

Chern-Weil Theory and Equivariant de Rham Cohomology

I Motivation

To begin, let's recall

Theorem (Gauss-Bonnet)

Let (Σ, g) be a compact, oriented, Riemannian 2-manifold without boundary.
Then

$$\int_{\Sigma} K dA = 2\pi \chi(\Sigma)$$

◊

Here K is the Gaussian curvature. If $R_{ij} dx_i dx_j$ is the Riemann curvature tensor, locally

$$R = \begin{pmatrix} 0 & R_{12} \\ -R_{21} & 0 \end{pmatrix} dx_1 dx_2$$

and $K = R_{12}$. So we can rewrite the above

$$\int_{\Sigma} \sqrt{\det R} = 2\pi \chi(\Sigma)$$

$$\langle [\sqrt{\det R}], [\Sigma] \rangle = \langle 2\pi e(T\Sigma), [\Sigma] \rangle$$

where the latter uses brackets to denote the pairing $H^2(\Sigma; \mathbb{R}) \otimes H_2(\Sigma; \mathbb{R}) \rightarrow \mathbb{R}$.

⇒ Thus we observe $\sqrt{\det R}$, a polynomial in the curvature, captures information about the topology of $\Sigma, T\Sigma$. Chern-Weil theory (which was actually the original formulation/theory of characteristic classes) generalizes the above to higher dimension and arbitrary bundles.

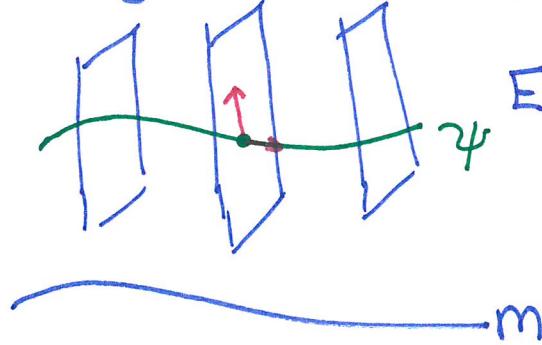
II Connections + Curvature

In order to formulate things correctly, we will need to recall some facts about connections and curvature.

Throughout, let M be a closed n -manifold, $E \rightarrow M$ a rank k real or complex vector bundle with structure group $O(k), U(k) = G$. Denote the real (or Hermitian) inner product by

$$\langle , \rangle$$

If $\psi: M \rightarrow E$ is a section, we want to differentiate it. The problem is $\psi_{x(t)}$ for a path $x(t) \in M$ all live in different vector spaces: $E_{x(t)}$ respectively so we must find a way to "connect" them.



View $\psi_{x(t)}$ as a path in the total space. The derivative (intuitively) is the vertical component of $\frac{d\psi}{dt}$ (think of $f: \mathbb{R} \rightarrow \mathbb{R}$, then df is the y-coord of graph $\in \mathbb{R}^2$). To define this precisely we need to choose a splitting

$$TE \approx VE \oplus HE \quad (\text{"vertical" and "horizontal" subbundles. } VE = \text{Ker } d\pi \text{ is canonical, } HE \text{ is not.})$$

Such a splitting is called a connection. Once we choose one, we get an isomorphism $d\pi: HE \rightarrow TM$ and so given $e \in E_{x(t)}$, we can lift \dot{x} (a vector field along $x(t)$) to one $X_H \in HE$. Then the flow is a path in E projecting to $x(t)$, which is the parallel transport, denoted $\psi_E e \in E_{x(t+I)}$. Then

$$\psi_{-t} \psi(t) \in E_{x(0)} \text{ for all } t,$$

so we can differentiate: $\left. \frac{d}{dt} \right|_{t=0} \psi_{-t} \psi(t)$ is the (covariant) derivative with respect to our chosen connection) in the $\dot{x}(0)$ direction at $x(0)$. Thus we get an operator

$$d_A \text{ or } \nabla_A : \Gamma(E) \rightarrow \Gamma(T^*M \otimes E) \quad (\text{here } \nabla_A \text{ eats } X \in \Gamma(TM) \text{ and gives the derivative in that direction at each point}).$$

Associated to a connection A , called the covariant derivative. It satisfies

- 1) $\nabla_{fx}^A \psi = f \nabla_x^A \psi \quad (C^\infty\text{-linear in direction of derivative})$
- 2) $\nabla^A f \psi = df \otimes \psi + f \nabla \psi \quad (\text{Leibniz rule})$

Next a few standard things.

