EXAMPLES OF DIFFERENTIAL COHOMOLOGY THEORIES

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1. Review: Differential Refinements

Let Man denote the category of manifolds and smooth maps.

Definition 1.1. A *differential cohomology theory* is a sheaf of spectra on the category Man.

Remark. We can also consider sheaves on Man valued in other categories. Of particular interest will be the category sSet of simplicial sets, the category Ch of chain complexes, and the category Grpd of groupoids.

Recall that, as in any category of sheaves, we have an adjunction

$$\Gamma^* : \mathsf{Shv}(\mathsf{Man}; \mathsf{Sp}) \rightleftarrows \mathsf{Sp} : \Gamma_*$$

given by taking the constant sheaf and global sections. For sheaves on manifolds, we have an extra functor, the left adjoint to Γ^*,

$$\Gamma_! : \mathsf{Shv}(\mathsf{Man}; \mathsf{Sp}) \rightleftarrows \mathsf{Sp} : \Gamma^*$$

We refer to the composite $\Gamma^*\Gamma_!$ as the *homotopification* functor. In Peter's talk this was denoted $\mathsf{hi}_! = \Gamma^*\Gamma_!$.

Definition 1.2. Let E be a spectrum. A *differential refinement* of E is a differential cohomology theory $\hat{E} \in \mathsf{Shv}(\mathsf{Man}; \mathsf{Sp})$ together with an isomorphism $\Gamma_!\hat{E} \simeq E$.

We saw last time that one should think of a differential cohomology theory as the data of a homotopy invariant piece, a "pure" piece, and some gluing data.

Definition 1.3. A sheaf $\mathcal{F} \in \mathsf{Shv}(\mathsf{Man}; \mathsf{Sp})$ is *pure* if the global sections of $\mathcal{F}$ is 0,

$$\Gamma_*\mathcal{F} \simeq 0$$

A differential cohomology theory $\hat{E}$ is a differential refinement of $E \in \mathsf{Sp}$ if the homotopy invariant piece of $\hat{E}$ is the constant sheaf on E.

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Lemma 1.4 (from last time). *A differential refinement of $E \in \mathsf{Sp}$ is equivalent to the data of a pure sheaf $\hat{P} \in \mathsf{Shv}(\mathsf{Man};\mathsf{Sp})$ and a map $f\colon E \to \Gamma_! \hat{P}$. The differential cohomology theory is then the homotopy pullback*

(1)
$$\begin{tikzcd}
\hat{E} \arrow[r] \arrow[d] & \Gamma^* E \arrow[d, "\Gamma^* f"] \\
\hat{P} \arrow[r] & \Gamma^* \Gamma_! \hat{P}
\end{tikzcd}$$

1.0.1. Note that given a spectrum E, there are possibly many differential refinements of E. We will construct differential cohomology theories by the following process:

- Choose a pure sheaf $\hat{P}$
- Compute $\Gamma_! \hat{P}$ using the formula

$$\Gamma_! \hat{P} = \operatorname*{colim}_{\Delta^{\mathrm{op}}} \hat{P}(\Delta^n)$$

- Find spectra E that map to $\Gamma_! \hat{P}$
- Take $\hat{E}$ as in the pullback diagram 1

Remark. Let $\hat{E}$ be the differential refinement of E using the pure sheaf $\hat{P}$ and the map $f\colon E \to \Gamma_! \hat{P}$. The differential cohomology diagram for $\hat{E}$ looks like

(2)
$$\begin{tikzcd}
 & \Gamma^*\Gamma_* \hat{E} \arrow[rr] \arrow[dr] & & \Gamma^* E \arrow[dr] & \\
\Sigma^{-1}\Gamma^*\Gamma_! Z(\hat{E}) \arrow[ur] \arrow[dr] & & \hat{E} \arrow[ur] \arrow[dr] & & \Gamma^*\Gamma_! Z(\hat{E}) \\
 & A(\hat{E}) \arrow[rr] \arrow[ur] & & Z(\hat{E}) \arrow[ur] &
\end{tikzcd}$$

Last time, we saw that $Z(\hat{E}) \simeq \hat{P}$ recovers the pure sheaf. Thus the differential cohomology diagram 2 can be constructed from the data of the pullback square 1, since the diagonals must be fiber sequences.

