

The Morava K -theory of Eilenberg-MacLane spaces

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1. Introduction

Our goal in these two lectures is to describe the computation of the Morava K -theory of Eilenberg-MacLane spaces, which is originally due to Ravenel and Wilson in [RW80]. We will follow the argument provided in [HL13, §2.4], which is — unavoidably — technical, but unfortunately also contains numerous minor errors and typos, and lacks details in a few places. We have attempted to make the exposition as clear as possible, and add in missing details.

We begin by stating the main result which we will be working towards. Let us fix a perfect field κ of characteristic p and a one-dimensional formal group \mathbf{G}_0 over κ of finite height n . Let $K(n)$ denote the associated 2-periodic Morava K -theory. Then:

Theorem 1.1 (Ravenel-Wilson). *The graded κ -vector space $K(n)_*(K(\mathbf{Z}/p^k, m))$ is concentrated in even degrees. Moreover, the group $K(n)_0(K(\mathbf{Z}/p^k, m))$ is a connected Hopf algebra over κ , with associated Dieudonné module $M(m)$ satisfying the property that $M(m) \cong \bigwedge^m M(1)$, where F and V act as¹*

$$\begin{aligned} F(Vx_1 \wedge \cdots \wedge Vx_{i-1} \wedge x_i \wedge Vx_{i+1} \wedge \cdots \wedge Vx_m) &= x_1 \wedge \cdots \wedge x_{i-1} \wedge Fx_i \wedge x_{i+1} \wedge \cdots \wedge x_m, \\ V(x_1 \wedge \cdots \wedge x_m) &= Vx_1 \wedge \cdots \wedge Vx_m. \end{aligned}$$

The strategy of proof is simple: we induct on m , and prove the following three statements.

- (1) $K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ is isomorphic to a formal power series ring over κ ;
- (2) The formal group $\mathrm{Spf} K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ is p -divisible and has Dieudonné module given by the m th exterior power of the Dieudonné module associated to the formal group $\mathrm{Spf} K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, 1))$;
- (3) The scheme $\mathrm{Spec} K(n)^0(K(\mathbf{Z}/p^k, m))$ is the p^k -torsion in the connected p -divisible group $\mathrm{Spf} K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$.

We begin with a general discussion. The Morava K -theory spectrum $K(n)$ at height n is an \mathbf{E}_1 -algebra object of $\mathrm{Mod}(E)$, where E is the n th Lubin-Tate theory. However, it is *never* an \mathbf{E}_∞ -ring — in fact, it is never an \mathbf{E}_2 -ring². In fact, $K(n)$ is

¹We will explain below why it is sufficient to describe the action of F and V in the below manner.

²One proof of this result goes as follows. The coefficient ring of $K(n)$ is $\pi_* K(n) = \kappa[u^{\pm 1}]$, so $p = 0$ in $\pi_0 K(n)$. A theorem of Hopkins and Mahowald states that \mathbf{F}_p (as a discrete \mathbf{E}_2 -ring) is the universal \mathbf{E}_2 -ring with $p = 0$; it follows that if $K(n)$ is an \mathbf{E}_2 -ring, it would be a \mathbf{F}_p -algebra. However, the homotopy $\pi_*(\mathbf{F}_p \otimes K(n))$ of the complex oriented smash product $\mathbf{F}_p \otimes K(n)$ carries

not even homotopy commutative at the prime 2. In particular, if G is a topological group, it is not clear that $K(n)_*(X)$ is a Hopf algebra over $K(n)_*$. Nonetheless, since $K(n)$ is the quotient of E by its maximal ideal (which is generated by a regular sequence of elements, so quotienting is a legal operation), one might want to use the existing Hopf algebra structure on $E_*^\vee(X) = \pi_*(L_{K(n)}(E \otimes \Sigma_+^\infty X))$ to obtain one on $K(n)_*(X)$. This is possible if $K(n)_*(X)$ is concentrated in even degrees, in which case $K(n)_*(X)$ is the quotient of $E_*^\vee(X)$ by its maximal ideal. This discussion motivates the following definition.

Definition 1.2. A topological abelian group G is said to be $K(n)$ -good (although we will often simply write “good”) if:

- (1) $K(n)_*(X)$ is concentrated in even degrees; this implies that $K(n)_0(X)$ is a Hopf algebra over $K(n)_0 = \kappa$.
- (2) $K(n)_0(X)$ is the colimit of a sequence of Hopf algebras $H(t)$ which are killed by p^t .

