Characteristic Forms and Geometric Invariants
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Characteristic forms and
geometric invariants

By Shiing-shen Chern and James Simons*

1. Introduction

This work, originally announced in [4], grew out of an attempt to derive a purely combinatorial formula for the first Pontrjagin number of a 4-manifold. The hope was that by integrating the characteristic curvature form (with respect to some Riemannian metric) simplex by simplex, and replacing the integral over each interior by another on the boundary, one could evaluate these boundary integrals, add up over the triangulation, and have the geometry wash out, leaving the sought after combinatorial formula. This process got stuck by the emergence of a boundary term which did not yield to a simple combinatorial analysis. The boundary term seemed interesting in its own right and it and its generalization are the subject of this paper.

The Weil homomorphism is a mapping from the ring of invariant polynomials of the Lie algebra of a Lie group, G, into the real characteristic cohomology ring of the base space of a principal G-bundle, cf. [5], [7]. The map is achieved by evaluating an invariant polynomial P of degree l on the curvature form Ω of a connection, θ, on that bundle, and obtaining a closed form on the base, $P(\Omega^l)$. Because the lift of a principal bundle over itself is trivial, the forms $P(\Omega^l)$ are exact in the bundle. Moreover, in a way that is canonical up to an exact remainder one can construct a form $TP(\theta)$ on the bundle such that $dTP(\theta) = P(\Omega^l)$. Under some circumstances, e.g., $\dim P(\Omega^l) > \dim$ base, $P(\Omega^l) = 0$ and $TP(\theta)$ defines a real cohomology class in the bundle. Our object here is to give some geometrical significance to these classes.

In § 2 we review standard results in connection theory. In § 3 we construct the forms $TP(\theta)$ and derive some basic properties. In particular we show that if $\deg P = n$ and the base manifold has $\dim 2n - 1$ that the forms $TP(\theta)$ lead to real cohomology classes in the total space, and, in the case that $P(\Omega^l)$ is universally an integral class, to R/Q characteristic numbers. Both the class above and the numbers depend on the connection.

In § 4 we restrict ourselves to the principal tangent bundle of a

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manifold and show that if θ, θ', Ω, Ω' are the connection and curvature forms of conformally related Riemannian metrics then $P(\Omega^l) = P(\Omega'^l)$. Moreover, if $P(\Omega^l) = 0$ then $TP(\theta)$ and $TP(\theta')$ determine the same cohomology class and thus define a conformal invariant of M. In § 5 we examine the question of conformal immersion of an n-dim manifold into R^{n+k}. We show that a necessary condition for such an immersion is that in the range $i > [k/2]$ the forms $P_i^{\perp}(\Omega^{2i}) = 0$, and the classes $\{(1/2)\,TP_i^{\perp}(\theta)\}$ be integral classes in the principal bundle. Here $P_i^{\perp}$ is the ith inverse Pontrjagin polynomial. In § 6 we apply these results to 3-manifolds.

In a subsequent paper, [3], by one of the present authors and J. Cheeger, it will be shown that the forms $TP(\theta)$ can be made to live on the manifold below in the form of "differential characters". These are homomorphisms from the group of smooth singular cycles into R/Z, subject to the restriction that on boundaries they are the mod Z reduction of the value of a differential form with integral periods evaluated on a chain whose boundary is the given one. These characters form a graded ring, and this ring structure may be exploited to perform vector bundle calculations of geometric interest.

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2. Review of connection theory*

Let G be a Lie group with finitely many components and Lie algebra $\mathcal{G}$. Let M be a C^{∞} oriented manifold, and $\{E, M\}$ a *principal G-bundle* over M with projection $\pi\colon E \to M$. $\mathrm{R}g\colon E \to E$ will denote right action by $g \in G$. If $\{E', M'\}$ is another principal G-bundle and $\varphi\colon E \to E'$ is a C^{∞} map commuting with right action, φ is called a *bundle map*. Such a map defines $\varphi\colon M \to M'$, and the use of the same symbol should lead to no confusion.

Let $\{E_G, B_G\}$ be a universal bundle and classifying space for G. B_G is not a manifold. Its key feature is that every principal G-bundle over M admits a bundle map into $\{E_G, B_G\}$, and any two such maps of the same bundle are homotopic. If Λ is any coefficient ring, $u \in H^k(B_G, \Lambda)$, and $\alpha = \{E, M\}$, then the *characteristic class*

$$u(\alpha) \in H^k(M, \Lambda)$$

is well-defined by pulling back u under any bundle map. Since G is assumed to have only finitely many components it is well-known that

(2.1) $$H^{2l-1}(B_G, R) = 0 \ \text{ all } l \ .$$

We finally recall that E_G is *contractible*.

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* This chapter summarizes material presented in detail in [7].

Let $\mathcal{G}^l = \overbrace{\mathcal{G} \otimes \mathcal{G} \otimes \cdots \otimes \mathcal{G}}^{l}$. Polynomials of degree l are defined to be symmetric, multilinear maps from $\mathcal{G}^l \to R$. G acts on $\mathcal{G}^l$ by inner automorphism, and polynomials invariant under this action are called *invariant polynomials* of degree l, and are denoted by $I^l(G)$. These multiply in a natural way, and if $P \in I^l(G)$, $Q \in I^{l'}(G)$ then $PQ \in I^{l+l'}(G)$. We set $I(G) = \sum \otimes I^l(G)$, a graded ring.

These polynomials give information about the real cohomology of B_G. In fact, there is a *universal Weil homomorphism*

(2.2) $$W: I^l(G) \longrightarrow H^{2l}(B_G, R)$$

such that $W: I(G) \to H^{\text{even}}(B_G, R)$ is a ring homomorphism.

If $\{E, M\}$ is a principal G-bundle over M we denote by $\Lambda^{k,l}(E)$ k-forms on E taking values in $\mathcal{G}^l$. We have the usual exterior differential $d: \Lambda^{k,l}(E) \to \Lambda^{k+1,l}(E)$. If $\varphi \in \Lambda^{k,l}(E)$ and $\psi \in \Lambda^{k',l'}(E)$ define

$$\varphi \wedge \psi \in \Lambda^{k+k',l+l'}(E)$$

$$\varphi \wedge \psi(x_1, \cdots, x_{k+k'}) = \sum_{\pi \text{ shuffle}} \sigma(\pi)\varphi(x_{\pi_1}, \cdots, x_{\pi_k}) \otimes \psi(x_{\pi_{k+1}}, \cdots, x_{\pi_{k+k'}}) .$$

If $\varphi \in \Lambda^{k,l}(E)$ and $\psi \in \Lambda^{k',l}(E)$ define

$$[\varphi, \psi] \in \Lambda^{k+k',l}(E)$$

$$[\varphi, \psi](x_1, \cdots, x_{k+k'}) = \sum_{\pi \text{ shuffle}} \sigma(\pi)[\varphi(x_{\pi_1}, \cdots, x_{\pi_k}), \psi(x_{\pi_{k+1}}, \cdots, x_{\pi_{k+k'}})] .$$

Let P be a polynomial of degree l and $\varphi \in \Lambda^{k,l}(E)$. Then $P(\varphi) = P \circ \varphi$ is a *real valued* k-form on E. The following are elementary

(2.3) $[\varphi, \psi] = (-1)^{kk'+1}[\psi, \varphi]$

(2.4) $[[\varphi, \varphi], \varphi] = 0$

(2.5) $d[\varphi, \psi] = [d\varphi, \psi] + (-1)^k[\varphi, d\psi]$

(2.6) $d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^k \varphi \wedge d\psi$

(2.7) $d(P(\varphi)) = P(d\varphi)$

(2.8) $P(\varphi \wedge \psi \wedge \rho) = (-1)^{kk'}P(\psi \wedge \varphi \wedge \rho)$

where $\varphi \in \Lambda^{k,l}$, $\psi \in \Lambda^{k',l'}$, $\rho \in \Lambda^{k'',l''}$ and in the first three lines $l = l' = 1$. If $P \in I^l(G)$ then differentiating the invariance condition shows

(2.9) $\sum_{i=1}^l (-1)^{k_1+\cdots+k_i}P(\psi_1 \wedge \cdots \wedge [\psi_i, \varphi] \wedge \cdots \wedge \psi_l) = 0$

where $\psi_i \in \Lambda^{k_i,1}(E)$ and $\varphi \in \Lambda^{1,1}(E)$.