2. Examples

2.1. **Stupidest Example.** We need a sheaf whose global sections is 0. The easiest example of such a sheaf is the constant sheaf, $\hat{P} = \Gamma^* 0$. In this case, we have an equivalence

$$\Gamma_! \Gamma^* 0 \simeq 0$$

Indeed, $\Gamma_! \Gamma^*$ is the left adjoint of $\Gamma_* \Gamma^*$ which is the identity on Sp. Any spectrum E maps uniquely to the identity 0. Thus for any spectrum E we have a differential refinement $\hat{E}$ by the (homotopy) pullback

$$\begin{tikzcd}
\hat{E} \arrow[r] \arrow[d] & \Gamma^* E \arrow[d] \\
\Gamma^* 0 \arrow[r, equal] & \Gamma^* 0
\end{tikzcd}$$

Since the bottom horizontal arrow is an equivalence, the top horizontal arrow is as well. Thus $\hat{E} = \Gamma^* E$. The rest of the differential cohomology diagram looks as follows,

$$\begin{tikzcd}
 & \Gamma^* E \arrow[rr] \arrow[dr] & & \Gamma^* E \arrow[dr] & \\
\Sigma^{-1}\Gamma^* 0 \arrow[ur] \arrow[dr] & & \Gamma^* E \arrow[ur] \arrow[dr] & & \Gamma^* 0 \\
 & A(\Gamma^* E) \arrow[rr] \arrow[ur] & & \Gamma^* 0 \arrow[ur] &
\end{tikzcd}$$

Since the upwards diagonal is a fiber sequence, we must have $A(\Gamma^* E) \simeq \Sigma^{-1}\Gamma^* 0$.

In particular, this example shows that E-cohomology is a special case of differential cohomology.

2.2. **Stupidest Example, but with a filtration.** We give an alternative description of $\Gamma^* 0$ which comes with a natural filtration.

First note that there is a functor $H \colon \mathsf{Ch} \to \mathsf{Sp}$ from chain complexes of abelian groups to spectra, called the Eilenberg-MacLane functor. For example, if A is an abelian group and $A[0]$ is the chain complex with A in degree 0, then $H(A[o]) = HA$, the Eilenberg-MacLane spectrum. Note also that H takes quasi-isomorphic chain complexes to the same spectrum.

Let $\Omega^\bullet_{dR} \in \mathsf{Shv}(\mathsf{Man}; \mathsf{Ch})$ the sheaf of deRham forms with cohomological grading; so Ω^k_{dR} is in degree $-k$. Consider the resulting functor of spectra, $H\Omega^\bullet_{dR}$. By the Poincaré Lemma, $\Omega^\bullet_{dR}$ is quasi-isomorphic to the constant sheaf at $\mathbb{R}[0]$. Thus $H\Omega^\bullet_{dR} \simeq \Gamma^* H\mathbb{R}$. In particular, $H\Omega^\bullet_{dR}$ is not pure. However, since $\Omega^\bullet_{dR}$ is homotopy invariant (alternatively, since $\Gamma^* H\mathbb{R}$ is homotopy invariant), the purification $Z(H\Omega^\bullet_{dR})$, is equivalent to $\Gamma^* 0$ (this was shown last time),

$$\Gamma^* 0 \simeq Z(H\Omega^\bullet_{dR})$$

Now $\Omega^\bullet_{dR}$ comes has a filtration by degree. For $k \in \mathbb{N}$, let $\Omega^{\geq k}_{dR}$ denote the stunted piece of the chain complex $\Omega^\bullet_{dR}$ where we have replaced everything in degree $< k$ by 0. We get induced filtrations of $H\Omega^\bullet_{dR}$ and of $Z(H\Omega^\bullet_{dR}) \simeq \Gamma^* 0$.