Since $K(n)_0(X)$ is a Hopf algebra over k , we can use the constructions described in Morgan’s and Dexter’s lectures to construct an associated Dieudonné module³. This Dieudonné module $D(X) = \text{DM}_+(K(n)_0(X))$ is defined by $D(X) = (W(\bar{\kappa}) \otimes_{\mathbf{Z}} \text{GLike}(K(n)_0(X)_{\bar{\kappa}}))^{\text{Gal}_{\kappa}}$, where the action of F is induced by the map $x \mapsto p\phi(x)$ of $W(\bar{k})$ and the action of V is by the map $x \mapsto \phi^{-1}(x)$ of $W(\bar{k})$. By [HL13, Corollary 1.4.17], the data of the Dieudonné module $D(X)$ is equivalent to the data of the Hopf algebra $K(n)_0(X)$ over κ . Since $K(n)_0(X)$ is a Hopf algebra over κ , the formal scheme $\text{Spf } K(n)_0(X)$ acquires the structure of a formal group over κ , and this data is equivalent to the Hopf algebra $K(n)_0(X)$.

Definition 1.3. Let $H = K(n)_0(\mathbf{C}P^\infty)$, and let $H[p^k]$ denote the kernel of the map $[p^k] : H \rightarrow H$ (again, all kernels are in the category of Hopf algebras over κ); this is a connected Hopf algebra of dimension p^{nk} over κ . Define M to be the Dieudonné module defined by $\lim \text{DM}(H[p^k])$.

The Dieudonné module M is free of rank n over $W(\kappa)$. We begin with an lemma, whose (easy) proof we leave to the reader.

Lemma 1.4. *There is a stable equivalence $\Sigma_+^\infty K(\mathbf{Q}_p/\mathbf{Z}_p, n) \simeq \Sigma_+^\infty K(\mathbf{Z}_p, n+1)$, which exists after p -completion.*

Recall that our goal is to understand the Morava K -theory of the spaces $K(\mathbf{Z}/p^k, m)$. We shall do this by induction on m . The base case is when $m = 1$, which we shall now address.

Proposition 1.5. *The space $B\mathbf{Z}/p^k$ is good. The fiber sequence $B\mathbf{Z}/p^k \rightarrow \mathbf{C}P^\infty \xrightarrow{p} \mathbf{C}P^\infty$ gives a short exact sequence of Hopf algebras*

$$\kappa \rightarrow K(n)_0(B\mathbf{Z}/p^k) \rightarrow H \xrightarrow{[p^k]} H \rightarrow \kappa,$$

so that $K(n)_0(B\mathbf{Z}/p^k) = H[p^k]$ as Hopf algebras. This also gives an equivalence $D(B\mathbf{Z}/p^k) = M/p^k$.

both the additive formal group and our fixed formal group \mathbf{G}_0 of height n , so these two formal groups must be isomorphic. This is impossible in characteristic p .

³Recall that a Dieudonné module is a module over the Cartier-Dieudonné ring $\text{Cart}_\kappa = D_\kappa = W(\kappa)\langle F, V \rangle / (FV = VF = p, Fx = x^\phi F, Vx^\phi = xV)$.

Proof. This can be proved by the Gysin sequence of the fibration $S^1 \rightarrow B\mathbf{Z}/p^k \rightarrow \mathbf{C}P^\infty$. Indeed, choose a complex orientation η of $K(n)$, and let β be an invertible element in $K(n)_2$. Define $x = \beta^{-1}\eta$, so that $K(n)_0(\mathbf{C}P^\infty) = \kappa[[x]]$. The Gysin sequence is a long exact sequence

$$\cdots \rightarrow K(n)^{* - 2}(\mathbf{C}P^\infty) \xrightarrow{\phi} K(n)^*(\mathbf{C}P^\infty) \rightarrow K(n)^*(B\mathbf{Z}/p^k) \rightarrow K(n)^{* - 1}(\mathbf{C}P^\infty) \xrightarrow{\phi} K(n)^{* + 1}(\mathbf{C}P^\infty) \rightarrow \cdots,$$

where ϕ is the map which is multiplication by $\beta[p^k](x)$, where $[p^k](x)$ is the p^k -series of the formal group law defined by \mathbf{G}_0 along with the coordinate coming from the chosen complex orientation of $K(n)$. Since \mathbf{G}_0 has finite height, the element $[p^k](x)$ is a nonzero divisor in $K(n)_0(\mathbf{C}P^\infty)$, so we find that $K(n)^*(B\mathbf{Z}/p^k)$ vanishes in odd degrees, and that $K(n)_0(B\mathbf{Z}/p^k) \cong \kappa[[x]]/[p^k](x)$. We obtain the desired result after taking κ -linear duals. \square