For $e \in E$, let $T(E)_e$ denote the tangent space of E at e and $V(E)_e = \{x \in T(E)_e \mid d\pi(x) = 0\}$. $V(E)_e$ is called the *vertical space*, and may be canonically identified with $\mathcal{G}$. If $x \in V(E)_e$ we let $\bar{x} \in \mathcal{G}$ denote its image

under this identification. A *connection* on $\{E, M\}$ in a $\mathcal{G}$ valued 1-form, θ, on E satisfying $R_g^*(\theta) = \mathrm{ad}_g^{-1} \circ \theta$, and $\theta(v) = \bar{v}$ for vertical v. If θ is a connection, setting $H(E)_e = \{x \in T(E)_e \mid \theta(x) = 0\}$ defines a complement to $V(E)_e$ called the *horizontal space*; i.e., $T(E)_e \cong V(E)_e \oplus H(E)_e$ and $dR_g\big(H(E)_e\big) = H(E)_{R_g(e)}$. The *structural equation* states

$$(2.10) \qquad\qquad\qquad d\theta = \Omega - \frac{1}{2}[\theta, \theta]$$

where Ω is the *curvature form*. $\Omega \in \Lambda^{2,1}(E)$ and is horizontal, i.e., $\Omega(x, y) = \Omega\big(H(x), H(y)\big)$, $H(x)$, and $H(y)$ denoting the horizontal projections of x and y. (2.4) and (2.5) show

$$(2.11) \qquad\qquad\qquad d\Omega = [\Omega, \theta].$$

An element $\varphi \in \Lambda^{k,l}$ is called *equivariant* if $R_g^*(\varphi) = \mathrm{ad}\, g^{-1} \circ \varphi$. A connection is equivariant by definition, and so is its curvature by (2.10), as equivariance is preserved under d, wedge products, and brackets. If $\varphi \in \Lambda^{k,l}(E)$ is equivariant and $P \in I^l(G)$ then $P(\varphi)$ is a real valued *invariant* k-form on E. In particular, $\Omega^l = \overbrace{\Omega \wedge \cdots \wedge \Omega}^{l}$ is equivariant, and so $P(\Omega^l)$ is real valued, invariant and horizontal, and so uniquely defines a $2l$-form on M whose lift is $P(\Omega^l)$. We will also denote this form on M by $P(\Omega^l)$. Formulae (2.11) and (2.9) immediately show this form is *closed*.

THEOREM 2.12 (*Weil homomorphism*). *Let* $\alpha = \{E, M, \theta\}$ *be a principal G-bundle with connection, and let* $P \in I^l(G)$. *Then*

$$P(\Omega^l) \in W(P)(\alpha) \; ;$$

i.e., $P(\Omega^l)$ *represents the characteristic class corresponding to the universal Weil image of* P.

For some of the calculations in the sections that follow it will be convenient to have classifying bundles equipped with connections. To do this we use a theorem of Narasimhan-Ramanan [10]. We introduce the category $\varepsilon(G)$. *Objects* in $\varepsilon(G)$ are triples $\alpha = \{E, M, \theta\}$ where $\{E, M\}$ is a principal G-bundle with connection θ. *Morphisms* are connection-preserving bundle maps; i.e., if $\alpha = \{E, M, \theta\}$ and $\hat{\alpha} = \{\hat{E}, \hat{M}, \hat{\theta}\}$, and $\varphi: \{E, M\} \to \{\hat{E}, \hat{M}\}$ is a bundle map, then $\varphi: \alpha \to \hat{\alpha}$ is a morphism if $\varphi^*(\hat{\theta}) = \theta$. An object $A \in \varepsilon(G)$ is called *n-classifying* if two conditions hold: First for every $\alpha \in \varepsilon(G)$ with $\dim M \leq n$ there exists a morphism $\varphi: \alpha \to A$. Second, any two such morphisms are homotopic through bundle maps. We do not require the homotopy to be via morphisms.

THEOREM 2.13 (*Narasimhan-Ramanan*). *For each integer n there exists an n-classifying* $A \in \varepsilon(G)$.

3. The forms $TP(\theta)$

Let $\alpha = \{E, M, \theta\} \in \varepsilon(G)$. The bundle $\{\pi^*(E), E\}$ is trivial as a principal G-bundle, and so all of its characteristic cohomology vanishes. Thus $\pi^*\big(P(\Omega^l)\big) = P(\Omega^l)$ is exact when considered as a form on E. Set $\varphi_t = t\Omega + 1/2(t^2 - t)[\theta, \theta]$, and set

$$(3.1) \qquad\qquad TP(\theta) = l \int_0^1 P(\theta \wedge \varphi_t^{l-1}) dt \ .$$

$P \in I^l(G)$, and $TP(\theta)$ is a real-valued invariant $(2l-1)$-form on E. It is of course not horizontal.

PROPOSITION 3.2. $dTP(\theta) = P(\Omega^l)$.

Proof. Set $f(t) = P(\varphi_t^l)$. Then $f(0) = 0$ and $f(1) = P(\Omega^l)$. Thus

$$(3.3) \qquad\qquad P(\Omega^l) = \int_0^1 f'(t) dt \ .$$

We claim

$$(3.4) \qquad\qquad f'(t) = l dP(\theta \wedge \varphi_t^{l-1}) \ .$$

We first observe

$$\frac{d}{dt}(\varphi_t) = \Omega + \left(t - \frac{1}{2}\right)[\theta, \theta] \ .$$

Using (2.3)–(2.8) we have

$$f' = l P\left(\frac{d}{dt}(\varphi_t) \wedge \varphi_t^{l-1}\right)$$

$$= l P(\Omega \wedge \varphi_t^{l-1}) + l\left(t - \frac{1}{2}\right) P([\theta, \theta] \wedge \varphi_t^{l-1}) \ .$$

On the other hand,

$$l dP(\theta \wedge \varphi_t^{l-1}) = l P(d\theta \wedge \varphi_t^{l-1}) - l(l-1) P(\theta \wedge d\varphi_t \wedge \varphi_t^{l-2})$$

$$= l P(\Omega \wedge \varphi_t^{l-1}) - \frac{1}{2} l P([\theta, \theta] \wedge \varphi_t^{l-1}) - l(l-1) P(\theta \wedge d\varphi_t \wedge \varphi_t^{l-2})$$

by the structural equation (2.10). Now using (2.10), (2.11), and (2.4)

$$d\varphi_t = t[\varphi_t, \theta] \ .$$

Plugging this into the formula above and using the invariance formula (2.9) on the last piece we get

$$ldP(\theta \wedge \varphi_t^{l-1}) = lP(\Omega \wedge \varphi_t^{l-1}) - \frac{1}{2}lP\big([\theta, \theta] \wedge \varphi_t^{l-1}\big) + ltP\big([\theta, \theta] \wedge \varphi_t^{l-1}\big) = f'$$

by the computation above. This shows (3.4) and the proposition follows from (3.3).

The form $TP(\theta)$ can of course be written without the integral, and, in fact, setting

$$A_i = (-1)^i l! \, (l-1)!/2^i (l+i)! \, (l-1-i)!$$

one computes

(3.5) $$TP(\theta) = \sum_{i=0}^{l-1} A_i P\big(\theta \wedge [\theta, \theta]^i \wedge \Omega^{l-i-1}\big) \, .$$

The operation which associates to $\alpha \in \varepsilon(G)$ the form $TP(\theta)$ is natural; i.e., if $\varphi \colon \alpha \to \hat{\alpha}$ is a morphism, since $\varphi^*(\hat{\theta}) = \theta$ and thus $\varphi^*(\hat{\Omega}) = \Omega$, clearly $\varphi^*\big(TP(\hat{\theta})\big) = TP(\theta)$. This naturality characterizes T up to an exact remainder:

PROPOSITION 3.6. *Given* $P \in I^l(G)$, *let* S *be another functor which associates to each* $\alpha \in \varepsilon(\theta)$ *a* $(2l-1)$-*form in* E, $SP(\theta)$, *which satisfies* $dSP(\theta) = P(\Omega^l)$. *Then* $TP(\theta) - SP(\theta)$ *is exact.*

