Differential cohomology seminar overview

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Contents

1 Motivation for differential cohomology

1.1 Observation (Simons–Sullivan [9, §1]). Let M be a manifold. Then we have exact sequences

(1.2)
$$\begin{tikzcd}
 & \mathrm{H}^{k-1}(M;\mathbf{R}/\mathbf{Z}) \arrow[rr, "-\beta"] \arrow[dr] & & \mathrm{H}^{k}(M;\mathbf{Z}) \arrow[dr] & \\
\mathrm{H}^{k-1}_{dR}(M) \arrow[ur] \arrow[dr] & & \hat{\mathrm{H}}^k(M;\mathbf{Z}) \arrow[ur] \arrow[dr] & & \mathrm{H}^k_{dR}(M) \\
 & \Omega^{k-1}(M)/\operatorname{im}(d) \arrow[rr, "d"'] \arrow[ur] & & \Omega^k_{c\ell}(M) \arrow[ur] &
\end{tikzcd} ,$$

where the top sequence is the Bockstein sequence associated to the short exact sequence

$$0 \longrightarrow \mathbf{Z} \hookrightarrow \mathbf{R} \twoheadrightarrow \mathbf{R}/\mathbf{Z} \longrightarrow 0 ,$$

and we are identifying singular and de Rham cohomology via the de Rham isomorphism $\mathrm{H}^\star_{dR}(M) \cong \mathrm{H}^\star(M;\mathbf{R})$.

The top sequence is 'purely homotopy-theoretic' in nature, while the bottom sequence is 'purely geometric' in nature (e.g., the functor $\Omega^k_{c\ell}$ is not homotopy-invariant).

1.3 Question. Can we fill (1.2) in with an invariant $\hat{\mathrm{H}}^k(M;\mathbf{Z})$ in red that better blends homotopy theory and geometry, and makes the diagonals exact?

Now let us attempt to provide a satisfactory answer to Question 1.3 when $k = 1$.

1.4 Attempt (for $k = 1$). Let M be a manifold. Consider the abelian group $\mathrm{C}^\infty(M, \mathbf{R}/\mathbf{Z})$ of smooth functions to the circle (with the group structure defined pointwise). Recall that the inclusion $\mathrm{C}^\infty(M, \mathbf{R}/\mathbf{Z}) \subset \mathrm{Map}(M, \mathbf{R}/\mathbf{Z})$ from the space of smooth maps to the space of all maps is a homotopy equivalence. So since the circle is 1-truncated, $\mathrm{C}^\infty(M, \mathbf{R}/\mathbf{Z})$ is also 1-truncated.

Since $\mathbf{R}/\mathbf{Z}$ is a $\mathrm{K}(\mathbf{Z}, 1)$, we see that

$$\pi_0 \mathrm{C}^\infty(M, \mathbf{R}/\mathbf{Z}) \cong \mathrm{H}^1(M; \mathbf{Z}) \,.$$

In particular, we have a surjection $\pi_0 \colon \mathrm{C}^\infty(M, \mathbf{R}/\mathbf{Z}) \twoheadrightarrow \mathrm{H}^1(M; \mathbf{Z})$. Also notice that

$$\begin{aligned} \pi_1 \mathrm{C}^\infty(M, \mathbf{R}/\mathbf{Z}) &\cong \pi_0 \mathrm{Map}_*(S^1, \mathrm{C}^\infty(M, \mathbf{R}/\mathbf{Z})) \\ &\cong \pi_0 \mathrm{Map}_*(S^1, \mathrm{Map}(M, \mathbf{R}/\mathbf{Z})) \\ &\cong \pi_0 \mathrm{Map}(M, \mathrm{Map}_*(S^1, \mathbf{R}/\mathbf{Z})) \\ &\cong \pi_0 \mathrm{Map}(M, \Omega(\mathbf{R}/\mathbf{Z})) \\ &\cong \mathrm{H}^0(M; \mathbf{Z}) \,. \end{aligned}$$

1.5 Construction. Define a ***curvature*** map $\mathrm{curv} \colon \mathrm{C}^\infty(M, \mathbf{R}/\mathbf{Z}) \to \Omega^1_{c\ell}(M)$ by

$$\mathrm{curv}(f) \coloneqq f^\star(\mathrm{vol}) \,,$$

where vol is the standard volume form on $S^1 \cong \mathbf{R}/\mathbf{Z}$.