We would now like to understand the Morava K -theory of the spaces $K(\mathbf{Z}/p^k, m)$ for $m > 1$. We will do so by studying the Morava K -theory of the spaces $K(\mathbf{Q}_p/\mathbf{Z}_p, m)$. Since we understand the Morava K -theory of $K(\mathbf{Q}_p/\mathbf{Z}_p, 1)$ well (this is the Morava K -theory of $\mathbf{C}P^\infty$), we would like to describe $K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ in terms of $K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, 1))$. To do so, note that we have a cup product map

$$K(\mathbf{Z}/p^k, 1)^m \rightarrow K(\mathbf{Z}/p^k, m) \simeq K(p^{-k}\mathbf{Z}_p/\mathbf{Z}_p, m) \rightarrow K(\mathbf{Q}_p/\mathbf{Z}_p, m).$$

If each $K(p^{-k}\mathbf{Z}_p/\mathbf{Z}_p, m)$ is good, then $K(\mathbf{Q}_p/\mathbf{Z}_p, m)$ is also good, since it is a filtered colimit of the spaces $K(p^{-k}\mathbf{Z}_p/\mathbf{Z}_p, m)$. This suggests understanding $K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ in terms of its associated Dieudonné module. The map above induces a map of Hopf algebras

$$K(n)_0(K(\mathbf{Z}/p^k, 1))^{\boxtimes m} \rightarrow K(n)_0(K(\mathbf{Z}/p^k, m)) \rightarrow K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m)),$$

and hence a map of Dieudonné modules (using the fact that the Dieudonné module functor is symmetric monoidal)

$$\theta_k^m : D(K(\mathbf{Z}/p^k, 1))^{\times m} = M/p^k \times \cdots \times M/p^k \rightarrow D(K(\mathbf{Z}/p^k, m)) \rightarrow D(K(\mathbf{Q}_p/\mathbf{Z}_p, m)).$$

Lemma 1.6. *The map θ_k^m is strictly alternating, i.e., $\theta_k^m(x_1, \dots, x_m) = 0$ if $x_i = x_j$ for $i \neq j$.*

Proof. The cup product is antisymmetric; therefore, if $\sigma \in \Sigma_m$, we have

$$\theta_k^m(x_1, \dots, x_m) = \text{sign}(\sigma)\theta_k^m(x_{\sigma(1)}, \dots, x_{\sigma(m)}).$$

It follows that if $x_i = x_j$ for $i \neq j$, we have $2\theta_k^m(x_1, \dots, x_m) = 0$, so the desired result follows if p is odd. It remains to prove the result when $p = 2$. There is a commutative diagram

$$\begin{array}{ccc} K(\mathbf{Z}/p^{k+1}, 1)^{m-1} \times K(\mathbf{Z}/p^k, 1) & \longrightarrow & K(\mathbf{Z}/p^k, 1)^{m-1} \times K(\mathbf{Z}/p^k, 1) \\ \downarrow 1 \times p & & \downarrow \\ K(\mathbf{Z}/p^{k+1}, 1)^{m-1} \times K(\mathbf{Z}/p^{k+1}, 1) & \longrightarrow & K(\mathbf{Q}_p/\mathbf{Z}_p, m). \end{array}$$

This yields the relation

$$\theta_k^m(\bar{x}_1, \dots, \bar{x}_{m-1}, y) = \theta_{k+1}^m(x_1, \dots, x_{m-1}, py),$$

where $x_1, \dots, x_{m-1} \in M/p^{k+1}$ with images $\bar{x}_1, \dots, \bar{x}_{m-1} \in M/p^k$, and $y \in M/p^k$. In the situation above, suppose $x_1, \dots, x_m \in M/p^k$ are such that $x_i = x_j$ for $i \neq j$. Choose lifts $y_1, \dots, y_m \in M/p^{k+1}$ such that $y_i = y_j$; then

$$\theta_k^m(x_1, \dots, x_m) = \theta_{k+1}^m(y_1, \dots, y_{m-1}, py_m) = p\theta_{k+1}^m(y_1, \dots, y_{m-1}).$$

But if $p = 2$, then

$$p\theta_k^m(x_1, \dots, x_m) = 2\theta_k^m(x_1, \dots, x_m) = 0,$$

as was argued above, so $\theta_k^m(x_1, \dots, x_m) = 0$, as desired. \square

The upshot of Lemma 1.6 is that we obtain a map $\bigwedge^m M/p^k \rightarrow D(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ of $W(\kappa)$ -modules. Since $\bigwedge^m M/p^k \cong \mathbf{Z}/p^k \otimes \bigwedge^m M$ as $W(\kappa)/p^k$ -modules, and the diagram

$$\begin{array}{ccc} \mathbf{Z}/p^k \otimes \bigwedge^m M & \xrightarrow{\theta_k^m} & D(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \\ p \downarrow & \nearrow \theta_{k+1}^m & \\ \mathbf{Z}/p^{k+1} \otimes \bigwedge^m M & & \end{array}$$

commutes (by the proof of Lemma 1.6), we obtain a map

$$\theta^m : \mathbf{Q}_p/\mathbf{Z}_p \otimes \bigwedge^m M \rightarrow D(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$$

of $W(\kappa)$ -modules. We are now in a position to state the version of Theorem 1.1 which we shall prove.