For $k \geq 1$, there is an equivalence $\Omega^{\geq k}_{dR}(*) \simeq 0$ of chain complexes. Thus the global sections of $H\Omega^{\geq k}_{dR}$ is 0,

$$\Gamma_* H\Omega^{\geq k}_{dR} = H\Omega^{\geq k}_{dR}(*) = H(0) = 0$$

By definition, this means that $H\Omega^{\geq k}_{dR}$ is a pure sheaf if (and only if) $k \geq 1$. The purification functor Z is the identity on pure sheaves, so we obtain a filtration of the pure sheaf $\Gamma^* 0$ by pure sheaves

$$\Gamma^* 0 \to H\Omega^{\geq 1}_{dR} \to \cdots \to H\Omega^{\geq k}_{dR} \to \cdots$$

Now for each $k \geq 1$, we can choose the pure sheaf $H\Omega^{\geq k}_{dR}$ and follow our procedure, 1.0.1.

We need to compute the homotopification of our chosen pure sheaf.

Lemma 2.1. *For any $k \in \mathbb{N}$, there is an equivalence $\Gamma_! H\Omega^{\geq k}_{dR} \simeq H\mathbb{R}$.*

Proof. For $k = 0$, we have seen that $H\Omega^{\geq 0}_{dR} \simeq \Gamma^* H\mathbb{R}$, which is already homotopy invariant. Thus $\Gamma_! \Gamma^* H\mathbb{R} \simeq H\mathbb{R}$.

For $k \geq 1$, see [1, Lem. 7.15]. $\square$

Thus for any spectrum E together with a map $E \to H\mathbb{R}$, we can define a family of differential refinement of E by the pullbacks

$$\begin{tikzcd} \hat{E}(k) \arrow[r] \arrow[d] & \Gamma^* E \arrow[d] \\ H\Omega_{dR}^{\geq k} \arrow[r] & H\mathbb{R} \end{tikzcd}$$

for $k \geq 1$. This family of differential refinements was introduced by Hopkins-Singer, [2].

The differential cohomology diagram 2 for $\hat{E}(k)$ looks like

$$\begin{tikzcd} & \Gamma^*\Gamma_*\hat{E} \arrow[rr] \arrow[dr] & & \Gamma^* E \arrow[dr] & \\ \Sigma^{-1}\Gamma^* H\mathbb{R} \arrow[ur] \arrow[dr] & & \hat{E}(k) \arrow[ur] \arrow[dr] & & \Gamma^* H\mathbb{R} \\ & H\Omega_{dR}^{\leq k-1}[-1] \arrow[rr] \arrow[ur] & & H\Omega_{dR}^{\geq k} \arrow[ur] & \end{tikzcd}$$

2.2.1. *Cheeger-Simons Differential Characters.* Take $E = H\mathbb{Z}$ and the map $H\mathbb{Z} \to H\mathbb{R}$ induced from the inclusion $\mathbb{Z} \subset \mathbb{R}$.

Definition 2.2. The kth *ordinary differential cohomology group* of a manifold M, denoted $\hat{H}^k(M)$ is the kth homotopy group

$$\check{H}^k(M) = \pi_{-k}\widehat{H\mathbb{Z}}(k)(M)$$

where $\widehat{H\mathbb{Z}}(k)$ is defined by the homotoyp pullback square

$$\begin{tikzcd} \widehat{H\mathbb{Z}}(k) \arrow[r] \arrow[d] & \Gamma^* H\mathbb{Z} \arrow[d] \\ Z(H\Omega_{dR}^{\geq k}) \arrow[r] & \Gamma^* H\mathbb{R} \end{tikzcd}$$

Note that $Z(H\Omega_{dR}^{\geq k} \simeq H\Omega_{dR}^{\geq k}$ if $k \geq 1$ and is $H\mathbb{R}$ if $k = 0$.

The group $\hat{H}^k(M)$ is also known as the Cheeger-Simons differential characters, or the smooth Deligne cohomology.

We give a more explicit description of $\check{H}^k$, which agrees with Deligne's original construction.

Lemma 2.3. *Let $k \geq 1$. The sheaf of spectra $\widehat{H\mathbb{Z}}(k)$ is given by applying the Eilenberg-MacLane functor H to the sheaf of chain complexes*

$$\left(\Gamma^*\mathbb{Z} \to \Omega_{dR}^0 \to \Omega_{dR}^1 \to \cdots \to \Omega_{dR}^{k-1} \to 0 \to \cdots\right)$$

where Ω_{dR}^i is in degree $-i-1$. Moreover, the group $\check{H}^k(M)$,for a manifold M, can be computed as the kth sheaf cohomology group of this sheaf of chain complexes.