The kernel of curv consists of the locally constant maps $M \to \mathbf{R}/\mathbf{Z}$, i.e.,

$$\ker(\mathrm{curv}) \cong \mathrm{H}^0(M; \mathbf{R}/\mathbf{Z}) \,.$$

Note that the curvature map is not surjective:

$$\mathrm{im}(\mathrm{curv}) = \left\{ \alpha \in \Omega^1_{c\ell}(M) \;\middle|\; \textstyle\int_{S^1} \alpha \in \mathbf{Z} \text{ for every embedding } S^1 \hookrightarrow M \right\} \,.$$

That is, the image of curv is the group of *closed 1-forms with integral periods*

1.6 Definition. Let M be a manifold and $k \geq 0$ an integer. A closed k-form ω on M ***has integral periods*** if for every smooth k-cycle c in M the integral $\int_c \omega$ is an integer. We write

$$\Omega^k_{c\ell}(M)_{\mathbf{Z}} \subset \Omega^k_{c\ell}(M)$$

for the subgroup of k-forms with integral periods.

1.7 Remark. A closed k-form ω has integral periods if and only if the class of ω lies in the image of the change-of-coefficients map

$$\mathrm{H}^k(M; \mathbf{Z}) \to \mathrm{H}^k(M; \mathbf{R}) \cong \mathrm{H}^k_{dR}(M) \,.$$

1.8. We also have a map

$$\iota \colon \Omega^0(M) = \mathrm{C}^\infty(M, \mathbf{R}) \to \mathrm{C}^\infty(M, \mathbf{R}/\mathbf{Z})$$

given by post-composition with the quotient map $\mathbf{R} \twoheadrightarrow \mathbf{R}/\mathbf{Z}$. The map ι has kernel the integer-valued smooth functions $M \to \mathbf{R}$, i.e., the locally constant functions with integer values. That is, $\mathrm{im}(\iota) = \Omega^0_{c\ell}(M)_{\mathbf{Z}}$.

1.9. These maps give rise to a commutative diagram with exact diagonals

$$\begin{tikzcd}
 & \mathrm{H}^0(M;\mathbf{R}/\mathbf{Z}) \arrow[rr, "-\beta"] \arrow[dr, hook] & & \mathrm{H}^1(M;\mathbf{Z}) \arrow[dr] & \\
\mathrm{H}^0_{dR}(M) \arrow[ur] \arrow[dr] & & \mathrm{C}^\infty(M,\mathbf{R}/\mathbf{Z}) \arrow[ur, two heads, "\pi_0"] \arrow[dr, two heads, "\mathrm{curv}"] & & \mathrm{H}^1_{dR}(M) \\
 & \Omega^0(M) \arrow[rr, "d"'] \arrow[ur, "\iota"] & & \Omega^1_{c\ell}(M) \arrow[ur] &
\end{tikzcd}$$

The diagonals become short exact sequences if we replace $\Omega^0(M)$ by $\Omega^0(M)/\Omega^0_{c\ell}(M)_{\mathbf{Z}}$ and $\Omega^1_{c\ell}(M)$ by $\Omega^1_{c\ell}(M)_{\mathbf{Z}}$:

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
 & \mathrm{H}^0(M;\mathbf{R}/\mathbf{Z}) \arrow[rr, "-\beta"] \arrow[dr, hook] & & \mathrm{H}^1(M;\mathbf{Z}) \arrow[dr] \arrow[ur] & \\
\mathrm{H}^0_{dR}(M) \arrow[ur] \arrow[dr] & & \mathrm{C}^\infty(M,\mathbf{R}/\mathbf{Z}) \arrow[ur, two heads, "\pi_0"] \arrow[dr, two heads, "\mathrm{curv}"] & & \mathrm{H}^1_{dR}(M) \\
 & \Omega^0(M)/\Omega^0_{c\ell}(M)_{\mathbf{Z}} \arrow[rr, "d"'] \arrow[ur, tail, "\iota"] & & \Omega^1_{c\ell}(M)_{\mathbf{Z}} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0\,.
\end{tikzcd}$$

1.10. The takeaway is that in Question 1.3, we should really replace $\Omega^{k-1}(M)/\operatorname{im}(d)$ by $\Omega^{k-1}(M)/\Omega^{k-1}_{c\ell}(M)_{\mathbf{Z}}$ and $\Omega^k_{c\ell}(M)$ by $\Omega^0_{c\ell}(M)_{\mathbf{Z}}$ and ask for the diagonal sequences to be short exact.