Theorem 1.7. *Let $m > 0$. Then⁴:*

- (1) $K(\mathbf{Q}_p/\mathbf{Z}_p, m)$ is good, and $K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ is isomorphic to a power series ring over κ in $\binom{n-1}{m}$ generators.
- (2) The map θ^m described above is an isomorphism of $W(\kappa)$ -modules, and $\mathrm{Spf} K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ is a p -divisible formal group of height $\binom{n}{m}$ and dimension $\binom{n-1}{m}$.
- (3) The space $K(\mathbf{Z}/p^k, m)$ is good, and the induced map $K(n)_0(K(\mathbf{Z}/p^k, m)) \rightarrow K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ is an injection which exhibits $K(n)_0(K(\mathbf{Z}/p^k, m))$ as the kernel of the map $[p^k] : K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \rightarrow K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$.

We already proved Theorem 1.7 in the base case $m = 1$ in Proposition 1.5. We will prove Theorem 1.7 using induction on m (for the last part of Theorem 1.7, we will still induct on m , while keeping k fixed); the inductive step will be broken into three parts.

Proposition A. *Let $m > 1$, and assume that the conclusions of Theorem 1.7 are true for $K(\mathbf{Q}_p/\mathbf{Z}_p, m-1)$. Then $K(\mathbf{Q}_p/\mathbf{Z}_p, m)$ is good, and $K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ is isomorphic to a power series ring over κ in $\binom{n-1}{m}$ generators⁵.*

Proposition B. *Let $m > 1$, and assume that the conclusions of Theorem 1.7 are true for $K(\mathbf{Q}_p/\mathbf{Z}_p, m-1)$ and $K(\mathbf{Z}/p, m-1)$. Then*

⁴There are typos in [HL13]: $K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ is a power series ring in $\binom{n-1}{m}$ generators, not $\binom{n-1}{m-1}$ generators; therefore, $\mathrm{Spf} K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ is a p -divisible formal group of dimension $\binom{n-1}{m}$, not $\binom{n-1}{m-1}$.

⁵See the previous footnote.

- (1) θ^m is an isomorphism.
- (2) $\mathrm{Spf} K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ is a p -divisible formal group of height $\binom{n}{m}$ and dimension $\binom{n-1}{m}$.

Proposition C. *Let $m > 1$, and assume that the conclusions of Theorem 1.7 are true for $K(\mathbf{Q}_p/\mathbf{Z}_p, m-1)$ and $K(\mathbf{Z}/p^k, m-1)$. Then the space $K(\mathbf{Z}/p^k, m)$ is good, and the induced map $K(n)_0(K(\mathbf{Z}/p^k, m)) \rightarrow K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ is an injection which exhibits $K(n)_0(K(\mathbf{Z}/p^k, m))$ as the kernel of the map $[p^k] : K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \rightarrow K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$.*

It is clear that these three results are equivalent to Theorem 1.7, so we shall prove each of them in order. They are arranged in the order of increasingly technical proofs, so we shall prove Proposition A and Proposition B in the first lecture, and leave Proposition C to the second lecture.

2. The proof of Proposition A

In the last lecture, Andy told us about the Bousfield-Kan spectral sequence. In this case, we can use the fact that $BK(\mathbf{Q}_p/\mathbf{Z}_p, m-1) = K(\mathbf{Q}_p/\mathbf{Z}_p, m)$ to write $\mathrm{Map}(K(\mathbf{Q}_p/\mathbf{Z}_p, m), K(n))$ as the totalization $\mathrm{Tot} \underline{\mathrm{Map}}(K(\mathbf{Q}_p/\mathbf{Z}_p, m-1)^\bullet, K(n))$. It follows that

$$\underline{\mathrm{Map}}(K(\mathbf{Q}_p/\mathbf{Z}_p, m), K(n)) \simeq \lim \mathrm{Tot}^t \underline{\mathrm{Map}}(K(\mathbf{Q}_p/\mathbf{Z}_p, m-1)^\bullet, K(n)).$$

The Bousfield-Kan spectral sequence has E_2 -page given by the cohomology of the cochain complex associated to the cosimplicial graded abelian group given by $\pi_* \underline{\mathrm{Map}}(K(\mathbf{Q}_p/\mathbf{Z}_p, m-1)^\bullet, K(n))$. We showed last time that this spectral sequence runs:

$$E_2^{s,t} = \pi_t K(n) \otimes_\kappa \mathrm{Ext}_{K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m-1))}^s(\kappa, \kappa) \Rightarrow K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m)).$$