Proof. Let $\alpha = \{E, M, \theta\}$ with $\dim M = n$. Choose $\hat{\alpha} = \{\hat{E}, \hat{M}, \hat{\theta}\} \in \varepsilon(G)$ so that $\hat{\alpha}$ is m classifying with m sufficiently greater than n. Let $\varphi \colon \alpha \to \hat{\alpha}$ be a morphism. Now in $\hat{E}$ we have $dSP(\hat{\theta}) = dTP(\hat{\theta}) \Rightarrow SP(\hat{\theta}) - TP(\hat{\theta})$ is closed. But since $\hat{E}$ is an approximation to E_G its $2l-1$ cohomology vanishes for sufficiently large m. Thus $SP(\hat{\theta}) = TP(\hat{\theta}) + \text{exact}$. So by the naturality assumption on S, $SP(\theta) = \varphi^* SP(\hat{\theta}) = \varphi^* TP(\hat{\theta}) + \varphi^* \text{ exact} = TP(\theta) + \text{exact}$. $\square$

PROPOSITION 3.7. *Let* $P \in I^l(G)$ *and* $Q \in I^s(G)$.
(1) $PQ(\Omega^{l+s}) = P(\Omega^l) \wedge Q(\Omega^s)$
(2) $TPQ(\theta) = TP(\theta) \wedge Q(\Omega^s) + \text{exact} = TQ(\theta) \wedge P(\Omega^l) + \text{exact}.$

Proof. (1) is immediate. To prove (2) we may use naturality and work in a classifying bundle. But there, $d\big(TP(\theta) \wedge Q(\Omega^s)\big) = P(\Omega^l) \wedge Q(\Omega^s) = PQ(\Omega^{l+s}) = d\big(TPQ(\theta)\big)$. Similarly $d\big(TQ(\theta) \wedge P(\Omega^l)\big) = d\big(TPQ(\theta)\big)$. (2) then follows by low dimensional acyclicity of the total space of the classifying bundle. $\square$

We are interested in how the forms $TP(\theta)$ change as the connection changes.

PROPOSITION 3.8. *Let* $\theta(s)$ *be a smooth* 1-*parameter family of connections on* $\{E, M\}$ *with* $s \in [0, 1]$. *Set* $\theta = \theta(0)$ *and* $\theta' = (d/ds)\big(\theta(s)\big)|_{s=0}$. *For* $P \in I^l(G)$

$$\frac{d}{ds}\big(TP(\theta(s))\big)\,|_{s=0} = lP(\theta' \wedge \Omega^{l-1}) + exact \; .$$

Proof. Building on the theorem of Narasimhan-Ramanan it is not difficult to show that one can find a principal G-bundle $\{\hat{E}, \hat{M}\}$ which classifies bundles over manifolds of dim $\geqq m \geqq$ dim M, and to equip this bundle with a smooth family of connections $\hat{\theta}(s)$, and to find a bundle map $\varphi \colon \{E, M\} \rightarrow \{\hat{E}, \hat{M}\}$ such that $\varphi^*(\hat{\theta}(s)) = \theta(s)$ $s \in [0, 1]$. It thus suffices to prove the theorem in $\{\hat{E}, \hat{M}\}$, and by choosing m large enough $\hat{E}$ will be acyclic in dimensions $\leqq 2l-1$. We now drop all "hats" and simply assume $H^{2l-1}(E, R) = 0$. Thus it is sufficient to prove both sides of the equation have the same differential. Now

$$d\Big(\frac{d}{ds}\big(TP\theta(s)\big)\,|_{s=0}\Big) = \frac{d}{ds}\big(dTP(\theta(s))\big)\,|_{s=0}$$

$$= \frac{d}{ds}\big(P(\Omega(s)^l)\,|_{s=0}\big) = lP(\Omega' \wedge \Omega^{l-1})$$

where $\Omega' = (d/ds)\big(\Omega(s)\big)\,|_{s=0}$. Also

$$d\big(lP(\theta' \wedge \Omega^{l-1})\big) = lP(d\theta' \wedge \Omega^{l-1}) - l(l-1)P(\theta' \wedge d\Omega \wedge \Omega^{l-2})$$
$$= lP(d\theta' \wedge \Omega^{l-1}) - l(l-1)P(\theta' \wedge [\Omega, \theta] \wedge \Omega^{l-2}) \;\; \text{by (2.11)}$$
$$= lP(d\theta' \wedge \Omega^{l-1}) + lP([\theta', \theta] \wedge \Omega^{l-1}) \;\; \text{by (2.9)} \; .$$

Now $d\theta' = d\big((d/ds)\big(\theta(s)\big)\,|_{s=0}\big) = (d/ds)\big(d\theta(s)\big)\,|_{s=0} = (d/ds)\big(\Omega(s) - (1/2)[\theta(s), \theta(s)]\big)\,|_{s=0} = \Omega' - [\theta', \theta]$. Putting this in the calculation above shows

$$dlP(\theta' \wedge \Omega^{l-1}) = lP(\Omega' \wedge \Omega^{l-1})$$

and this with the first calculation completes the proof. □

If $P \in I^l(G)$ and $P(\Omega^l) = 0$ then $TP(\theta)$ is closed in E and so defines a cohomology class in E. We denote this class by $\{TP(\theta)\} \in H^{2l-1}(E, R)$.

THEOREM 3.9. *Let* $\alpha = \{E, M, \theta\}$ *with* dim $M = n$. *If* $2l - 1 = n$ *then* $TP(\theta)$ *is closed and* $\{TP(\theta)\} \in H^n(E, R)$ *depends on* θ. *If* $2l - 1 > n$ *then* $TP(\theta)$ *is closed and* $\{TP(\theta)\} \in H^{2l-1}(E, R)$ *is independent of* θ.

Proof. $P(\Omega^l)$ is a horizontal $2l$-form. If $2l - 1 \geqq n$ then $2l > n$ and since the dimension of the horizontal space is exactly n, $P(\Omega^l) = 0$. Thus $TP(\theta)$ is closed, and $\{TP(\theta)\}$ is defined. We will see in a later section that when $2l - 1 = n$, $\{TP(\theta)\}$ depends on the connection. However, suppose $2l - 1 > n$. Since any two connections may be joined by a smooth 1-parameter family, it is sufficient to show, using the notation of the previous proposition that

$$\frac{d}{ds}\big(TP(\theta(s))\big)\,|_{s=0} = \text{exact} .$$

By that proposition it is sufficient to show $P(\theta' \wedge \Omega^{l-1}) = 0$. Since θ' is the derivative of a family of connections, all of which must agree on vertical vectors, $\theta'(v) = 0$ for v vertical. Thus $P(\theta' \wedge \Omega^{l-1})$ is a horizontal $(2l-1)$-form, and thus must vanish for $2l - 1 > n$. $\qquad\square$

The equation in E, $dTP(\theta) = P(\Omega^l)$, implies that $TP(\theta)\,|\,E_m$ is a closed form, where E_m is the fibre over $m \in M$. Formula (3.5) shows that $TP(\theta)\,|\,E_m$ is expressed purely in terms of $\theta\,|\,E_m$, which is independent of the connection. More specifically, let ω denote the Maurer-Cartan form on G, which assigns to each tangent vector the corresponding Lie algebra element. Set

$$(3.10) \qquad TP = \frac{(-1)^{l-1}}{2^l \binom{2l-1}{l}} P(\omega \wedge [\omega, \omega]^{l-1}) .$$

TP is a real valued, bi-invariant $(2l-1)$-form on G. It is closed and represents an element of $H^{2l-1}(G, R)$. For $m \in M$ and $e \in E_m$ let $\lambda \colon G \to E_m$ by $\lambda(g) = R_g(e)$. Then $\lambda^*(\theta) = \omega$, and (3.5) shows

$$(3.11) \qquad \lambda^*\big(TP(\theta)\big) = TP .$$

The class $\{TP\} \in H^{2l-1}(G, R)$ is universally transgressive in the sense of [1]. In fact, recalling Borel's definition of transgressive ([1], p. 133), a class $h \in H^k(G, \Lambda)$ is called transgressive in the fibre space $\{E, M\}$ if there exists $c \in C^k(G, \Lambda)$ so that $c\,|\,G \in h$ and δc is a lift of a cochain (and thus a cocycle) from the base. It is called universally transgressive if this happens in the classifying bundle. In this case the transgression goes from $\{TP\}$ via $TP(\theta)$ to $P(\Omega^l)$. One can do this over the integers as well as the reals, and if we set

$$I_0^l(G) = \{P \in I^l(G) \mid W(P) \in H^{2l-1}(B_G, Z)\}$$

one can easily show

$$(3.12) \qquad P \in I_0^l(G) \implies \{TP\} \in H^{2l-1}(G, Z)$$

and (3.11) shows this is equivalent to

$$(3.13) \qquad P \in I_0^l(G) \implies TP(\theta)\,|\,E_m \in H^{2l-1}(E_m, Z)$$

where in all these equations we mean the real image of the integral cohomology. The following proposition will provide a proof of this, but also will give us some extra understanding of the form $TP(\theta)$ when $P \in I_0^l(G)$.