Proposition 1: Given ∇^A on E , ∇^B on F we get connections

- i) ∇^{A*} on E^*
- ii) ∇^{AB} on $E \otimes F$ ($\nabla^{AB}(\psi \otimes \varphi) = \nabla^A \psi \otimes \varphi + \psi \otimes \nabla^B \varphi$)

- iii) If $f: M \rightarrow N$, $E \rightarrow N$ then $f^*(\nabla^A)$ on f^*E by

$$(f^*\nabla^A)_x \psi(m) := \nabla_{f_x X}^A \psi(f(m)) \in E_{f(m)} = f^*E_m.$$

Proposition 2: Two connections differ by a 1-form valued in $\text{End}(E)$.
In particular, they are an affine space, hence contractible.

Remark: thus one might expect invariants defined using them (if discrete) to not depend on the choice of connection.

Proof: $(\nabla^A - \nabla^{A'})_x \psi = df \otimes \psi + f \nabla^A \psi - df \otimes \psi - f \nabla^{A'} \psi$
 $= f(\nabla^A - \nabla^{A'})\psi$

is C^∞ -linear w/ values in $\Gamma(T^*M \otimes E)$, so $\nabla^A - \nabla^{A'} \in \mathcal{L}^1(\text{End}(E))$. \square

Ex 1: On the trivial bundle on \mathbb{R}^n , $d: \Gamma(\mathbb{R}^k) \rightarrow \mathcal{L}^1(\mathbb{R}^k)$ is a connection.

Ex 2: In a local trivialization (by Proposition 2) we can always write

technically $\nabla^A \rightarrow \nabla = d + A$ where $A \in \mathcal{L}^1(\text{End}(E))$ Of course, here
for ∇ the matrix
depends on the
trivialization.
That is $A = A_1 dx_1 + \dots + A_n dx_n$ for A_i matrices, and
 $\nabla_i \psi = \frac{\partial \psi}{\partial x_i} + A_i \psi$.

Ex 3: On $\text{End}(E) = E^* \otimes E$, the induced ∇ from Proposition 1 is

$$\nabla B = dB + [A, B]$$

in a trivialization.

Define a connection as compatible with $\langle \quad \rangle$ if for compatible ∇, A
will be in $\mathcal{L}^1(SO(E))$
or $Sp(E)$

$$d\langle \psi, \varphi \rangle = \langle \nabla \psi, \varphi \rangle + \langle \psi, \nabla \varphi \rangle$$

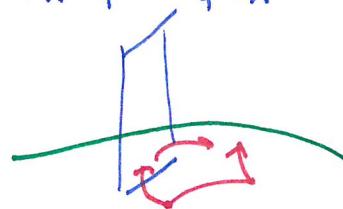
Remark: a fancy way of saying this is $\langle \quad \rangle \in E^* \otimes E^*$ has $\nabla = 0$.

Fact: E has a connection compatible with $\langle \quad \rangle$.

Proof: Locally, they are $d+A$ (so exist), and then use a partition of unity.

iii) Curvature

Given two vectors (or vector fields) $X, Y \in \Gamma(TM)$, $\nabla_{X,Y}$ need not commute, i.e.

$$\nabla_X \nabla_Y - \nabla_Y \nabla_X \neq 0.$$


Geometrically, since these were defined by flowing along horizontal lifts \tilde{X}, \tilde{Y} , this is a question about non-commuting flows, i.e. $[\tilde{X}, \tilde{Y}]$. In particular, if HE is integrable, $[\tilde{X}, \tilde{Y}] = 0$ so the flows (hence ∇_X, ∇_Y) commute. Thus the curvature is a measure of the integrability of $HE \subseteq E$.