The convergence of this spectral sequence is guaranteed by the fact that it is the κ -linear dual of the Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{s,t}^{K(n)_*(K(\mathbf{Q}_p/\mathbf{Z}_p, m-1))}(\pi_* K(n), \pi_* K(n)) \Rightarrow K(n)_{t+s}(K(\mathbf{Q}_p/\mathbf{Z}_p, m)).$$

We will exploit this spectral sequence to prove Proposition A. Let us write $A = K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m-1))$, and Ext_A^s to denote $\mathrm{Ext}_A^s(\kappa, \kappa)$. Last time, Andy showed that if A is a (connected) p -divisible Hopf algebra over κ , then there is a canonical map $\psi : \mathrm{Ext}_{A[p]}^1 \rightarrow \mathrm{Ext}_A^2$, which extends to a map $\mathrm{Ext}_{A[p]}^1 \rightarrow \mathrm{Ext}_A^*$ that factors through a canonical *isomorphism* $\psi : \mathrm{Sym}^*(\mathrm{Ext}_{A[p]}^1) \rightarrow \mathrm{Ext}_A^*$ of κ -algebras. In particular, Ext_A^* is concentrated in even degrees. We conclude that $E_2^{s,t}$ vanishes unless s and t are both even, and in this case

$$E_2^{2s,2t} = \pi_t K(n) \otimes_\kappa \mathrm{Sym}^s(\mathrm{Ext}_{A[p]}^1).$$

There can be no nontrivial differentials in this spectral sequence, so $E_2^{*,*} \cong E_\infty^{*,*}$. The spectral sequence is therefore strongly convergent, and there is a decreasing filtration $F^* K(n)^{-d}(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ such that

$$K(n)^{-d}(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \cong \lim_s K(n)^{-d}(K(\mathbf{Q}_p/\mathbf{Z}_p, m))/F^s K(n)^{-d}(K(\mathbf{Q}_p/\mathbf{Z}_p, m)).$$

The associated graded is given by

$$F^s K(n)^{-d}(K(\mathbf{Q}_p/\mathbf{Z}_p, m))/F^{s+1} K(n)^{-d}(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \cong E_\infty^{s,d+s} \cong \mathrm{Ext}_A^s \otimes_\kappa \pi_{d+s} K(n).$$

Since this is zero unless s and d are both even, we conclude that $K(n)^{-d}(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ vanishes unless d is even, i.e., $K(n)^*(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ is even.

Let us now concentrate on $K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$, the degree 0 component. Let $I(s) = F^s K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$, so that we can phrase the multiplicativity of the filtration by $I(s)I(t) = I(s+t)$. Each $I(s)$ is an ideal in $K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$, and $K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \cong \lim K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))/I(s)$; endow $K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ with the inverse limit topology. Then, the associated graded of this filtration is

$$\mathrm{gr}(K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))) = \bigoplus_{s \geq 0} I(s)/I(s+1) \cong \bigoplus_{s \geq 0} \mathrm{Ext}_A^s \otimes_{\kappa} \pi_s K(n) \cong \mathrm{Sym}^*(\mathrm{Ext}_{A[p]}^1 \otimes_{\kappa} \pi_2 K(n)).$$

We now need to understand $\mathrm{Ext}_{A[p]}^1$; this is the content of the following lemma.

Lemma 2.1. *We have*

$$\dim_{\kappa} \mathrm{Ext}_{A[p]}^1 = \binom{n-1}{m}.$$

Proof. Using the cobar complex, we showed last time that if $\mathfrak{m}_{A[p]}$ denotes the augmentation ideal, then $\mathrm{Ext}_{A[p]}^1$ is canonically isomorphic to the κ -linear dual $(\mathfrak{m}_{A[p]}/\mathfrak{m}_{A[p]}^2)^{\vee} \cong (\mathrm{DM}(A[p])/F \mathrm{DM}(A[p]))^{\vee}$. In particular, we find that

$$\dim_{\kappa} \mathrm{Ext}_{A[p]}^1 = \dim_{\kappa} \mathrm{DM}(A[p])/F \mathrm{DM}(A[p]).$$

Since $[p] = VF$, there is an exact sequence of finite group schemes

$$0 \rightarrow \ker F \rightarrow \ker [p] \rightarrow \ker V \rightarrow 0.$$

The scheme $\ker [p]$ is the p -torsion in the p -divisible group $\mathrm{Spf} A^{\vee}$, so it is a finite group scheme of rank $p^{\mathrm{height}} = p^{\binom{n-1}{m}}$. Similarly, F raises each coordinate to the p th power, and is an isomorphism on the étale part; therefore, $\ker F$ is the connected portion of the p -torsion in the p -divisible group $\mathrm{Spf} A^{\vee}$. It is therefore a finite group scheme of rank $p^{\mathrm{dimension}} = p^{\binom{n-1}{m}}$. It follows that $\ker V$ must be of rank $p^{\binom{n-1}{m} - \binom{n-1}{m-1}} = p^{\binom{n-1}{m}}$. The Dieudonné module of $\ker V$ is precisely $\mathrm{DM}(A[p])/F \mathrm{DM}(A[p])$, so we conclude that $\dim_{\kappa} \mathrm{Ext}_{A[p]}^1 = \binom{n-1}{m}$, as desired. \square