For a real number a let $\tilde{a} \in R/Z$ denote its reduction, and similarly for

any real cochain or cohomology class $\sim$ will denote its reduction mod Z. The Bockstein exact sequence

$$(3.14) \quad \longrightarrow H^i(X, Z) \xrightarrow{r} H^i(X, R) \rightsquigarrow H^i(X, R/Z) \xrightarrow{b} H^{i+1}(X, Z) \longrightarrow$$

shows that a real class, U, is an integral class if and only if $\tilde{U} = 0$. For X any manifold and Λ any coefficient group we let $C(X, \Lambda)$ denote the cochain group with respect to the group of *smooth singular* chains. If φ is a differential form on X then $\varphi \in C(X, R)$, and by $\tilde{\varphi} \in C(X, R/Z)$ we mean its reduction mod Z as a real cochain.

PROPOSITION 3.15. *Let* $\alpha = \{E, M, \theta\} \in \varepsilon(G)$. *Then if* $P \in I_0^l(G)$ *there exists* $u \in C^{2l-1}(M, R/Z)$ *so that*

$$\widetilde{TP(\theta)} = \pi^*(u) + coboundary \ .$$

Proof. Let $\hat{\alpha} = \{\hat{E}, \hat{M}, \hat{\theta}\} \in \varepsilon(G)$ be k-classifying with k sufficiently large. Since $P \in I_0^l$ we know that $P(\hat{\Omega}^l)$ represents an integral class in $\hat{M}$. Thus the R/Z cocycle $\widetilde{P(\hat{\Omega}^l)}$ vanishes on all cycles in $\hat{M}$, and so is an R/Z coboundary; i.e., there exists $\hat{u} \in C^{2l-1}(\hat{M}, R/Z)$ such that $\delta\hat{u} = \widetilde{P(\hat{\Omega}^l)}$. Thus

$$\delta\pi^*(\hat{u}) = \pi^*(\delta\hat{u}) = \pi^*\big(\widetilde{P(\hat{\Omega}^l)}\big)$$
$$= \widetilde{\pi^*\big(P(\hat{\Omega}^l)\big)} = d\widetilde{TP(\hat{\theta})} = \delta\widetilde{TP(\hat{\theta})} = \delta\big(\widetilde{TP(\hat{\theta})}\big) \ .$$

So $\delta\pi^*(\hat{u}) = \delta\big(\widetilde{TP(\hat{\theta})}\big)$. Since we have chosen k large, $\hat{E}$ is acyclic in dim $2l-1$, and so

$$\widetilde{TP(\hat{\theta})} = \pi^*(\hat{u}) + \text{coboundary} \ .$$

The proposition then follows in general by choosing a morphism $\varphi: \alpha \to \hat{\alpha}$ and taking $u = \varphi^*(\hat{u})$. $\qquad\square$

We note that (3.13) and hence (3.12) follow directly from this. We also note that for these special polynomials, the classes $\{TP(\theta)\}$, when they exist, have the property that their mod Z reductions are already lifts. That is

THEOREM 3.16. *Let* $\alpha = \{E, M, \theta\} \in \varepsilon(G)$ *and let* $P \in I_0^l(G)$. *Suppose* $P(\Omega^l) = 0$. *Then there exists* $U \in H^{2l-1}(M, R/Z)$ *so that*

$$\{\widetilde{TP(\theta)}\} = \pi^*(U) \ .$$

Proof. Choose $u \in C^{2l-1}(M, R/Z)$ as in Proposition 3.14. The assumption $P(\Omega^l) = 0$ implies $\pi^*(\delta u) = 0$. Since every chain in M comes from one in E this means $\delta u = 0$. Thus u is an R/Z cocycle in M, and Proposi-

tion 3.14 shows $\pi^*(u) \sim \widetilde{TP(\theta)}$. Letting $U \in H^{2l-1}(M, R/Z)$ denote the class represented by u the theorem follows.

Characteristic numbers in R/Q. An interesting special case of this theorem occurs when M is compact, oriented, and dim $M = 2l - 1$. Then for each $P \in I_0^l(G)$ we know that $P(\Omega^l) = 0$ and $\{TP(\theta)\} \in H^{2l-1}(E, R)$ depends on the connection. On the other hand, reducing mod Z, $\{\widetilde{TP(\theta)}\} = \pi^*(U)$ for some $U \in H^{2l-1}(M, R/Z) \cong R/Z$. Thus U is determined up to an element of ker π^*. Now, either ker $\pi^* = H^{2l-1}(M, R/Z)$, or ker π^* is a finite subgroup of $H^{2l-1}(M, R/Z)$. In the second case, since all finite subgroups of R/Z lie in Q/Z, U is determined uniquely in $R/Z/Q/Z \cong R/Q$. Let μ denote the fundamental cycle of M. Define $SP(\theta) \in R/Q$ by

$$SP(\theta) = 0 \qquad \text{if ker } \pi^* = H^{2l-1}(M, R/Z)$$
$$SP(\theta) = u(\mu)/Q \quad \text{otherwise .}$$

Examples in the last section will show that these numbers are nontrivial invariants.*

COROLLARY 3.17. *Suppose* dim $M < 2l - 1$. *Then for* $P \in I_0^l(G)$

$$\{TP(\theta)\} \in H^{2l-1}(E, Z) .$$

Proof. Since dim $M < 2l - 1$, $H^{2l-1}(M, R/Z) = 0$ and so $\{\widetilde{TP(\theta)}\} = 0$. Thus from (3.14) $\{TP(\theta)\}$ is the image of an integral class.

4. Conformal invariance

In this section we suppose $G = Gl(n, R)$. $\mathcal{G}$ consists of all $n \times n$ matrices, and we define the basic invariant polynomials $Q_1, \cdots, Q_n$

$$Q_l(A_1 \otimes \cdots \otimes A_l) = \frac{1}{l!} \sum_\pi \text{tr } A_{\pi_1} A_{\pi_2} \cdots A_{\pi_l} .$$

It is well known that the Q_i generate the ring of invariant polynomials on $\mathcal{G}$. If $\alpha = \{E, M, \theta\}$ is a principal G bundle then $\theta = \{\theta_{ij}\}$ and $\Omega = \{\Omega_{ij}\}$, matrices of real valued 1 and 2-forms respectively. One verifies directly that for any $\varphi = \{\varphi_{ij}\} \in \Lambda^{k,1}(E)$

(4.1) $\qquad Q_l(\Omega^l) = \sum_{i_1, \cdots, i_l=1}^n \Omega_{i_1, i_2} \wedge \Omega_{i_2, i_3} \wedge \cdots \wedge \Omega_{i_l, i_1}$

(4.2) $\qquad Q_l(\varphi \wedge \Omega^{l-1}) = \sum_{i_1, \cdots, i_l=1}^n \varphi_{i_1, i_2} \wedge \Omega_{i_2, i_3} \wedge \cdots \wedge \Omega_{i_l, i_1} .$

These polynomials have different properties. In particular the Weil map

---

* This construction was made in discussion with J. Cheeger. It is an easy way of producing the mod Q reductions of R/Z invariants developed in [3].

(see (2.2)) takes the ring generated by $\{Q_{2l}\}$ isomorphically onto the real cohomology of $B_{Gl(n,R)} = B_{O(n)}$, while the kernel of the Weil map is the ideal generated by the $\{Q_{2l+1}\}$.

