) = \exp(2\pi i \int_M \sigma^*CS_A(\rho)),$$

or its more familiar form

$$\log(\widehat{CW}_A(\rho,u)([M])) = \int_M \sigma^*CS_A(\rho) \mod \mathbb{Z}.$$

This is conceptually nice, but how do we obtain computable topological invariants from this formula?

3 Chern-Simons invariants for 3-manifolds

As an example, we examine the case where P is a principal $SU(2)$-bundle over a path-connected 3-manifold M and $\rho(A) = \frac{1}{8\pi^2}\mathrm{tr}(F_A \wedge F_A)$ is the second chern class in $H^4(M;\mathbb{R})$. This is the classical Chern-Simons theory. We mostly follow the exposition in [5].

The advantage we have in this setting is that every such P is trivial, since $BSU(2)$ is 3-connected. Fixing a trivialization, we have $\mathscr{A}_P \cong \Omega(M)\otimes\mathfrak{g}$ and there is a trivial (flat) connection $A_0 = 0$. Recall that $SU(2)$ acts on $\mathscr{A}_P$ by

$$g\cdot A = gAg^{-1} - dg\, g^{-1}.$$

This action preserves flatness since $F^{g\cdot A} = gF^Ag^{-1}$. Now the gauge group of P (bundle automorphisms that cover the identity on M) is $\mathscr{G} \cong \mathrm{Maps}(M,SU(2))$ and it acts on $P \cong M\times SU(2)$ by left multiplication, so $\mathscr{G}$ preserves flat connections.

On the other hand, each flat connection A gives rise to a *holonomy representation* $\pi_1(M) \to G$: parallel transport along a loop γ at m_0 gives an autmorphism of the fiber G at m_0, which depends only on the homotopy class $[\gamma] \in \pi_1(M,m_0)$. (Roughly speaking, flat connections $\approx$ closed 1-forms.) With a bit of work, one can recover the well-known fact that

$$\{\text{Flat connections on } P\}/\mathscr{G} \hookrightarrow R(M) = \mathrm{Hom}(\pi_1(M),SU(2))/\text{conjugation}.$$

Since P is trivial, this injection becomes a bijection. In fact, this can be upgraded to a homeomorphism, with the right hand side the character variety of M.

Now let's look at the 3-form

$$CS_\rho(A) = CS_\rho(A, A_0) = \int_{[0,1]} \frac{1}{8\pi^2}\mathrm{tr}(F_A \wedge F_A).$$

Integrating over M gives us the *Chern-Simons functional* on $\mathscr{A}_P$, i.e.,

$$\tilde{cs}(A) = \int_{M\times[0,1]} \frac{1}{8\pi^2}\mathrm{tr}(F_A \wedge F_A) = \frac{1}{8\pi^2}\int_M \mathrm{tr}(dA\wedge A + \frac{2}{3}A\wedge A\wedge A).$$

This map is in fact smooth and functorial in $P \to M$. In particular, it is independent of the trivialization up to $\mathbb{Z}$, i.e., it descends to a functional $cs : R(M) \cong \mathscr{A}_P/\mathscr{G} \to \mathbb{R}/\mathbb{Z}$. The reason is as follows: for any gauge transformation $\sigma \in \mathscr{G}$, let $\bar{A}$ be the linear path in $\mathscr{A}$ connecting A and $\sigma \cdot A$, which descends to a loop in $\mathscr{A}_P/\mathscr{G}$. Then

$$cs(\sigma\cdot A) - cs(A) = \int_{M\times S^1} \frac{1}{8\pi^2}\mathrm{tr}(F_{\bar{A}} \wedge F_{\bar{A}})$$

is an integer because we are integrating the second chern class against the top dimensional integral homology class $[M \times S^1]$. Thus we obtain a collection of homotopy invariants of M. In practice, they are relatively computable, as we will see for Lens spaces.

3.1 Chern-Simons invariants of Lens spaces

Theorem 3.1. *[5, 5.1] The set of Chern-Simons invariants of the Lens space $L(p,q)$ are*

$$\Big\{ -\frac{n^2 r}{p} \Big| n = 0, 1, \ldots, \lfloor \frac{p}{2} \rfloor \Big\}.$$

Remark 3.2. Two Lens spaces $L(p,q)$ and $L(p',q')$ have the same set of Chern-Simons invariants if and only if $p = p'$ and $q'q^{-1} \equiv a^2 \mod p$ for some $a \in \mathbb{Z}$, i.e., there is an orientation preserving homotopy equivalence between the two. Hence Chern-Simons invariants detect the homotopy type of Lens spaces.

Sketch of proof: The Lens space $L(p,q)$ can be obtained by gluing the boundary of two solid tori X, K together via an element $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in SL_2(\mathbb{Z})$. Let $x = S^1 \times \{1\}$ represent a generator of $\pi_1(X)$ and y a meridian of ∂X. Let μ, λ be the corresponding generators of ∂K, so $\mu = px + qy$, $\lambda = rx + sy$.

Now we utilize some general results about 3-manifolds with a single torus boundary in [5]. Suppose we have a path ρ_t in $\mathrm{Hom}(\pi_1(X), SU(2))$ with

$$\rho_t(\mu) = \begin{bmatrix} e^{2\pi i\alpha(t)} & \\ & e^{-2\pi i\alpha(t)} \end{bmatrix}, \rho_t(\lambda) = \begin{bmatrix} e^{2\pi i\beta(t)} & \\ & e^{-2\pi i\beta(t)} \end{bmatrix}$$

where $\alpha, \beta : I \to \mathbb{R}$. The corresponding path of flat connections takes the form

$$A_t = \begin{bmatrix} i\alpha(t) & \\ & -i\alpha(t) \end{bmatrix} dx + \begin{bmatrix} i\beta(t) & \\ & -i\beta(t) \end{bmatrix} dy$$

near the torus boundary. If ρ_0, ρ_1 send μ to 1, then

$$cs(\rho_1) - cs(\rho_0) = -2\int_0^1 \beta\alpha' dt \mod \mathbb{Z}.$$

On the other hand, a holonomy representation on X extends to one on the Dehn filling M (in our case, the Lens space itself) if and only if it sends μ to 1.