Proof. By construction, $\widehat{H\mathbb{Z}}(k)$ comes from H of the sheaf of chain complexes F given by the (homotopy) pullback

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \Gamma^*\mathbb{Z}[0] \arrow[d] \\ \Omega_{dR}^{\geq k} \arrow[r] & \Omega_{dR}^{\bullet} \end{tikzcd}$$

Since the bottom horizontal arrow is an inclusion, its cofiber is given by the cokernel. We have a cofiber sequence

$$\Omega^{\geq k}_{dR} \to \Omega^{\bullet}_{dR} \to \Omega^{\leq k-1}_{dR}$$

where $\Omega^{\leq k-1}_{dR}$ has Ω^i_{dR} in degree $-i$, and 0 above $k-1$. The cofiber of the top horizontal map is equivalent to the cofiber of the bottom horizontal map. Since we are in a stable setting, these cofiber sequences are also fiber sequences. Thus, we have a fiber sequence

$$F \to \Gamma^*\mathbb{Z}[0] \to \Omega^{\leq k-1}_{dR}$$

where $\mathbb{Z}[0] \to \Omega^{\leq k-1}_{dR}$ includes $\mathbb{Z}$ into Ω^0_{dR}. The fiber of this inclusion is a shift of the mapping cone, which is

$$\left(\Gamma^*\mathbb{Z} \to \Omega^0_{dR} \to \Omega^1_{dR} \to \cdots \to \Omega^{k-1}_{dR} \to 0 \to \cdots\right)$$

Finally, note that $\pi_{-k}HF = H^kF$. □

Example 2.4. Take $k = 0$. Then $\widehat{H\mathbb{Z}}(k) \simeq \Gamma^*H\mathbb{Z}$ and

$$\Gamma^*H\mathbb{Z}(M) = [M, H\mathbb{Z}]$$

has 0th homotopy group $H^0(M;\mathbb{Z})$.

We stole the following two computations from Nilay Kumar's notes, [3].

Example 2.5. Take $k = 1$. We compute $\check{H}^1(M)$. By Lemma 2.3, we can compute $\check{H}^1(M)$ as the 1st sheaf cohomology group of the sheaf of chain complexes $(\Gamma^*\mathbb{Z} \to \Omega^0_{dR})$. After choosing a good cover of M, we can compute this sheaf cohomology as Cech cohomology. The Cech cohomology will be the cohomology of the total complex of the following bicomplex,

$$\begin{array}{ccc}
\check{C}^0(\Gamma^*\mathbb{Z}) & \longrightarrow & \check{C}^0(\Omega^0_{dR}) \\
\downarrow & & \downarrow \\
\check{C}^1(\Gamma^*\mathbb{Z}) & \longrightarrow & \check{C}^1(\Omega^0_{dR}) \\
\downarrow & & \downarrow \\
\check{C}^2(\Gamma^*\mathbb{Z}) & \longrightarrow & \check{C}^1(\Omega^0_{dR}) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

with $\check{C}^i(\Gamma^*\mathbb{Z})$ in bidegree $(0,-i)$ and $\check{C}^i(\Omega^0_{dR})$ in bidegree $(-1,-i)$. The differential on this bicomplex is $D = d^{\text{hor}} + (-1)^p d^{\text{ver}}$ where p is the horizontal degree. The piece of the total complex that we are interested looks like

$$\check{C}^0(\Gamma^*\mathbb{Z}) \xrightarrow{D_0} \check{C}^0(\Omega^0_{dR}) \oplus \check{C}^1(\Gamma^*\mathbb{Z}) \xrightarrow{D_1} \check{C}^1(\Omega^0_{dR}) \oplus \check{C}^2(\Gamma^*\mathbb{Z})$$

If our good cover of M is $\{U_\alpha\}$ with intersections $U_{\alpha\beta}$, then an element of $\check{C}^0(\Omega^0_{dR}) \oplus \check{C}^1(\Gamma^*\mathbb{Z})$ looks like a collection of smooth maps $f_\alpha\colon U_\alpha \to \mathbb{R}$ and integers $n_{\alpha\beta} \in \mathbb{Z}$. The map D_1 sends

$$D_1(f_\alpha, n_{\alpha\beta}) = (f_\alpha - f_\beta + n_{\alpha\beta}, n_{\beta\gamma} - n_{\alpha\gamma} + n_{\alpha\beta})$$

In particular, an element of $\ker D_1$ consists of maps f_α that agree on intersections up to an integer. These glue together to give a (smooth) map $f\colon M \to S^1 = U(1)$.