PROPOSITION 4.3. *Let* $\alpha = \{E, M, \theta\} \in \varepsilon\big(Gl(n, R)\big)$. *Suppose* θ *restricts to a connection on an* $O(n)$ *subbundle of* E. *Then* $Q_{2l+1}(\Omega^{2l+1}) = 0$, *and* $TQ_{2l+1}(\theta)$ *is exact.*

Proof. The first fact is well known and is one way to prove $Q_{2l+1} \in \mathrm{Ker}\, W$. Our assumption on θ is that there is an $O(n)$ subbundle $F \subsetneq E$ such that at each $f \in F$, $H(E)_f \subsetneq T(F)_f$, or equivalently that at all x tangent to F, $\theta_{ij}(x) = -\theta_{ji}(x)$. It easily implies that at all points of F, $\Omega_{ij} = -\Omega_{ji}$ as a form on E. Now if A is a skew symmetric matrix then $\mathrm{tr}\,(A^{2l+1}) = 0$ and by polarization we see $Q_{2l+1}(A_1 \otimes \cdots \otimes A_{2l+1}) = 0$ when all A_i are skew symmetric. Since $Q_{2l+1}(\Omega^{2l+1})$ is invariant, we need only show it vanishes at points in F, but at these points the range of Ω^{2l+1} lies in the kernel of Q_{2l+1}. Thus $Q_{2l+1}(\Omega^{2l+1}) = 0$. The same argument shows that $TQ_{2l+1}(\theta)\,|\,F = 0$. (Here we mean the form restricted to the submanifold, F, and not simply as a form on E considered at points of F.) Thus $TQ_{2l+1}(\theta)$ is a closed form in E whose restriction to F is 0. Since E is contractible to F, $TQ_{2l+1}(\theta)$ can carry no cohomology on E and hence must be exact. $\qquad\square$

Let us now specialize to the case where $E = E(M)$, the bundle of bases of the tangent bundle of M. Points in E are $(n + 1)$-tuples of the form $(m; e_1, \cdots, e_n)$ where $m \in M$ and $e_1, \cdots, e_n$ is a basis of $T(M)_m$. E comes equipped with a natural set of horizontal, real valued forms $\omega_1, \cdots, \omega_n$, defined by

$$d\pi(x) = \sum_{i=1}^{n} \omega_i(x)e_i$$

where $x \in T(E)_e$, and $e = (m; e_1, \cdots, e_n)$. Now let g be a Riemannian metric on M, and let θ be the associated Riemannian connection of $E(M)$. Let $E_1, \cdots, E_n$ be horizontal vector fields which are a dual basis to $\omega_1, \cdots, \omega_n$. Let $F(M)$ denote the orthonormal frame bundle. $F(M) \subsetneq E(M)$ is the $0(n)$ subbundle consisting of orthonormal bases, and since θ is the Riemannian connection, θ restricts to a connection on $F(M)$.

Let h be a C^∞ function on M, and consider the curve of conformally related metrics

$$g(s) = e^{2sh}g , \qquad\qquad s \in [0, 1] .$$

Let $\theta(s)$ denote the curve of associated Riemannian connections on $E(M)$. Let $\theta = \theta(0)$, $\theta' = (d/ds)\big(\theta(s)\big)\,|_{s=0}$, and $F(M)$ the frame bundle with respect to $g = g(0)$.

LEMMA 4.4. *At points in* $F(M)$

$$\theta'_{ij} = \delta_{ij}d(h \circ \pi) + E_i(h \circ \pi)\omega_j - E_j(h \circ \pi)\omega_i .$$

Proof. This is a standard computation, and is perhaps most easily done by using the formula for the Riemannian connection in terms of covariant differentiation (cf. [7]). It is easily seen how the connection changes under conformal change of metric, and one then translates this result back into bundle terminology.

THEOREM 4.5. *Let g and $\hat{g}$ be conformally related Riemannian metrics on M, and let θ, Ω, $\hat{\theta}$, $\hat{\Omega}$ denote the corresponding connection and curvature forms. Then for any $P \in I^l(Gl(n, R))$*
  (1) $TP(\hat{\theta}) = TP(\theta) + exact$,
  (2) $P(\hat{\Omega}^l) = P(\Omega^l)$.

COROLLARY. $P(\Omega^l) = 0$ *implies that the cohomology class* $\{TP(\theta)\} \in H^{2l-1}(E(M), R)$ *is a conformal invariant.*

The corollary follows immediately from the theorem, and (2) follows immediately from (1) and Proposition 3.2. So it remains to prove (1). Since the Q_i generate $I(Gl(n, R))$ we can assume P is a monomial in the Q_i. Using Proposition 3.7, an inductive argument shows that it is sufficient to prove (1) only in the case $P = Q_l$. Proposition 4.3 shows that for any Riemannian connection $Q_{2l+1}(\Omega^{2l+1}) = 0$ and $TQ_{2l+1}(\theta)$ is exact, so we can assume l is even.

Any two conformally related metrics can be joined by a curve of such metrics, with associated connections $\theta(s)$. By integration it is sufficient to prove

(*) $$\frac{d}{ds}\big(TQ_{2l}\big(\theta(s)\big)\big) = \text{exact} .$$

Since each point on the curve is the initial point of another such curve, it is enough to prove (*) at $s = 0$. By Proposition 3.8 it will suffice to prove

(**) $$Q_{2l}(\theta' \wedge \Omega^{2l-1}) = 0 .$$

We use the notation and formula of Lemma 4.4, and work at $f \in F(M)$. Set

$$\alpha = \big(\delta_{ij}d(f \circ \pi)\big)$$
$$\beta = \big(E_i(f \circ \pi)\omega_j - E_j(f \circ \pi)\omega_i\big) .$$

Then $\theta' = \alpha + \beta$. Now (4.2) shows

$$Q_{2l}(\alpha \wedge \Omega^{2l-1}) = d(f \circ \pi) \wedge Q_{2l-1}(\Omega^{2l-1}) = 0$$

by Proposition 4.3. Also using (4.2),

$$Q_{2l}(\beta \wedge \Omega^{2l-1}) = \sum_{i_1, \cdots, i_{2l}} \frac{E_{i_1}(f \circ \pi)\omega_{i_2} \wedge \Omega_{i_2,i_3} \wedge \cdots \wedge \Omega_{i_{2l},i_1}}{-E_{i_2}(f \circ \pi)\omega_{i_1} \wedge \Omega_{i_2,i_3} \wedge \cdots \wedge \Omega_{i_{2l},i}} .$$

But, since θ is a Riemannian connection, the Jacobi identity holds. This states

$$\sum_{i=1}^{n} \omega_i \wedge \Omega_{ij} = 0 ,$$

and shows $Q_{2l}(\beta \wedge \Omega^{2l-1}) = 0$. Thus at points in $F(M)$, $Q_{2l}(\theta' \wedge \Omega^{2l-1}) = 0$, and (**) follows by invariance. $\qquad \square$

5. Conformal immersions

Let $G = U(n)$. Let A be a skew Hermitian matrix and define the ith Chern polynomial $C_i \in I_0^i(U(n))$

$$(5.1) \qquad \det\left(\lambda I - \frac{1}{2\pi\sqrt{-1}}A\right) = \sum_{i=0}^{n} C_i(\overbrace{A \otimes \cdots \otimes A}^{i})\lambda^{n-i}$$

where C_i is extended by polarization to all tensors. Let c_i denote the ith integral Chern class in $B_{U(n)}$. Then $c_i \in H^{2i}(B_G, Z)$, and letting $r(c_i) \in H^{2i}(B_G, R)$ denote its real image,

$$(5.2) \qquad\qquad\qquad W(C_i) = r(c_i) .$$

We also define the inverse Chern polynomials and classes $C_i^{\perp}$ and $c_i^{\perp}$

$$(5.3) \qquad \begin{aligned} (1 + C_1^{\perp} + \cdots + C_i^{\perp} + \cdots)(1 + C_1 + \cdots + C_n) &= 1 \\ (1 + c_1^{\perp} + \cdots + c_i^{\perp} + \cdots) \cup (1 + c_1 + \cdots + c_n) &= 1 . \end{aligned}$$

