DIFFERENTIAL COHOMOLOGY IN FIELD THEORY

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There are a few different applications of differential cohomology to quantum physics; today, we'll focus on charge quantization, using Maxwell theory as an example.

1. Classical Maxwell theory

Let (N, g_N) be a Riemannian 3-manifold without boundary and $M \coloneqq \mathbb{R} \times N$. Let t be the $\mathbb{R}$ coordinate, so we give M the Lorentz metric

$$g_M \coloneqq \mathrm{d}t^2 - g_N. \tag{1.1}$$

Choose differential forms $E \in \Omega^1(N)$ and $B \in \Omega^2(N)$, respectively the electric and magnetic fields; also choose the *charge density* $\rho_E \in \Omega^3_c(N)$, and the *current* $J_E \in \Omega^2_c(N)$.[1] If $\star_N$ denotes the Hodge star on N, then *Maxwell's equations*, as you might see them on a t-shirt, are

$$\begin{aligned} \mathrm{d}B &= 0 & \qquad \frac{\partial B}{\partial t} + \mathrm{d}E &= 0 \\ \mathrm{d}{\star_N} E &= \rho_E & \qquad \star_N \frac{\partial E}{\partial t} - \mathrm{d}{\star_N} B &= J_E. \end{aligned} \tag{1.2}$$

Writing $F \coloneqq B - \mathrm{d}t \wedge E \in \Omega^2(M)$ and $j_E \coloneqq \rho_E + \mathrm{d}t \wedge J_E \in \Omega^3(M)$, we obtain a more concise form of Maxwell's equations:

$$\mathrm{d}F = 0, \qquad \mathrm{d}{\star_M} F = j_E. \tag{1.3}$$

Now we include topology. We just saw that j_E is exact, so it cannot define an interesting de Rham cohomology class, but F is closed, so may be interesting. Define the *charge* at time t to be the de Rham class

$$Q_E \coloneqq [j_E|_{\{t\}\times N}] \in H^3_c(N;\mathbb{R}). \tag{1.4}$$

This is in the kernel of the map $H^3_c(N;\mathbb{R}) \to H^3(N;\mathbb{R})$; hence, on a compact manifold, $Q_E = 0$.

Let W be the worldline of a charged particle with electric charge $q_E \in \mathbb{R}$. Then $j_E = q_E \cdot \delta_W$, where δ_W is the "current sitting at W." We have two ways of making sense of this.

- First, we could take δ_W to be a current in the de Rham sense, akin to a differential form but built with distributions instead of smooth functions. Amusingly, this is a current in both the Maxwell and de Rham senses. This is a typical example of a current in electromagnetism.
- Alternatively, we could take δ_W to be an honest 3-form Poincaré dual to W. In this case we can choose δ_W to be supported in an arbitrary neighborhood of W.

One more ingredient in Maxwell theory, though not strictly necessary, is an action principle. This follows the Lagrangian formulation of physics: we aim to find a variational problem whose solutions are the Maxwell equations. We add an assumption from classical physics: that $[F] = 0$ in $H^2_{\mathrm{dR}}(M)$; this means there are no magnetic monopoles.

This assumption also implies $F = \mathrm{d}A$ for some 1-form A called the *electromagnetic potential.* This is not unique, but its class in $\Omega^1(M)/\Omega^1(M)_{c\ell}$ (i.e. up to closed 1-forms) is unique. Then, the *classical action* of Maxwell theory is

$$S \coloneqq \int_M -\frac{1}{2}\mathrm{d}A \wedge \star\mathrm{d}A + A \wedge j_E. \tag{1.5}$$

Since M is noncompact, this could be infinite, but we're just interested in its first variation anyways, which is well-behaved.

[1] Here $\Omega^k_c(X)$ denotes the space of compactly supported k-forms on X.

Exercise 1.6. Show that the Euler-Lagrange equation for (1.5) is $\mathrm{d}\star F = j_E$. (We already assumed $\mathrm{d}F = 0$, the other half of Maxwell's equations.)

One caveat: defining the action requires A to be in $\Omega^1(M)$, not $\Omega^1(M)/\Omega^1(M)_{c\ell}$. This ends up not a problem; adding a closed form to A does not change the Euler-Lagrange equation.

2. Quantum Maxwell theory

In the quantum theory, we allow magnetic monopoles. Dirac argues that this forces electric and magnetic charges to be *quantized*, i.e. taking values in a discrete subgroup of $\mathbb{R}$. This is how differential cohomology enters the picture.