The κ -vector space $\pi_2 K(n)$ is one-dimensional, so $\mathrm{Ext}_{A[p]}^1 \otimes_{\kappa} \pi_2 K(n)$ is a $\binom{n-1}{m}$ -dimensional κ -vector space. Let $\ell = \binom{n-1}{m}$, and let t_1, \dots, t_{ℓ} be a basis for $I(2)/I(3) \cong \mathrm{Ext}_{A[p]}^1 \otimes_{\kappa} \pi_2 K(n)$ as a κ -vector space, and choose elements $\bar{t}_i \in I(2) = F^2 K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ having image t_i in $I(2)/I(3)$. Then, we have a continuous homomorphism $k[[x_1, \dots, x_{\ell}]] \rightarrow K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ sending x_i to \bar{t}_i . This map is an isomorphism on associated graded, and hence is an isomorphism itself.

3. The proof of Proposition B

Statement (2) of Proposition B is a consequence of Proposition A and statement (1) of Proposition B (the only nontrivial part is deducing the statement about the height of $\mathrm{Spf} K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$, but this is just the rank of $D(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ as a $W(\kappa)$ -module, which can be deduced from statement (1)). Recall that our goal is to prove that the map $\theta^m : \mathbf{Q}_p/\mathbf{Z}_p \otimes \bigwedge^m M \rightarrow D(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ is an isomorphism.

We begin with a slight digression. There is a Verschiebung map defined on $\bigwedge^m M$ as follows. If we base change to an algebraic closure of κ , we can assume that M is generated over D_k by an element x satisfying $Fx = V^{n-1}x$ (this is because the Dieudonné module of the formal group $\mathbf{G}_0 = \mathrm{Spf} K(n)^0(\mathbf{C}P^{\infty})$ over

an algebraically closed field is $W(\kappa)/(Fx = x^\phi F, F^n = p)$). It follows that we can assume that M is generated as a $W(\kappa)$ -module by the elements $x, Vx, \dots, V^{n-1}x$ (so that $V^n x = px$). Let $I = \{i_1 < \dots < i_m\}$ be an ordered subset of $\{0, \dots, n-1\}$; then, we can define an element $V^I x \in \bigwedge^m M$ by $V^I x = V^{i_1} x \wedge \dots \wedge V^{i_m} x$. The collection of such $V^I x$ generate $\bigwedge^m M$ as a $W(\kappa)$ -module. This allows us to construct a Verschiebung map $V : \bigwedge^m M \rightarrow \bigwedge^m M$ by

$$(1) \quad V(\lambda V^I x) = \begin{cases} \phi^{-1}(\lambda) V^{i_1+1} x \wedge \dots \wedge V^{i_m+1} x & \text{if } i_m < n-1 \\ (-1)^{m-1} p \phi^{-1}(\lambda) x \wedge V^{i_1+1} x \wedge \dots \wedge V^{i_{m-1}+1} x & \text{if } i_m = n-1. \end{cases}$$

This formula is not very surprising; it's what one would expect.

One thing which is immediate from this description of the action of V is that the induced map $V_{\mathbf{Q}} : \mathbf{Q} \otimes \bigwedge^m M \rightarrow \mathbf{Q} \otimes \bigwedge^m M$ is an isomorphism (since p is now invertible). In order to understand the map V , we therefore split the map up using the following diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigwedge^m M & \longrightarrow & \mathbf{Q} \otimes \bigwedge^m M & \longrightarrow & \mathbf{Q}_p/\mathbf{Z}_p \otimes \bigwedge^m M & \longrightarrow & 0 \\ & & \downarrow V & & \downarrow V_{\mathbf{Q}} & & \downarrow V_{\mathbf{Q}_p/\mathbf{Z}_p} & & \\ 0 & \longrightarrow & \bigwedge^m M & \longrightarrow & \mathbf{Q} \otimes \bigwedge^m M & \longrightarrow & \mathbf{Q}_p/\mathbf{Z}_p \otimes \bigwedge^m M & \longrightarrow & 0. \end{array}$$

Note that $\mathbf{Q} \otimes \bigwedge^m M \cong \mathbf{Q}_p \otimes \bigwedge^m M$. The snake lemma now shows that

- The map $V_{\mathbf{Q}_p/\mathbf{Z}_p}$ is surjective.
- There is an isomorphism $\ker V_{\mathbf{Q}_p/\mathbf{Z}_p} \rightarrow \text{coker } V$.