2 Differential characters

We now present a unified approach to defining the 'differential cohomology' groups $\hat{\mathrm{H}}^*(M;\mathbf{Z})$ due to Cheeger–Simons [3]. We follow Bär and Becker's exposition on *differential characters* [1, Part I, §5].

2.1 Notation. Let M be a manifold and $i \geq 0$ an integer. We write $\mathrm{C}^{sm}_i(M;\mathbf{Z})$ for the abelian group of smooth (integer-valued) chains on M. We write $\mathrm{Z}^{sm}_i(M;\mathbf{Z}) \subset \mathrm{C}^{sm}_i(M;\mathbf{Z})$ for the subgroup of smooth cycles.

2.2 Definition (Cheeger–Simons [3, §1])**.** Let $k \geq 1$ be an integer and M a manifold. A ***degree*** k ***differential character*** on M is a homomorphism $\chi\colon \mathrm{Z}^{sm}_{k-1}(M;\mathbf{Z}) \to \mathbf{R}/\mathbf{Z}$ such that there exists a k-form $\omega(\chi) \in \Omega^k(M)$ with the property that

$$\chi(\partial c) = \int_c \omega(\chi) \mod \mathbf{Z}$$

for every $c \in \mathrm{C}_k^{sm}(M;\mathbf{Z})$. We write

$$\hat{\mathrm{H}}^k(M;\mathbf{Z}) \subset \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Z}_{k-1}^{sm}(M;\mathbf{Z}), \mathbf{R}/\mathbf{Z})$$

for the abelian group of degree k differential characters on M.

It follows that $\omega(\chi)$ is unique and closed. Morevoer, $\omega(\chi)$ has integral periods. The form $\omega(\chi)$ is called the ***curvature*** of χ, and we have a curvature map

$$\begin{aligned} \mathrm{curv}\colon \hat{\mathrm{H}}^k(M;\mathbf{Z}) &\to \Omega^k(M) \\ \chi &\mapsto \omega(\chi) \end{aligned}$$

with image $\Omega^k_{c\ell}(M)_{\mathbf{Z}}$ those closed k-forms with integral periods.

2.3 Warning. The indexing convention used here is off by 1 from the indexing convention in [3, §1]. However, this indexing convention is better and is what was later adopted by Simons–Sullivan [9, §1].

2.4 Remark. When $k = 0$, the diagram (1.2) is quite degnerate, and it will be convenient to define $\hat{\mathrm{H}}^0(M;\mathbf{Z}) \coloneqq \mathrm{H}^0(M;\mathbf{Z})$.

Now let us constrct maps to fill in the 'differential cohomology' diagram (1.2).

2.5 Construction. There is a ***characteristic class*** map $\mathrm{ch}\colon \hat{\mathrm{H}}^k(M;\mathbf{Z}) \to \mathrm{H}^k(M;\mathbf{Z})$ defined as follows. Since $\mathrm{Z}_{k-1}^{sm}(M;\mathbf{Z})$ is a free $\mathbf{Z}$-module and the quotient map $\mathbf{R} \twoheadrightarrow \mathbf{R}/\mathbf{Z}$ is an epimorphism, any homomorphism $\chi\colon \mathrm{Z}_{k-1}^{sm}(M;\mathbf{Z}) \to \mathbf{R}/\mathbf{Z}$ lifts to a homomorphism $\tilde{\chi}\colon \mathrm{Z}_{k-1}^{sm}(M;\mathbf{Z}) \to \mathbf{R}$. Now define a homomorphism $I(\tilde{\chi})\colon \mathrm{C}_k^{sm}(M;\mathbf{Z}) \to \mathbf{Z}$ by the assignment

$$c \mapsto -\tilde{\chi}(\partial c) + \int_c \mathrm{curv}(\chi)\,.$$

Since $\mathrm{curv}(\chi)$ is closed, $I(\tilde{\chi})$ defines a cocycle. Moreover, $I(\tilde{\chi})$ takes integral values, and the cohomology class $[I(\tilde{\chi})] \in \mathrm{H}^k(M;\mathbf{Z})$ does not depend on the choice of lift $\tilde{\chi}$. We define ch by the assignment

$$\begin{aligned} \mathrm{ch}\colon \hat{\mathrm{H}}^k(M;\mathbf{Z}) &\to \mathrm{H}^k(M;\mathbf{Z}) \\ \chi &\mapsto [I(\tilde{\chi})] \end{aligned} \quad .$$