Back to the sketch. We take γ_t to be a path sending x to $e^{2\pi i\theta}$ with $\theta \in [0, \frac{1}{2}]$. (Note that every representation of $\pi_1(X)$ is conjugate to one of these along the path.) Then γ_{t_1} extends to a representation ρ_t of $\pi_1(L(p,q)) = \mathbb{Z}_p$ if and only if $pt_1 \in \mathbb{Z}$, so there are $\lfloor \frac{p}{2} \rfloor + 1$ conjugacy classes. On the other hand, $\alpha(t) = pt$ and $\beta(t) = rt$, so

$$cs(\rho_{t_1}) = -2\int_0^{t_1} \beta\alpha' dt = -rpt_1^2.$$

3.2 Application to configuration spaces of Lens spaces

Let's see how the Chern-Simons invariants above are used to produce homotopy invariants of two-point configuration spaces of Lens spaces in [4]. Recall that Longoni and Salvatore found a pair of homotopy equivalent Lens spaces $L(7,1), L(7,2)$ two-point configuration spaces are not homotopy equivalent. Now the question is: does the homotopy type of two-point configuration space distinguish Lens spaces up to homeomorphism?

Fix a Lens space $L = L(p,q)$ and a CW strucure with one cell e_i in the i-th dimension. Let $X_0 = \mathrm{Conf}_2(L) \cong L \times L \backslash \Delta$. The inclusion induces an isomorphism of fundamental groups, hence it makes sense to extend the Chern-Simons invariants to X instead.

The generators of $H_3(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_p$ are $[e_0 \times e_3] = [e_0 \times L]$, $[e_3 \times e_0] = [L \times e_0]$ and $[e_1 \times e_2 + e_2 \times e_1]$. Note that $\Omega_n(X) \to \Omega_n(X)$ is a bijection for $n = 3$. In particular, there is a smooth manifold S that represents $[e_1 \times e_2 + e_2 \times e_1]$ up to sign. (The explicit construction is carried out in great detail in [4, Section 3].) Furthermore, we can choose the free generators so that the inclusion $H_3(X_0) = \mathbb{Z} \oplus \mathbb{Z}_p \hookrightarrow H_3(X)$ sends the free generator to $(1,1,0)$ and the torsion to $[S] = (0,0,1)$.

Given a holonomy representation $\alpha : \pi_1(X) = \mathbb{Z}_p \oplus \mathbb{Z}_p \to SU(2)$ and smooth 3-cycle $f : M \to X$, we get a holonomy representation $f^*\alpha$ of M. Hence we can define an extension of the Chern-Simons invariants

$$cs_X : R(X) \to \mathrm{Hom}(H_3(X), \mathbb{R}/\mathbb{Z})$$

by

$$cs_X(\alpha)(\xi) = cs_M(f^*\alpha) = \frac{1}{8\pi^2} \int_M \mathrm{tr}(dA \wedge A + \frac{2}{3} A \wedge A \wedge A).$$

This map is functorial in X and independent of the choice of homology representatives. It produces a homotopy invariant for each pair of conjugacy class of representation and third homology class.

Let's compute these invariants for X. Fix an $SU(2)$ representation α, which is conjugate to one sending the generators of $\pi_1(X)$ to $e^{2\pi ik/p}$ and $e^{2\pi il/p}$. We will write $\alpha = \alpha(k,l)$. Then $\alpha(k,l)$ pulls back to representations sending the generator of $\pi_1(L)$ to $e^{2\pi ik/p}$ and $e^{2\pi ik/p}$ on the two free homology class. By Theorem 3.1, their Chern-Simons invariants are $-\frac{k^2 r}{p}$ and $-\frac{l^2 r}{p}$. The manifold S is Seifert fibered over S^2, and the Chern-Simons invariants for Seifert fibered spaces are well studied in [1]. Here we simply cite the result that the Chern-Simons invariant is $\frac{2kl}{p}$. These pullback to invariants $-\frac{r(k^2+l^2)}{p}$ and $\pm\frac{2kl}{p}$ on X_0.

Now suppose that $f : X_0 \to X_0'$ is a homotopy equivalence, where $X_0 = \mathrm{Conf}_2(L(p,q))$ and $X_0' = \mathrm{Conf}_2(L(p,q'))$. Then the induced isomorphism on the fundamental groups $\mathbb{Z}_p \oplus \mathbb{Z}_p$ corresponds to a matrix

$$f_1 = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p).$$

The induced isomorphism on $H_3 = \mathbb{Z} \oplus \mathbb{Z}_p$ has the form $h_3 = \begin{bmatrix} \varepsilon & 0 \\ \alpha & \beta \end{bmatrix}$, where $\varepsilon = \pm 1$ and $\beta \in \mathbb{Z}_p^\times$. Using naturality of CS_X we can deduce the following numerical constraints:

Proposition 3.3. *[4, 5.2] If f is a homotopy equivalence, then $\varepsilon q' = qa^2 (\mathrm{mod}\ p)$ and*

$$f_1 = \begin{bmatrix} a & 0 \\ 0 & \pm a \end{bmatrix}, \begin{bmatrix} 0 & a \\ \pm a & 0 \end{bmatrix}; \; h_3 = \begin{bmatrix} \varepsilon & 0 \\ 0 & \pm a^2 \end{bmatrix}.$$

References

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