Since $p = 0$ in $\text{coker } V$, it is a κ -vector space. It follows from Equation (1) that $\text{coker } V$ is generated by elements of the form $V^I x$ with $i_1 = 0$. In other words, $\dim_{\kappa} \text{coker } V$ is given by the number of ordered subsets $\{0 < i_2 < \dots < i_m\}$ of $\{0, \dots, n-1\}$. This number is $\binom{n-1}{m-1}$, so $\dim_{\kappa} \ker V_{\mathbf{Q}_p/\mathbf{Z}_p} = \binom{n-1}{m-1}$.

We now proceed to the proof of Proposition B. By naturality, we obtain a commutative diagram

$$\begin{array}{ccc} \mathbf{Q}_p/\mathbf{Z}_p \otimes \bigwedge^m M & \longrightarrow & D(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \\ \downarrow V_{\mathbf{Q}_p/\mathbf{Z}_p} & & \downarrow V \\ \mathbf{Q}_p/\mathbf{Z}_p \otimes \bigwedge^m M & \longrightarrow & D(K(\mathbf{Q}_p/\mathbf{Z}_p, m)). \end{array}$$

Therefore, we have a map $\ker V_{\mathbf{Q}_p/\mathbf{Z}_p} \rightarrow \ker(V : D(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \rightarrow D(K(\mathbf{Q}_p/\mathbf{Z}_p, m)))$. We claim that it suffices to prove the following lemma.

Lemma 3.1. *The map $\ker V_{\mathbf{Q}_p/\mathbf{Z}_p} \rightarrow \ker(V : D(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \rightarrow D(K(\mathbf{Q}_p/\mathbf{Z}_p, m)))$ is a surjection.*

Let us first show that Lemma 3.1 is sufficient to prove Proposition B. We first show that θ^m is an injection. By Proposition A, the κ -vector space $\ker(V : D(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \rightarrow D(K(\mathbf{Q}_p/\mathbf{Z}_p, m)))$ has dimension

$$\text{height} - \text{dimension} = \binom{n}{m} - \binom{n-1}{m} = \binom{n-1}{m-1}$$

as a κ -vector space. It follows from the above discussion that the map of Lemma 3.1 is in fact an isomorphism. In particular, $\theta^m|_{\ker V_{\mathbf{Q}_p/\mathbf{Z}_p}}$ is an injection. To show that θ^m is an injection, suppose $0 \neq z \in \mathbf{Q}_p/\mathbf{Z}_p \otimes \bigwedge^m M$. Then $V_{\mathbf{Q}_p/\mathbf{Z}_p}^N z = 0$ for

$N \gg 0$, so let N be the minimal integer for which $V_{\mathbf{Q}_p/\mathbf{Z}_p}^N z = 0$. Note that $N > 0$, since $z \neq 0$. It follows that $V_{\mathbf{Q}_p/\mathbf{Z}_p}^{N-1} z \in \ker V$, and $V_{\mathbf{Q}_p/\mathbf{Z}_p}^{N-1} z \neq 0$. Therefore,

$$V^{N-1}\theta^m(z) = \theta^m(V_{\mathbf{Q}_p/\mathbf{Z}_p}^{N-1} z) \neq 0,$$

where we used the injectivity of $\theta^m|_{\ker V_{\mathbf{Q}_p/\mathbf{Z}_p}}$. Therefore, $\theta^m(z) \neq 0$.

We now prove that θ^m is a surjection. Let $y \in D(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$. Since the formal group $\mathrm{Spf} K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ is connected, the action of V is locally nilpotent (by general properties of Dieudonné modules of connected p -divisible formal groups). In particular, there is some N such that $V^N y = 0$; again, let N be a minimal such integer. We shall show that y is in the image of θ^m by induction on N . If $N = 0$, then $y = 0$, so the claim is trivial in this case. Assume the statement is true for $N - 1$. Then $V^{N-1}y \in \ker V$, so by Lemma 3.1, we may find $z \in \mathbf{Q}_p/\mathbf{Z}_p \otimes \bigwedge^m M$ such that $\theta^m(z) = V^{N-1}y$. We proved above that $V_{\mathbf{Q}_p/\mathbf{Z}_p}$ is surjective⁶, so there is $\bar{z} \in \mathbf{Q}_p/\mathbf{Z}_p \otimes \bigwedge^m M$ such that $V_{\mathbf{Q}_p/\mathbf{Z}_p}^{N-1} \bar{z} = z$. It follows that $\theta^m(\bar{z}) - y$ is killed by V^{N-1} , since

$$V^{N-1}(\theta^m(\bar{z}) - y) = \theta^m V_{\mathbf{Q}_p/\mathbf{Z}_p}^{N-1} \bar{z} - V^{N-1}y = \theta^m z - V^{N-1}y = 0.$$

We conclude by the inductive hypothesis that the element $\theta^m(\bar{z}) - y$ is in the image of θ^m , so that y is itself in the image of θ^m , as desired.