The curvature is defined as

$$F_A \gamma := [\nabla^A, \nabla^A]$$

Claim : $F_A(f\gamma) = f F_A \gamma$ is C^∞ -linear.

Pf : Write $\nabla = d + A$. Since $d^2 = 0$ and the Leibniz rule, f pulls through.

Moreover, one sees locally

$$F_A = d_A A + A \wedge A,$$

i.e. $F_A = F_A^{ij} dx^i \wedge dx^j$ with $F_A^{ij} = \partial_i A_j - \partial_j A_i + A_i A_j - A_j A_i$

$\Rightarrow F_A \in \mathcal{S}^2(\text{End } E)$. We have $\star \stackrel{d_A}{\circ} : \mathcal{S}^2(\text{End } E) \rightarrow \mathcal{S}^3(\text{End } E)$
by $\alpha \otimes B \mapsto d\alpha \otimes B + \alpha \otimes \nabla B$.

Theorem (Bianchi Identity)

$$d_A F_A = 0.$$

III Invariant Polynomials

In Gauss-Bonnet we used $\sqrt{\det}$ to turn the $R \in \mathcal{S}^2(\text{so}(T\Sigma))$ into an \mathbb{R} -valued form to integrate. In general, since F_A isn't basis-invariant we want a map

$$P : g \rightarrow \mathbb{R}$$

$$(G = SO(k), SU(k))$$

invariant under Ad . If P is a polynomial, we say it is an invariant polynomial.

Ex 4 : Tr, Det are Ad-invariant

Thus given P, A we get an IR-valued form $P(F_A) \in \mathcal{Z}^*(M; IR)$

Proposition : $dP(F_A) = 0$, hence we get the (degree preserving) Weil Homomorphism

$$\text{Sym}^*(\mathfrak{g}^*)^{\text{Ad}} \rightarrow H_{dR}^*(M; IR)$$

Proof : Write $P(\xi) = \sum_I P_I \xi_{i_1} \dots \xi_{i_N}$

i) Ad-invariance says for $g_t = \exp(t\eta)$

$$P(\xi) = P(\text{Ad}_{g_t} \xi)$$

So

$$\begin{aligned} 0 &= \frac{d}{dt} P(\xi) = \frac{d}{dt} \sum_I P_I (\text{Ad}_{g_t} \xi)_{i_1} \dots (\text{Ad}_{g_t} \xi)_{i_N} \\ &= \sum_{I,K} P_I \xi_{i_1} \dots \xi_{i_{K-1}} [\eta, \xi]_{i_K} \dots \xi_{i_N} \end{aligned}$$

i) $F_A = [F_A]^i$

$$\begin{aligned} dP(F_A) &= d\left(\sum_I P_I F^{i_1} \wedge \dots \wedge F^{i_N}\right) \\ &= \sum_{I,K} P_I F^{i_1} \wedge \dots \wedge dF^{i_K} \wedge \dots F^{i_N} + \sum_{I,K} P_I F^{i_1} \wedge \dots \wedge [A, F_A]_{i_K} \dots \\ &= \sum_{I,K} P_I F^{i_1} \wedge \dots \wedge (d_A F_A)_{i_K} \wedge \dots \wedge F^{i_N} \\ &= 0 \quad (\text{Bianchi}). \end{aligned}$$

□

Proposition (Invariance)

i) $[P(F_A)]$ is independent of A.

ii) $[P(F_A)]$ is independent of $\langle \quad \rangle$

iii) If $E \cong E'$ then $[P(F_A)] = [P(F_{A'})]$. $[P(F_A)] \in H^*$ is a characteristic class of E.