So assume $N = \mathbb{R}^3$ with the usual Euclidean metric, and introduce a magnetic monopole of charge $q_B \in \mathbb{R}$ at the origin. Then we have a *magnetic current* $j_B \coloneqq q_B \cdot \delta_0$. The condition that $\mathrm{d}F = 0$ is modified to

$$\mathrm{d}F = q_B \cdot \delta_0. \tag{2.1}$$

The input to the path integral is the exponentiated action $\exp(iS/\hbar)$ (where S is as in (1.5)). However, this is not quite consistent with (2.1) — there is a problem at the origin. On $\mathbb{R} \times (\mathbb{R}^3 \setminus 0)$, we can write $F = \mathrm{d}A$, and therefore realize F as the curvature of a connection A on a principal $\mathbb{R}/q_B\mathbb{Z}$-bundle P. The characteristic class of P is

$$[P] \in H^2(\mathbb{R} \times (\mathbb{R}^3 \setminus 0); q_B\mathbb{Z}) \cong H^2(S^2; q_B\mathbb{Z}) = q_B\mathbb{Z}, \tag{2.2}$$

and $[P]$ is a generator of this abelian group.

The space of fields in the quantum theory is the groupoid of principal $\mathbb{R}/q_B\mathbb{Z}$-bundles with connection. Now we can revisit the action (1.5) — it doesn't have to make sense as is (e.g. A isn't exactly a 1-form), but we do want $\exp(iS/\hbar)$ to make sense.

Let's work on a general 4-manifold X. To avoid causality issues, let's make X a Riemannian manifold, rather than a Lorentz one. Assume j_E is Poincaré dual to some loop $\gamma \subset X$. If there is a q_E charge moving along this loop, then

$$\int_M A \wedge j_E = \oint_\gamma q_E A = q_E \operatorname{Hol}_\gamma(A). \tag{2.3}$$

Now $\operatorname{Hol}_\gamma(A) \in \mathbb{R}/q_B\mathbb{Z}$, so the quantity

$$\exp\left(\frac{i}{\hbar} q_E \operatorname{Hol}_\gamma(A)\right) \tag{2.4}$$

is well-defined iff

$$\frac{1}{\hbar} q_E q_B \in 2\pi\mathbb{Z}. \tag{2.5}$$

This is Dirac's quantization condition. Thus integrality enters a story told with differential forms; this is already suggestive of differential cohomology!

To say it more explicitly, the space of quantum fields is $\mathcal{B}un^\nabla_{\mathbb{R}/q_B\mathbb{Z}}(X) = \check{H}^2(X; q_B\mathbb{Z})$. The curvature map lands in those 2-forms with periods in $q_B\mathbb{Z}$, giving us a short exact sequence we've seen before:

$$0 \longrightarrow H^1(X; \mathbb{R}/q_B\mathbb{Z}) \longrightarrow \check{H}^2(X; q_B\mathbb{Z}) \xrightarrow{\text{curv}} \Omega^2(X)_{c\ell; q_B\mathbb{Z}} \longrightarrow 0. \tag{2.6}$$

The classical fields $\Omega^1(X)/\Omega^1(X)_{c\ell}$ sit as a subspace in $\check{H}^2(X)$; the cokernel is $H^2(X; q_B\mathbb{Z})$ modulo torsion, indicating the new information in the quantum theory.

Another interesting upshot is that since the kernel of the curvature map corresponds to the flat connections, i.e. those on which F is boring, the electric flux really lives in $\check{H}^2(X; q_B\mathbb{Z})$. This is new. The flat connections are new, too — even if you don't usually get to observe them, they manifest in the physics, e.g. through the Aharonov-Bohm effect. And all of this is still "semiclassical," i.e. about the input to the path integral, before we try to evaluate said path integral.

Remark 2.7. One important clarification: F is not a differential cohomology class; it's the curvature of an actual bundle with connection, not an equivalence class. So really we need a cochain model: bundles and connections glue, but equivalence classes don't. Cheeger-Simons characters aren't built in this way, so for physics applications one must do something different. ◀

Now we revisit the electric charge, a closed 3-form.[2] Because $(i/\hbar)j_E j_B \in 2\pi\mathbb{Z}$, we'd like to impose that $[j_E] \in H^3_{\mathrm{dR}}(X)$ is also in the image of the map $H^3(X; q_E\mathbb{Z}) \to H^3(X;\mathbb{R})$, i.e. that we're in the homotopy pullback, which is $\check{H}^3(X; q_E\mathbb{Z})$. Again, though, we want a local object in the end, not just its isomorphism class.