The map D_0 sends a collection (n_α) to

$$D_0(n_\alpha) = (c_{n_\alpha}, n_\alpha - n_\beta)$$

where c_{n_α} is the constant function $U_\alpha \to \mathbb{R}$ at the integer n_α. As a map $M \to S^1$, these glue together to the constant map at the base point.

Thus we have an equivalence

$$\check{H}^1(M) \simeq \mathsf{Maps}_{sm}(M, U(1))$$

In ordinary cohomology, we have

$$H^1(M;\mathbb{Z}) = [M, K(\mathbb{Z},1)] = [M, U(1)]$$

In this sense, differential cohomology replaced homotopy maps with smooth maps.

Example 2.6. Take $k = 2$. Then we have an equivalence

$$\check{H}^2(M) \simeq \{\text{line bundles on } M \text{ with connection}\}/\sim$$

In ordinary cohomology, we have

$$H^2(M;\mathbb{Z}) = [M, K(\mathbb{Z},2)] = [M, BU(1)] = \{\text{line bundles on } M\}/\sim$$

In this sense, the new geometric information encoded in differential cohomology is the connection.

2.2.2. *Differential K-Theory.* Consider de Rham forms with $\mathbb{C}[u, u^{-1}]$ with u in degree 2. We obtain a family of pure sheaves $H\Omega^{\geq k}_{dR}(-;\mathbb{C}[u,u^{-1}])$. As in Lemma 2.1, we have a equivalences

$$\Gamma_! H\Omega^{\geq k}_{dR}(-;\mathbb{C}[u,u^{-1}]) \simeq H\mathbb{C}[u,u^{-1}]$$

Take $E = \mathsf{ku}$ the spectrum defining complex K-theory. The Chern character defines a map $\mathsf{ch}\colon \mathsf{ku} \to H\mathbb{C}[u,u^{-1}]$. The resulting family of differential cohomology theories is what Hopkins-Singer refer to as differential K-theory. There are other interesting differential refinements of ku that do not arise from the pure sheaves $H\Omega^{\geq k}_{dR}(-;\mathbb{C}[u,u^{-1}])$.

2.3. **Classifying G-Bundles.** We will be interested in differential cohomology classes. As in ordinary homology theory, characteristic classes come from studying classifying spaces of bundles. Let G be a Lie group. View BG as a simplicial set,

$$BG = \operatorname*{colim}_{\Delta^{\mathrm{op}}}(\mathsf{sing} BG^\bullet)$$

We will be interested in a differential refinement of BG. This will be a sheaf on Man of simplicial sets.

Start with the sheaf of groupoids Bun_G or Bun_G^∇ on Man which assigns to a manifold M the groupoid of principal G-bundles on M (and connections) and bundle maps. The nerve defines a functor

$$\mathrm{N}\colon \mathsf{Grpd} \to \mathsf{Man}$$

We obtain sheaves of simplicial sets, $B_\bullet G = \mathrm{N}(\mathsf{Bun}_G)$ and $B_\nabla G = \mathrm{N}(\mathsf{Bun}_G^\nabla)$.

Lemma 2.7. *The sheaves $B_\bullet G$ and $B_\nabla G$ refine BG.*

This is [1, Lem. 5.2].

References

[1] Ulrich Bunke, Thomas Nikolaus, and Michael Völkl. Differential cohomology theories as sheaves of spectra. *arXiv e-prints*, page arXiv:1311.3188, Nov 2013.

[2] M. J. Hopkins and I. M. Singer. Quadratic functions in geometry, topology, and M-theory. *J. Differential Geom.*, 70(3):329–452, 2005.

[3] Nilay Kumar. Notes on differential cohomology and gerbes. 2018.