Proof : i) Take A, A' and set $\nabla_A - \nabla_{A'} = B$. Define A_t on $E \times I \rightarrow M \times I$

by $\nabla_A + tB$. $P(F_{A_t}) \in \mathcal{Z}^*(M \times I; IR)$, and $i_0: M \rightarrow M \times \{0\}$

has $i_0^* P(F_{A_t}) = P(F_A)$ same for i_1, A' . But i_0, i_1 are homotopic.

ii) Similar

iii) Use pullback connection + i).

□

IV Examples

Now the fun part: choose different P and see what we get.

i) Chern Classes

Consider $\text{Det}(\lambda \text{Id} - \frac{1}{2\pi i} X) : \mathfrak{u}(k) \rightarrow \mathbb{R}$
 $= \lambda^k - c_1(X)\lambda^{k-1} + c_2(X)\lambda^{k-2} + \dots$ for c_k polynomials in X .

Define c_k as the k^{th} chern class

(the characteristic class in H^{2k} obtained from $P =$)

$$\text{Explicitly } c_k(F_A) = \frac{\text{Tr}(\Lambda^k F_A)}{(2\pi i)^k}$$

$$= 1 - \frac{1}{2\pi i} \frac{\text{Tr}(F_A)}{\lambda^{k-1}} + \frac{\text{Tr}(F_A \wedge F_A) - \text{Tr}(F_A)^2}{8\pi^2 \lambda^{k-2}} - \dots$$

Remark 1 :- It's immediate $c_1 = 0$ for an $SU(n)$ -bundle since $\mathfrak{su}(k)$ is traceless
 - In fact, one can show c_k are a basis for Ad -invariant polynomials,
 so this is a complete list.

ii) Pontryagin Classes

$$\text{Det}(\lambda \text{Id} - \frac{1}{2\pi} X) : \mathfrak{o}(k) \rightarrow \mathbb{R}$$
 $= \lambda^k - g_1(X)\lambda^{k-1} + \dots$

Since $\mathfrak{o}(k)$ is skew-symmetric $g_{\text{odd}} = 0$ and $g_{2k} : P_k(E)$ is the k^{th} Pontryagin class.

$$P_1 = -\frac{\text{Tr}(F_A \wedge F_A)}{8\pi^2} \quad P_2 = \frac{\text{Tr}(F_A \wedge F_A)^2 - 2\text{Tr}(F_A \wedge \dots \wedge F_A)}{128\pi^4}$$

iii) Euler Class

If k is even there's the Pfaffian $\text{Pf} : \mathfrak{o}(2k) \rightarrow \mathbb{R}$ w/ $\text{Pf}(X)^2 = \det X$

$e(E) := \text{Pf}(F_A)$ is the Euler class

v) Other classes : If $g(X) = a_0 + a_1 X + a_2 X^2 \dots$ is a power series,
 $\det(g(X))$ is invariant

$$g = 1 + \frac{z}{2\pi i} ; \text{ total Chern class}$$

$$g = \frac{z}{\tanh z} ; \text{ L-genus}$$

$$g = \frac{z^2}{1 - \exp(-z^2)} ; \text{ Todd genus}$$

V Axioms

Grothendieck's axioms are

- i) $c_0(E) = 1$, $c_i(E) = 0$ for $i > \text{rank } E$ ($c_i = \frac{\text{Tr}}{i!} (\Lambda^i F_A)$ and $\Lambda^i = 0$ for $i \geq \text{rk } E + 1$)
- ii) Naturality w/ pullbacks (obvious: use pullback connection)
- iii) (Whitney Sum) $c(E \oplus F) = c(E) \cup c(F)$ (splitting principle)
- iv) (Normalization) $c_i(\mathcal{O}(1)) = -1$ on \mathbb{CP}^1 (direct computation)

So this Chern-Weil definition ~~stays~~ gives the same Chern classes as other definition.

Remark : Although, a priori, $c_k \in H_{dR}^{2k}(M; \mathbb{R})$, the normalization shows it's actually in the image of

$$H^{2k}(M; \mathbb{Z}) \rightarrow H^{2k}(M; \mathbb{R}).$$

VI An Application

Here's an application of Chern-Weil theory to something harder to see with other definitions of characteristic classes:

Let $E \rightarrow M$ be a complex vector bundle that admits a reduction of structure group to locally constant transition functions (if E is a local system with group \mathbb{C}^n), then $c_k(E) \in H^{2k}(M; \mathbb{Z})$ is torsion.