We can also rewrite the action in terms of differential cohomology, as

$$\exp\left(\frac{i}{\hbar}\int_X \check{F}\cdot \check{\jmath}_E\right). \tag{2.8}$$

Here $\check{F}$ and $\check{\jmath}_E$ are the differential cohomology refinements of F and j_E, respectively. The product $\cdot$ is the cup product in Deligne cohomology, which is a map

$$\check{H}^2(X; q_B\mathbb{Z}) \otimes \check{H}^3(X; q_E\mathbb{Z}) \longrightarrow \check{H}^5(X; q_E q_B\mathbb{Z}). \tag{2.9}$$

Since X is a 4-manifold, the integration map has degree -4, so is of the form

$$\int_X : \check{H}^5(X; q_E q_B\mathbb{Z}) \longrightarrow H^1(\mathrm{pt}; q_E q_B\mathbb{Z}) \cong \mathbb{R}/q_E q_B\mathbb{Z}. \tag{2.10}$$

Exercise 2.11. Show that if $\check{F}$ is topologically trivial (i.e. the curvature of a flat connection), then $\check{F}\cdot\check{\jmath}_E$ is also topologically trivial.

Remark 2.12. There are many variations of this story in field theory and string theory, generally for abelian gauge fields. For example, F might have some other degree, or even be inhomogeneous. Dirac charge quantization still applies, and will refine F to an appropriate differential cohomology group.

More recently, people realized that this story sometimes yields generalized differential cohomology theories. Understanding which cohomology theory one obtains is a bit of an art — physics tells you some constraints, but not an algorithm. For example, this happens in superstring theory: the Ramond-Ramond field is realized in differential K-theory, and the B-field in a differential refinement of (a truncation of) $gl_1 KU$. ◀

If we consider Maxwell theory with both electric and a magnetic currents, the theory has an "anomaly," meaning that some quantity that we'd like to obtain as a complex number is actually an element of a complex line that's not trivialized (and in some cases cannot be trivialized canonically for all manifolds of a given dimension). Differential cohomology also provides a perspective on the anomaly. The expression $\check{F}\cdot\check{\jmath}_E$ in (2.9) is valid if there's electric current but not magnetic current; if $j_B \neq 0$, then F isn't closed, hence isn't the curvature of a line bundle. But $\check{\jmath}_B$ is also quantized, hence represents a differential cohomology class, and we can ask for $\check{F}$ to trivialize $\check{\jmath}_B$. Now the action is

$$\exp\left(\frac{i}{\hbar}\int_X \check{F}\cdot \check{\jmath}_E\check{\jmath}_B\right). \tag{2.13}$$

Since $\check{F}\cdot\check{\jmath}_E\check{\jmath}_B \in \check{H}^6$, integrating brings us to $\check{H}^2(\mathrm{pt}; q_E q_B\mathbb{Z})$, yielding the complex line which signals the anomaly.

Example 2.14. In the last few minutes, we'll discuss a different example of differential cohomology in physics. Suppose M is an oriented Riemannian 3-manifold and $P \to M$ is a principal SU_2-bundle with connection Θ. The second Chern class admits a differential refinement $\check{c}_2(\Theta) \in \check{H}^4(M)$, and

$$\int_M \check{c}_2(\Theta) \in \check{H}^1(\mathrm{pt}) \cong \mathbb{R}/\mathbb{Z}. \tag{2.15}$$

Hence this is the sort of thing you can add to an action. It's an example of a *Chern-Simons term* in 3d QFT. The de Rham class underlying $\check{c}_2$ is sometimes called the level of the theory, and the fact that it must refine to differential cohomology is saying the level is quantized. In general, quantization of coupling constants provides another instance of differential cohomology in physics.

Chern-Simons terms are usually described without differential cohomology, using the Chern-Simons form associated to Θ, but writing that term on M, rather than on P, requires a choice of a section of P, and we don't always have that. ◀

[2]You might be wondering where the compact support condition went. To work in Euclidean signature, rather than Minkowski signature, we must *Wick-rotate* the theory, a nontrivial procedure which ultimately removes the requirement for compact supports.