These formulae uniquely determine $C_i^{\perp}$ and $c_i^{\perp}$, and since W is a ring homomorphism

$$(5.4) \qquad\qquad\qquad W(C_i^{\perp}) = c_i^{\perp} .$$

The inverse classes are so named because, for vector bundles, they are the classes of an inverse bundle. That is, if V, W are complex vector bundles over M with $V \oplus W$ trivial, then using the product formula for Chern class, cf. [9], one knows

$$(5.5) \qquad\qquad\qquad c_i(W) = c_i^{\perp}(V) .$$

Let $G_{n,k}(c)$ denote the Grassmann manifold of complex n-planes in C^{n+k}, and let $E_{n,k}(c)$ denote the Stiefel manifold of orthonormal n-frames in C^{n+k}, with respect to the Hermitian metric. Then $\{E_{n,k}(c), G_{n,k}(c)\}$ is a principal $U(n)$ bundle. There is a natural connection in this bundle most easily visualized by constructing it in the associated canonical n-dim vector bundle over $G_{n,k}(c)$. Let $\gamma(t)$ be a curve in $G_{n,k}(c)$ and let $\rho(t)$ be a curve in the

vector bundle with $\pi \circ \rho = \gamma$. So for each t, $\gamma(t)$ is an n-plane in C^{n+k}, and $\rho(t)$ is a vector in C^{n+k} with $\rho(t) \in \gamma(t)$. Then $\rho'(t) = (d/dt)(\rho(t))$ is a vector in C^{n+k}, and the covariant derivative of $\rho(t)$ along γ is obtained by orthogonally projecting $\rho'(t)$ into $\gamma(t)$. We let θ denote this connection and set

$$\alpha_{n,k}(c) = \{E_{n,k}(c), G_{n,k}(c), \theta\} \ .$$

PROPOSITION 5.6. *For $i > k$*

(1) $C_i^\perp(\Omega^i) = 0$

(2) $\{TC_i^\perp(\theta)\} \in H^{2i-1}(E_{n,k}(c), Z) \ .$

Proof. Since the n-dim vector bundle associated to $\{E_{n,k}(c), G_{n,k}(c)\}$ has a k-dim inverse, (5.5) shows that $c_i^\perp(\alpha_{n,k}(c)) = 0$ for $i > k$. Thus the form $C_i^\perp(\Omega^i)$ is exact on $G_{n,k}(c)$. Now $G_{n,k}(c)$ is a compact, irreducible Riemannian symmetric space, and it is easily checked that the forms $P(\Omega^i)$ are invariant under the isometry group. Thus $C_i^\perp(\Omega^i)$ is invariant and exact, and therefore must vanish. So the class $\{TC_i^\perp(\theta)\} \in H^{2i-1}(E_{n,k}(c), R)$ is defined. Since $W(C_i^\perp) = c_i^\perp \in H^{2i}(B_{U(n)}, Z)$ we see that $C_i^\perp \in I_0^i(U(n))$. Using Theorem 3.16 we see that $\{\widetilde{TC_i^\perp}(\theta)\}$ is a lift of a $2i - 1$ dimensional R/Z cohomology class of $G_{n,k}(c)$. But the odd dimensional cohomology of this space is zero (for any coefficient group), and thus $\{\widetilde{TC_i^\perp}(\theta)\} = 0$. The Bockstein sequence (3.14) then shows that $\{TC_i^\perp(\theta)\} \in H^{2i-1}(E_{n,k}(c), Z)$.   □

Now let $G = 0(n)$. Let A be a skew symmetric matrix and define for $i = 1, \cdots, [n/2]$ the ith Pontrjagin polynomial $P_i \in I_0^{2i}(0(n))$

$$(5.7) \qquad \det\left(\lambda I - (1/2\pi)A\right) = \sum_{i=0}^{[n/2]} P_i(\overbrace{A \otimes \cdots \otimes A}^{2i})\lambda^{n-2i} + Q(\lambda^{n-\text{odd}})$$

where we ignore the terms involving the n-odd powers of λ. Also let $p_i \in H^{4i}(B_{0(n)}, Z)$ denote the ith integral Pontrjagin class. Then

$$W(P_i) = r(p_i) \ .$$

Let $\rho: 0(n) \rightarrow U(n)$ be the natural map. Then ρ induces $\rho^*: I(U(n)) \rightarrow I(0(n))$, $\rho^*: H^*(U(n)) \rightarrow H^*(0(n))$, and $\rho: B_{0(n)} \rightarrow B_{U(n)}$. Using Theorem 2.12 one easily sees

$$W(\rho^*(Q)) = \rho^*(W(Q))$$

for any $Q \in I^l(U(n))$. The definitions of P_i and p_i are such that

$$(5.8) \qquad\qquad \rho^*(C_{2i}) = (-1)^i P_i \ , \qquad \rho^*(c_{2i}) = (-1)^i p_i \ .$$

We also define the inverse Pontrjagin polynomials $P_i^\perp$

$$(5.9) \qquad (1 + P_1 + \cdots + P_{[n/2]})(1 + P_1^\perp + \cdots + P_i^\perp + \cdots) = 1$$

and note that $P_i^\perp \in I_0^{2i}\big(0(n)\big)$ since $\rho^*(c_i^\perp) \in H^{2i}(B_{0(n)}, Z)$, and one easily sees that

$$W(P_i^\perp) = (-1)^i r\big(\rho^*(c_{2i}^\perp)\big) \ .$$

Formula (5.9) shows $P_i^\perp = -P_i - P_{i-1}P_1^\perp - \cdots - P_i P_{i-1}^\perp$. Proposition 3.7 shows that $TP_i^\perp(\theta) = -TP_i(\theta) +$ terms involving curvature. Thus for any $\alpha = \{E, M, \theta\} \in \varepsilon\big(0(n)\big)$

(5.10) $$TP_i^\perp(\theta) \mid E_m = -TP_i(\theta) \mid E_m \ .$$

We now define the real Grassmann manifold, $G_{n,k}$, the real Stiefel manifold $E_{n,k}$, and the canonical connection θ on $\{E_{n,k}, G_{n,k}\}$ exactly as in the complex case. We set $\alpha_{n,k} = \{E_{n,k}, G_{n,k}, \theta\} \in \varepsilon\big(0(n)\big)$.

PROPOSITION 5.11. *For* $i > [k/2]$

(1) $$P_i^\perp(\Omega^{2i}) = 0$$
(2) $$\{(1/2)TP_i^\perp(\theta)\} \in H^{4i-1}(E_{n,k}, Z) \ .$$

Proof. The natural map $R^n \to C^n$ induces the commutative diagram

$$
\begin{array}{ccc}
E_{n,k} & \xrightarrow{\ \varphi\ } & E_{n,k}(c) \\
\downarrow & & \downarrow \\
G_{n,k} & \xrightarrow{\ \varphi\ } & G_{n,k}(c) \ .
\end{array}
$$

It is straightforward to check that

$$P_i^\perp(\Omega^{2i}) = (-1)^i \varphi^*\big(C_{2i}^\perp(\Omega^{2i})\big) \ ,$$
$$TP_i^\perp(\theta) = (-1)^i \varphi^*\big(TC_{2i}^\perp(\theta)\big) \ .$$

Since $i > [k/2] \Rightarrow 2i > k$, (1) follows from Proposition 5.6, and from (2) of that proposition we see that

$$\{TP_i^\perp(\theta)\} = (-1)^i \varphi^*\{TC_{2i}^\perp(\Omega^{2i})\} \in H^{4i-1}(E_{n,k}, Z) \ .$$

We will be finished when we show

LEMMA 5.12. *Let* $\gamma \in H^*\big(E_{n,k}(c), Z\big)$. *Then* $\varphi^*(\gamma)$ *is an even integral class in* $H^*(E_{n,k}, Z)$.

Proof. For any Lie group G and any coefficient group Λ we want to consider the inverse transgression map $\tau: H^i(B_G, \Lambda) \to H^{i-1}(G, \Lambda)$. This map is defined as follows. Let $u \in H^i(B_G, \Lambda)$ be given and choose $\gamma \in Z^i(B_G, \Lambda)$ with $\gamma \in u$ and with $\gamma \mid \{m\} = 0$ for all $m \in B_G$. Letting $\pi: E_G \to B_G$ be the projection map, and recalling that E_G is acyclic, we see that $\pi^*(\gamma) = \delta\beta$, where $\beta \in C^{i-1}(E_G, \Lambda)$. Since $\gamma \mid \{m\} = 0$, $\beta \mid G$ is closed, and thus defines $\tau(u) \in H^{i-1}(G, \Lambda)$. Acyclicity of E_G guarantees the map independent of

choice of β, and it is easy to check it is also independent of choice of γ. Thus $\tau: H^i(B_G, \Lambda) \to H^{i-1}(G, \Lambda)$ is well-defined. τ is in fact the inverse of the transgression mapping considered in [1]. We remark that if Λ is a ring then $\tau(u \cup u) = 0$. This follows since if $\gamma \in u$ with $\pi^*(\gamma) = \delta\beta$, then $\pi^*(\gamma \cup \gamma) = \delta(\beta \cup \pi^*(\gamma))$, and $\beta \cup \pi^*(\gamma) \mid G = 0$.