Proof : E admits a flat connection.

$$(A \mapsto gAg^{-1} + g^{-1}dg)$$

so we can take $d+A$ with $A=0$, and this is preserved by changing trivializations. □

Part 2: Equivariant de Rham Cohomology

VII Motivation

Let $G \rightarrow M$ be a space with the action of a Lie group. We want a cohomology theory that takes into account the G -action. If the action is free, then we take

$$H_G^*(M) := H^*(M/G).$$

If the action is not free, we take the homotopy quotient $EG \times M/G$ and set

$$H_G^*(M) := H^*(EG \times M/G).$$

Here EG is a contractible space with a free G -action. (If $G = U(n)$, EG is the total space of the Tangential bundle $E \rightarrow \text{Gr}(n, \mathbb{C}^\infty)$).

Question: How do we do equivariant cohomology with differential forms?

Consider a free action, i.e. $P \rightarrow M$ is a principal bundle. We want to distinguish forms $\alpha \in \mathcal{Z}^*(P)$ that pull back from $M = P/G$. The linearized action gives $X_\xi \in \Gamma(TP)$ for $\xi \in \mathfrak{g}$, which together span $\text{Ker } \pi_{\text{v}}: TP \rightarrow TM$. (The vertical subbundle)

Locally if we write

$$\alpha = \sum_I \alpha_I dx_1 \wedge \dots \wedge dx_n$$

or α to come from M , we need none of the dx_i to be vertical, i.e.

i) $i_\xi \alpha = 0 \quad \forall \xi \in \mathfrak{g} \quad (i_\xi := i_{X_\xi} \text{ for shorter notation})$

and α_I doesn't depend on vertical coordinates, i.e.

ii) $i_\xi d\alpha = 0 \quad \forall \xi \in \mathfrak{g} \quad (\text{or } L_\xi \alpha = 0 \text{ if above holds, by Cartan's formula}).$

Define forms satisfying i) and ii) as basic. When the action is free

$$H_{\text{dr}}^*(M) = H_G^*(P/G) = H^*(\mathcal{Z}(P)_{\text{bas}}).$$

VIII G^* Algebras

There are multiple ways \mathfrak{g} acts on $\mathcal{Z}^*(M)$: ($G \rightarrow M$ again)

$$\xi \mapsto i_\xi \quad (\text{deg } -1)$$

$$\xi \mapsto L_\xi \quad (\text{deg } 0)$$

we also have d $(\text{deg } 1)$

We can phrase this as a representation of the Lie Superalgebra

$$\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathbb{R} := \tilde{\mathfrak{g}}$$

with generators i_3, L_3, d resp. and relations

$$[i_3, i_3] = 0$$

$$[L_3, i_3] = i [s, \eta]$$

$$[L_3, L_3] = L_{[3, \eta]}$$

$$[d, i_3] = L_3 \quad (\text{Cartan})$$

$$[d, L_3] = 0$$

$$[d, d] = 2d^2 = 0$$

(remember it's a superalgebra so $\deg!$ is anticommutator).

Define a G^* -module (resp algebra) as a super vector space (algebra) A with an action $G \rightarrow \text{Aut}(A)$ and a representation $\tilde{\mathfrak{g}} \rightarrow \text{End}(A)$ compatible in the sense that

$$\begin{aligned} \text{i)} \quad & \left. \frac{d}{dt} \right|_{t=0} \exp(t_3) = L_3 \quad \text{ii)} \quad g L_3 g^{-1} = L_{\text{Ad}_g 3} \quad \text{iii)} \quad g d = g d g \\ & g i_3 g^{-1} = i_{\text{Ad}_g s} \end{aligned}$$

Remark: Clearly, $\mathcal{S}^*(m)$ is a G^* -algebra under \wedge . That's the point of the definition.