2.6 Construction. Consider the universal coefficient sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbf{Z}}(\mathrm{H}_{i-1}(M;\mathbf{Z}), \mathbf{R}/\mathbf{Z}) \longrightarrow \mathrm{H}^i(M;\mathbf{R}/\mathbf{Z}) \xrightarrow{\langle -,- \rangle} \mathrm{Hom}_{\mathbf{Z}}(\mathrm{H}_i(M;\mathbf{Z}), \mathbf{R}/\mathbf{Z}) \longrightarrow 0\,,$$

where the morphism $\langle -,- \rangle$ is given by sending a the class of a cocycle u to the homomorphism

$$\begin{aligned} \langle u,- \rangle\colon \mathrm{H}_i(M;\mathbf{Z}) &\to \mathbf{R}/\mathbf{Z} \\ [z] &\mapsto u(z)\,. \end{aligned}$$

Since the circle $\mathbf{R}/\mathbf{Z}$ is an injective $\mathbf{Z}$-module, for any $\mathbf{Z}$-module A and integer $j > 0$, we have $\mathrm{Ext}^j_{\mathbf{Z}}(A, \mathbf{R}/\mathbf{Z}) = 0$. In particular, $\langle -,- \rangle$ is an isomorphism.

Setting $i = k-1$, precomposition with the quotient map $\mathrm{Z}^{sm}_{k-1}(M;\mathbf{Z}) \twoheadrightarrow \mathrm{H}_{k-1}(M;\mathbf{Z})$ defines an injection

$$\mathrm{H}^i(M;\mathbf{R}/\mathbf{Z}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}}(\mathrm{H}_i(M;\mathbf{Z}),\mathbf{R}/\mathbf{Z}) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Z}^{sm}_{k-1}(M;\mathbf{Z}),\mathbf{R}/\mathbf{Z}) .$$

It follows from the definitions that this factors through $\hat{\mathrm{H}}^k(M;\mathbf{Z})$. We simply denote this composite by $\langle -,- \rangle \colon \mathrm{H}^{k-1}(M;\mathbf{R}/\mathbf{Z}) \hookrightarrow \hat{\mathrm{H}}^k(M;\mathbf{Z})$.

2.7 Construction. Define a map $\iota \colon \Omega^{k-1}(M) \to \hat{\mathrm{H}}^k(M;\mathbf{Z})$ by setting

$$\iota(\omega)(z) \coloneqq \exp\left(2\pi i \textstyle\int_z \omega\right)$$

for every smooth $(k-1)$-cycle z. By Stokes' Theorem, we see that $\mathrm{curv}(\iota(\omega)) = d\omega$.

We have an $\mathbf{R}$-valued lift of $\iota(\omega)$ given by setting

$$\tilde{\iota}(\omega)(z) \coloneqq \int_z \omega$$

for every smooth $(k-1)$-cycle z. So by Stokes' Theorem we have

$$\begin{aligned} I(\tilde{\iota}(\omega))(c) &= -\tilde{\iota}(\omega)(\partial c) + \int_c \mathrm{curv}(\iota(\omega)) \\ &= -\int_{\partial c} \omega + \int_c d\omega = 0 \end{aligned}$$

for every smooth k-chain c. Hence $\mathrm{ch} \circ \iota = 0$.

We see that $\iota \colon \Omega^{k-1}(M) \to \hat{\mathrm{H}}^k(M;\mathbf{Z})$ has kernel those closed forms ω such that $\int_z \omega$ is an integer for all $z \in \mathrm{Z}^{sm}_{k-1}(M;\mathbf{Z})$. That is,

$$\ker(\iota) = \Omega^{k-1}_{c\ell}(M)_{\mathbf{Z}}$$

is the group of closed $(k-1)$-forms with integral periods. Hence ι descends to an injection

$$\iota \colon \Omega^{k-1}(M)/\Omega^{k-1}_{c\ell}(M)_{\mathbf{Z}} \rightarrowtail \hat{\mathrm{H}}^k(-;\mathbf{Z}) .$$