We first consider the case $k = 0$, i.e., $E_{n,0} = 0(n)$, $E_{n,0}(c) = U(n)$, and $\varphi: 0(n) \to U(n)$ is the natural map. We consider the diagram

$$
\begin{array}{ccc}
H^*(U(n)) & \xrightarrow{\varphi^*} & H^*(0(n)) \\
\big\uparrow{\scriptstyle \tau} & & \big\uparrow{\scriptstyle \tau} \\
H^*(B_{U(n)}) & \xrightarrow{\rho^*} & H^*(B_{0(n)})
\end{array}
$$

and note that it is commutative. The Bockstein exact sequence of cohomology corresponding to the coefficient sequence $0 \to Z \xrightarrow{2} Z \to Z_2 \to 0$ shows that an integral class is *even* if and only if its mod 2 reduction is zero. Thus it is sufficient to show that for any $u \in H^*(U(n), Z_2)$, $\varphi^*(u) = 0$. Let $\hat{c}_i \in H^{2i}(U(n), Z_2)$ denote the mod 2 reduction of c_i. Now it is well-known that

$$
\rho^*(\hat{c}_i) = W_i \cup W_i
$$

where W_i is the ith Stiefel-Whitney class. Thus

$$
\varphi^*(\tau(\hat{c}_i)) = \tau(\rho^*(\hat{c}_i)) = \tau(W_i \cup W_i) = 0 \ .
$$

On the other hand, $H^*(U(n), Z_2)$ is generated by the set $\{\tau(\hat{c}_i)\}$, and thus $\varphi^*(u) = 0$ for any $u \in H^*(U(n), Z_2)$.

To do the general case consider the commutative diagram

$$
\begin{array}{ccc}
H^*(E_{n,k}(c), Z_2) & \xrightarrow{\varphi^*} & H^*(E_{n,k}, Z_2) \\
\big\downarrow{\scriptstyle \pi^*} & & \big\downarrow{\scriptstyle \pi^*} \\
H^*(U(n+k), Z_2) & \xrightarrow{\varphi^*} & H^*(0(n+k), Z_2)
\end{array}
$$

where $\pi: U(n+k) \to U(n+1)/U(k) = E_{n,k}(c)$, and $\pi: 0(n+k) \to 0(n+k)/0(k) = E_{n,k}$ are the quotient maps. It is known, cf. [2], that the π^* on the right is injective. Thus, since the image of the lower φ^* is zero from our special case, so is that of the upper φ^*. This completes the proof of the lemma and the proposition follows. $\qquad\square$

By restricting this proposition to the fibre and using (5.10) and (3.11) we obtain the well-known fact that

(5.13) $$\frac{1}{2}\{TP_i\} \in H^{4i-1}(0(n), Z) \ .$$

The polynomials P_i and $P_i^{\perp}$ were considered on the Lie algebra of $0(n)$,

but they also live on that of $Gl(n, R)$, and pull back under $0(n) \to Gl(n, R)$. We will also denote these by P_i, $P_i^\perp \in I_0^{2i}(Gl(n, R))$.

THEOREM 5.14. *Let M^n be an n-dim Riemannian manifold. Let $\alpha(M) = \{E(M^n), M^n, \theta\}$ denote the $Gl(n, R)$ basis bundle of M equipped with the Riemannian connection θ. A necessary condition that M^n admit a global conformal immersion in R^{n+k} is that $P_i^\perp(\Omega^{2i}) = 0$ and $\{(1/2) TP_i^\perp(\theta)\} \in H^{4i-1}(E(M), Z)$ for $i > [k/2]$.*

Proof. Let $\varphi: M^n \to R^{n+k}$ be a conformal immersion. By Theorem 4.5 we may assume φ is an isometric immersion. Let $F(M^n)$ denote the orthonormal frame bundle of M^n, and consider the Gauss map Φ

$$
\begin{array}{ccc}
F(M^n) & \xrightarrow{\ \Phi\ } & E_{n,k} \\
\downarrow & & \downarrow \\
M^n & \xrightarrow{\ \Phi\ } & G_{n,k}
\end{array}
$$

which is defined as usual by mapping a point into the tangent plane at its image. Letting θ denote the canonical connection on $E_{n,k}$, it is a standard fact that $\Phi^*(\theta) = \theta$, the Riemannian connection on $F(M^n)$; i.e.,

$$\Phi: \{F(M^n), M^n, \theta\} \longrightarrow \alpha_{n,k}$$

is a morphism. Thus by naturality and the previous proposition, in $F(M^n)$, $P_i^\perp(\Omega^{2i}) = 0$ and $\{(1/2) TP_i^\perp(\theta)\} \in H^{4i-1}(F(M^n), Z)$ for $i > [k/2]$. By invariance, $P_i^\perp(\Omega^{2i}) = 0$ in all of $E(M^n)$, and since $\{(1/2) TP_i^\perp(\theta)\} \in H^{4i-1}(E(M^n), R)$ it must actually be an integral class there since its restriction to the retract $F(M^n)$ is integral.   □

Remark. This theorem is probably of interest only for the codimension $k \leqq n/2$. This is because if $k > n/2$ our condition $i > [k/2]$ already implies $P_i^\perp(\Omega^{2i}) = 0$ for dimension reasons, and the corresponding class, $\{TP_i^\perp(\theta)\}$, is independent of connection (see Theorem 3.9). At the same time Corollary 3.17 already shows that $\{TP_i^\perp(\theta)\} \in H^{4i-1}(E(M), Z)$, and it seems likely that the same is true for $\{(1/2) TP_i^\perp(\theta)\}$.

6. Applications to 3-manifolds

In this section M will denote a compact, oriented, Riemannian 3-manifold, and $F(M) \xrightarrow{\pi} M$ will denote its $SO(3)$ oriented frame bundle equipped with the Riemannian connection θ and curvature tensor Ω. For A, B skew symmetric matrices, the specific formula for P_1 shows $P_1(A \otimes B) = -(1/8\pi^2) \operatorname{tr} AB$. Calculating from (3.5) shows

$$(6.1) \qquad TP_1(\theta) = \frac{1}{4\pi^2}\{\theta_{12} \wedge \theta_{13} \wedge \theta_{23} + \theta_{12} \wedge \Omega_{12} + \theta_{13} \wedge \Omega_{13} + \theta_{23} \wedge \Omega_{23}\} \; .$$

Since $\dim M = 3$, $dTP_1 = 0$. By (5.13), $(1/2)TP_1(\theta) \mid F(M)_m \in H^3(F(M)_m, Z)$. We will thus be interested in the class

$$\left\{\frac{1}{2}TP_1(\theta)\right\} \in H^3(F(M), R) \; .$$

From the general considerations at the end of § 3 this data is enough to produce an R/Q invariant of M, but since M is an oriented 3-manifold, $F(M)$ is trivial; and we define the R/Z invariant,* $\Phi(M)$, as follows: Let $\chi: M \to F(M)$ be any cross-section. Then set

$$(6.2) \qquad\qquad \Phi(M) = \int_\chi \widetilde{\frac{1}{2}TP_1(\theta)} \in R/Z \; .$$

If χ' were another such, then homologically $\chi' = \chi + nF(M)_m +$ torsion, where n is an integer. Thus since $(1/2)TP_1(\theta) \mid F(M)_m$ is integral, and forms integrated over torsion classes give 0, $\Phi(M)$ is well-defined. Recalling that $P_1^\perp = -P_1$ we immediately get the following two special cases of Theorems 4.5 and 5.14.

THEOREM 6.3. $\Phi(M)$ *is a conformal invariant of* M.

THEOREM 6.4. *A necessary condition that* M *admit a conformal immersion in* R^4 *is that* $\Phi(M) = 0$.

Example 1. Let $M = RP^3 = SO(3)$ together with the standard metric of constant curvature 1. Let E_1, E_2, E_3 be an orthonormal basis of left invariant fields on $SO(3)$, oriented positively. Then it is easily seen that $\nabla_{E_1}E_2 = E_3$, $\nabla_{E_1}E_3 = -E_2$, and $\nabla_{E_2}E_3 = E_1$. Let $\chi: M \to F(M)$ be the cross-section determined by this frame. The above equations and (6.1) show

$$\chi^*\left(\frac{1}{2}TP_1(\theta)\right) = \frac{-1}{2\pi^2}\omega$$

where ω is the volume form on $SO(3)$. Thus from (6.1)

$$\Phi(SO(3)) = \widetilde{\frac{1}{2\pi^2}\operatorname{Vol}(SO(3))} = \widetilde{\frac{1}{2}}$$

since $\operatorname{Vol}(SO(3)) = (1/2)\operatorname{Vol}(S^3) = \pi^2$. Using Theorem 6.4 we see that $SO(3)$ admits no global conformal immersion in R^4. This is interesting since, being parallelizable, it certainly admits a C^∞ immersion in R^4, and locally it is isometrically imbeddable in R^4.

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* Atiyah has subsequently shown that $2\Phi(M)$ is the mod Z reduction of a real class. This will be discussed further in [3].