The following things work basically as normal:

$$\text{i)} \text{ Cohomology} \quad H^*(A) = H_*((A, d)).$$

$$\text{ii)} \text{ Chain homotopy}$$

The generalization of basic forms in this algebraic setting is

$$i_3 \alpha = L_3 \alpha = 0 \quad \forall \alpha.$$

Now let E be a G^* -algebra that is acyclic and satisfies "condition C"

Remark: Condition C is just a few other necessary conditions that EG should satisfy, e.g. the action is free.

Define the equivariant de Rham Cohomology by

$$H_{G, \text{dR}}^*(m) := H^*((\mathcal{S}^*(m) \otimes E)_{\text{bas}})$$

Theorem (Equivariant de Rham Theorem)

$$H_{G,dR}^*(M) = H_G^*(M) \quad (= H^*(M \times EG/G))$$

The proof isn't that bad: approximate EG w/ a finite model, and take sequence of

$$E = \varinjlim \mathcal{S}L(E_n). \quad (\text{intuitively } \mathcal{S}L^*(EG))$$

by free case $H^*(M \times E_k/G) = H^*(\mathcal{S}L(M \times E_k)_{\text{bas}})$ for $k \ll k$.

To finish, show $\mathcal{S}L(M \times E_k)_{\text{bas}} = \mathcal{S}L(M) \otimes \mathcal{S}L(E_k)_{\text{bas}}$ in the limit.

Proposition

The definition of $H_{G,dR}^*$ is independent of E satisfying the assumptions (acyclic + Condition C).

IX The Cartan Model

Now we can look for a specific E that gives a nice algebraic structure, so it might be more computable.

For a vector space V , the Koszul algebra is $(\Lambda^*(V) \otimes S^*(V), d)$ where $d(\alpha \otimes 1) = 1 \otimes \alpha$, $d(1 \otimes \alpha) = 0$ extended as a derivation. The Weil Algebra is

$$W = \Lambda^*(g^*) \otimes S^*(g^*)$$

as a G^* algebra: For a basis (as an algebra) Θ^i, z^j we have

$$i_a \Theta_b = \delta_{ab} \quad L_a \Theta_b = -[\Theta_a, \Theta_b] \quad i_a z_b = -C_{ab}^k \Theta_k \quad (\text{extended as derivations})$$

$$L_a z_b = -C_{ab}^k z_k$$

Proposition: W is acyclic and satisfies condition C.

Proof (of acyclic): $Q(\alpha \otimes 1) = 0$ $Q(1 \otimes \alpha) = d \otimes 1$ is a chain homotopy Id to 0.

In particular, we can use W as a model for E .

Proposition: $i_g = 0 \forall g$ on $1 \otimes \mathbb{C}[M, \dots, M_n] \subseteq W$ $M = d\Theta^a + \frac{1}{2} C_{ij}^a \Theta_i \wedge \Theta_j$ and the basic forms are

$$W_{\text{bas}} = \mathbb{C}[M, \dots, M_n]^G$$

and $d|_{W_{\text{bas}}} = 0$.

Thus $H_*(W \otimes \Omega^*(M))_{\text{bas.}} d|_{\text{bas.}}$ calculates $H_G^*(M)$.

Theorem (Cartan Model) For a G^* -algebra A , there is an isomorphism

$$\varphi : (W \otimes A)_{\text{bas.}} \xrightarrow{\sim} (S^*(\mathfrak{g}^r) \otimes A)^G \quad (\text{the Mathai-Quillen iso.})$$

sending

$$d|_{\text{bas.}} \mapsto d_G = 1 \otimes d_A - \mu^a \otimes i_a.$$

In particular, $H_G^*(M)$ can be calculated from

$$(S^*(\mathfrak{g}^r) \otimes \Omega^*(M))^G, d_G$$

G -equivariant forms valued in

\mathfrak{g}^r polynomials

$$(d_G \alpha)(z) = d(\alpha(z)) - i_z(\alpha(z))$$