2.8 Notation. Write **Man** for the cateogry of smooth manifolds and **GrAb** for the category of graded abelian groups.

2.9 Theorem (Simons–Sullivan [9, Theorem 1.1])**.** *There is an essentially unique functor* $\hat{\mathrm{H}}^*(-;\mathbf{Z}) \colon \mathbf{Man}^{op} \to \mathbf{GrAb}$ *equipped with natural transformations*

(2.9.1) $\langle -,- \rangle \colon \mathrm{H}^{*-1}(-;\mathbf{R}/\mathbf{Z}) \to \hat{\mathrm{H}}^*(-;\mathbf{Z})$,

(2.9.2) $\iota \colon \Omega^{*-1}(M)/\Omega^{*-1}_{c\ell}(M)_{\mathbf{Z}} \to \hat{\mathrm{H}}^*(-;\mathbf{Z})$,

(2.9.3) $\mathrm{ch} \colon \hat{\mathrm{H}}^*(-;\mathbf{Z}) \to \mathrm{H}^*(-;\mathbf{Z})$,

(2.9.4) *and* $\mathrm{curv} \colon \hat{\mathrm{H}}^*(-;\mathbf{Z}) \to \Omega^*_{c\ell}(-)_{\mathbf{Z}}$

filling in the 'differential cohomology diagram'

$$\begin{tikzcd}[column sep=small]
0 \arrow[dr] & & & & 0 \\
& \mathrm{H}^{*-1}(M;\mathbf{R}/\mathbf{Z}) \arrow[rr, "-\beta"] \arrow[dr] & & \mathrm{H}^{*}(M;\mathbf{Z}) \arrow[ur] \arrow[dr] & \\
\mathrm{H}^{*-1}_{dR}(M) \arrow[ur] \arrow[dr] & & \hat{\mathrm{H}}^{*}(M;\mathbf{Z}) \arrow[ur] \arrow[dr] & & \mathrm{H}^{*}_{dR}(M) \\
& \frac{\Omega^{*-1}(M)}{\Omega^{*-1}_{c\ell}(M)_{\mathbf{Z}}} \arrow[rr, "d"'] \arrow[ur] & & \Omega^{*}_{c\ell}(M)_{\mathbf{Z}} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

so that the diagonal sequences are exact.

Any functor $\hat{\mathrm{H}}^*(-;\mathbf{Z})\colon \mathbf{Man}^{op} \to \mathbf{GrAb}$ satisfying these properties is called ***ordinary differential cohomology***.

2.10 Remark (Deligne's model)**.** Motivated by Deligne cohomology in Hodge theory [12, §12.3], we can consider the smooth version of the Deligne complex on a manifold M. Write $\mathbf{Z}_D(k)$ for the complex of sheaves on M

$$0 \longrightarrow \mathbf{Z} \hookrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \longrightarrow \Omega^{k-1} \longrightarrow 0\,,$$

where Ω^i is in degree $i+1$. The k^{th} ***smooth Deligne cohomology group*** of M is the hypercohomology group $\mathbf{H}^k(M;\mathbf{Z}_D(k))$. We will see later in the seminar that smooth Deligne cohomology agrees with ordinary differential cohomology [2, §4.3; 6, §3].

2.11 Questions.

(2.11.1) Is there differential K-theory?

Yes! Simons–Sullivan tell a similar story, and define differential K-theory in terms of vector bundles with connection [10; 11].

(2.11.2) What about differential [favorite cohomology theory]?

Also yes, but the theory is more complicated. The fundamental obeservation is that everything we've considered comes from a sheaf of abelian groups or chain complexes (which we regard as spectra) on the category of *all* smooth manifolds.

The category $\mathbf{Sh}(\mathbf{Man};\mathbf{Sp})$ has rich structure that gives rise to a 'differential cohomology diagram' associated to every object (see [2, §3; 4]).

2.12 Remark. The category $\mathbf{Sh}(\mathbf{Man};\mathbf{Set})$ is really the right place for moduli spaces of manifolds to live, and both Banach [5] and Fréchet manifolds [7; 8, Theorem 3.1.1; 13, Theorem A.1.5] embed as full subcategories of $\mathbf{Sh}(\mathbf{Man};\mathbf{Set})$.

2.13 Applications. The following are some applications we will study throughout the semester.

(2.13.1) Lifting characteristic classes to differential cohomology. In particular, there are applications to the existence of conformal immersions [3, §6].

(2.13.2) Work of Freed–Hopkins–Teleman on the Virasoro group.

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