Example 2. Again let $M = SO(3)$, but this time with the left invariant metric, g_λ, with respect to which λE_1, E_2, E_3 is an orthonormal frame. Direct calculation shows

$$\Phi\big(SO(3), g_\lambda\big) = \frac{2\lambda^2 - 1}{2\lambda^4}$$

and this can take any value in R/Z.

Let M be a fixed 3-manifold and let $\mathcal{C}(M)$ denote the space of conformal structures on M. Since Φ is a conformal invariant we may regard

$$\Phi\colon \mathcal{C}(M) \longrightarrow R/Z .$$

If g_t is a C^∞ curve of conformal structures then $\Phi(g_t)$ is a C^∞ R/Z valued function (we shall see this below). We are interested in calculating the critical points of the map Φ.

Let $g = \langle\,,\,\rangle$ be a fixed metric on M. With respect to this we let $\nabla_X Y$ and $R_{x,y}Z$ denote covariant differentiation and curvature; i.e.,

$$R_{X,Y}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[x,y]}Z$$

where X, Y, Z are vector fields. The operator ∇_X extends as a derivation to all tensors, and all tensors have a natural inner product induced by $\langle\,,\,\rangle$. We make the usual identification of Λ^2 with skew symmetric linear transformations and so for $x, y \in T(M)_m$ we often regard

$$(6.5) \qquad\qquad R_{x,y} \in \Lambda^2 T(M)_m .$$

Because we are working on an oriented 3-manifold there is an identification of $T(M)_m$ with $\Lambda^2 T(M)_m$ given by the metric. We denote this by

$$(6.6) \qquad\qquad * : T(M)_m \longrightarrow \Lambda^2 T(M)_m .$$

Let e_1, e_2, e_3 be an orthonormal basis of $T(M)_m$ and define

$$(6.7) \qquad \begin{aligned} &\delta R\colon T(M)_m \longrightarrow \Lambda^2 T(M)_m , \\ &\delta R(x) = \sum_{i=1}^3 \nabla_{e_i}(R)_{e_i,x} . \end{aligned}$$

This definition is independent of choice of frame. Combining (6.6) and (6.7) we define the *symmetric bilinear form* $\widehat{\delta R}$ on $T(M)_m$ by

$$(6.8) \qquad\qquad \widehat{\delta R}(x, y) = \langle \delta R(x), y^* \rangle + \langle \delta R(y), x^* \rangle .$$

Now let $B = B(\,,\,)$ be a C^∞ field of symmetric bilinear forms on M and consider the curve of metrics

$$g_t(x, y) = \langle x, y \rangle + tB(x, y) .$$

For small t these are Riemannian.

THEOREM 6.9. *Let $M_t = \{M, g_t\}$. Then for small t, $\Phi(M_t) \in C^\infty(t)$ and*

$$\frac{d}{dt}(\Phi(M_t))\Big|_{t=0} = \frac{-1}{16\pi^2} \int_M \langle B, \widehat{\delta R} \rangle \ .$$

Proof. The invariant Φ was defined by choosing a cross-section in $F(M)$, but we would clearly get the same value by choosing one in $E(M)$, the full $Gl(3, R)$ basis bundle. This is more convenient. So let θ^t denote the curve of connections in $E(M)$ corresponding to the metrics g_t, and let $\theta = \theta^0$, $\Omega = \Omega^0$, and $\theta' = (d/dt)(\theta^t)\vert_{t=0}$. The general variation formula in Theorem 3.8 shows

(6.10) $$\frac{d}{dt}(\Phi(M_t))\Big|_{t=0} = \int_M P_1(\theta' \wedge \Omega)$$

where this makes sense since the forms $P(\theta' \wedge \Omega^{l-1})$ all are horizontal and invariant. The definition of P_i given in (5.7) works as well for general matrices and one easily sees

(6.11) $$P_1(A, B) = \frac{1}{8\pi^2}[\operatorname{tr} A \operatorname{tr} B - \operatorname{tr} AB] \ .$$

Now if we work at points in $F(M)$, range Ω is skew symmetric matrices, and so the first term vanishes, to give

(6.12) $$P_1(\theta' \wedge \Omega) = \frac{1}{8\pi^2} \sum_{i,j=1}^3 \theta'_{ij} \wedge \Omega_{ij}$$

at points of $F(M)$.

Let $x \in T(M)_m$, Y a local vector field, and let $\nabla^t_x Y$ denote covariant differentiation with respect to the connection at time t. Differentiating we get a tensor, A, defined by

(6.13) $$A_x y = \frac{d}{dt}(\nabla^t_x Y)\Big|_{t=0}(m)$$

where $y = Y(m)$. At $f = (m; f_1, f_2, f_3) \in F(M)$ the following hold

$$\theta'_{ij}(x) = -\langle A_{d\pi(x)} f_i, f_j \rangle$$
$$\Omega_{ij}(x, y) = -\langle R_{d\pi(x), d\pi(y)} f_i, f_j \rangle$$

where $x \in T(E(M))_f$ and R is the curvature of $\{M, g\}$. Now regarding $P_1(\theta' \wedge \Omega)$ as a form on M, (6.12) gives

$$P_1(\theta' \wedge \Omega)(x, y, z) = \frac{1}{8\pi^2}[\langle A_x, R_{y,z} \rangle - \langle A_y, R_{x,z} \rangle + \langle A_z, R_{x,y} \rangle]$$

$$= \frac{1}{8\pi^2}\langle A, R \circ * \rangle \omega(x, y, z)$$

$(\omega =$ volume form on $\{M, g\})$. Combining this with (6.10) gives

(6.13)
$$\frac{d}{dt}\big(\Phi(M_t)\big)\Big|_{t=0} = \frac{1}{8\pi^2}\int_M \langle A, R\circ *\rangle \;.$$

Since the range of $R\circ *$ is skew symmetric linear transformations, we may as well project A to have the same range; i.e., set

$$\langle \hat{A}_x y, z\rangle = \frac{1}{2}\langle A_x y, z\rangle - \frac{1}{2}\langle A_x z, y\rangle \;.$$

Then

(6.14)
$$\langle A, R\circ *\rangle = \langle \hat{A}, R\circ *\rangle \;.$$

Finally, using the definition of Riemannian connection in terms of covariant derivatives (cf. [7]), and setting

$$\langle DB_x y, z\rangle = \frac{1}{2}\big[\nabla_y(B)(z, x) - \nabla_z(B)(y, x)\big]$$

equation (6.13) shows

$$\hat{A} = DB$$

and thus from (6.13)

$$\frac{d}{dt}\big(\Phi(M_t)\big)\Big|_{t=0} = \frac{1}{8\pi^2}\int_M \langle DB, R\circ *\rangle = -\frac{1}{16\pi^2}\int_M \langle B, \widehat{\delta R}\rangle$$

where the last equation used Stokes' theorem and integration by parts.   ☐

We should note from the definition of $\widehat{\delta R}$ that $\operatorname{tr}\widehat{\delta R} = 0$, and this is as it should be since if $B = \lambda g$, where λ is a function on M, our metric is changing conformally, $\Phi(M_t)$ should be constant, and $\langle B, \widehat{\delta R}\rangle$ should vanish, which is implied by $\operatorname{tr}\widehat{\delta R} = 0$. More importantly, the bilinear form $\widehat{\delta R}$ is itself a conformal invariant (this can be directly checked), and it has been shown by Schouten, cf. [6], that $\widehat{\delta R} \equiv 0$ if and only if $\{M, g\}$ is *locally conformally flat*; i.e., if and only if each point of M has a neighborhood conformally equivalent to R^3. For example, S^3 is locally conformally flat. This fact is peculiar to 3-manifolds, since the integrability condition for local conformal flatness in higher dimensions involves no derivatives of curvature. We, therefore, conclude

COROLLARY 6.14. $g \in \mathcal{C}(M)$ *is a critical point of* Φ *if and only if* $\{M, g\}$ *is locally conformally flat.*

Kuiper, in [8], has shown that compact $\{M, g\}$ is locally conformally flat and simply connected if and only if $\{M, g\}$ is conformally equivalent to S^n with the usual metric. We, therefore, conclude

COROLLARY 6.15. *Suppose M is a simply connected compact oriented 3-manifold. Then either Φ has exactly one critical point and M is diffeomorphic to S^3 or Φ has no critical points and M is not diffeomorphic to S^3.*

We do not see how this helps to settle the Poincaré conjecture.

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