It remains to prove Lemma 3.1; this will occupy our attention for the rest of the section. The cup product induces a map $c : K(\mathbf{Z}/p, 1) \times K(\mathbf{Z}/p, m - 1) \rightarrow K(\mathbf{Q}_p/\mathbf{Z}_p, m)$. Our assumption in Proposition B states that the map

$$\theta^{m-1} : \mathbf{Q}_p/\mathbf{Z}_p \otimes \bigwedge^{m-1} M \rightarrow D(K(\mathbf{Q}_p/\mathbf{Z}_p, m - 1))$$

is an isomorphism. Let $B = K(n)_0(K(\mathbf{Z}/p, 1))$, and $B' = K(n)_0(K(\mathbf{Z}/p, m - 1))$, so that $\mathrm{DM}(B) = M/p$ (by Proposition 1.5) and B' is the kernel of the map $[p] : D(K(\mathbf{Q}_p/\mathbf{Z}_p, m - 1)) \rightarrow D(K(\mathbf{Q}_p/\mathbf{Z}_p, m - 1))$. In particular, $\mathrm{DM}(B') = D(K(\mathbf{Q}_p/\mathbf{Z}_p, m - 1))/p$. Since θ^{m-1} is an isomorphism, we obtain an isomorphism $\mathbf{Z}/p \otimes \bigwedge^{m-1} M \rightarrow \mathrm{DM}(B')$. The map $K(n)_0(c)$ induces a map $\mu : B \otimes_{\kappa} B' \rightarrow K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ of Hopf algebras over κ , and hence a pairing

$$\lambda : \mathrm{DM}(B) \times \mathrm{DM}(B') \cong M/p \times \mathbf{Z}/p \otimes \bigwedge^{m-1} M \rightarrow D(K(\mathbf{Q}_p/\mathbf{Z}_p, m)).$$

Since $V\lambda(x, y) = \lambda(Vx, Vy)$, we conclude that λ takes

$$\ker(V : M/p \rightarrow M/p) \times \mathrm{DM}(B') \rightarrow \ker(V : D(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \rightarrow D(K(\mathbf{Q}_p/\mathbf{Z}_p, m))).$$

Moreover, if $x \in M/p$ is such that $Vx = 0$, then

$$\lambda(x, Fy) = F\lambda(Vx, y) = F\lambda(0, y) = 0,$$

so λ in fact induces a map

$$\bar{\lambda} : \ker(V : M/p \rightarrow M/p) \times \mathrm{DM}(B') / F \mathrm{DM}(B') \rightarrow \ker(V : D(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \rightarrow D(K(\mathbf{Q}_p/\mathbf{Z}_p, m))).$$

Lemma 3.1 will follow if we show that $\bar{\lambda}$ is surjective.

We shall now make another reduction. Recall the bilinear pairing $\mu : B \otimes_{\kappa} B' \rightarrow K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ described above. If $x \in B$ is primitive, and y is in the

⁶There are a lot of typos in this part of the proof in [HL13].

augmentation ideal $\mathfrak{m}_{B'}$ of B' , then it is easy to see that $\mu(x \otimes y)$ is primitive in $K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$. If $y, y' \in \mathfrak{m}_{B'}$, then

$$\mu(x \otimes yy') = \mu(x \otimes y)\mu(1 \otimes y') + \mu(1 \otimes y)\mu(x \otimes y') = 0,$$

so μ induces a map

$$\bar{\mu} : \text{Prim}(B) \otimes \mathfrak{m}_{B'}/\mathfrak{m}_{B'}^2 \rightarrow \text{Prim}(K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))).$$

We claim that it suffices to show that $\bar{\mu}$ is surjective; this is because it is shown in [HL13, Proposition 1.3.33] that there is a commutative diagram

(2)

$$\begin{array}{ccc} \ker(V : M/p \rightarrow M/p) \times \text{DM}(B')/F & \text{DM}(B') & \xrightarrow{\bar{\lambda}} \ker(V : D(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \rightarrow D(K(\mathbf{Q}_p/\mathbf{Z}_p, m))) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{Prim}(B) \otimes \mathfrak{m}_{B'}/\mathfrak{m}_{B'}^2 & \xrightarrow{\bar{\mu}} & \text{Prim}(K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))) \end{array}$$

where the vertical maps are isomorphisms⁷.

We now show that $\bar{\mu}$ is surjective. The Atiyah-Hirzebruch spectral sequence $\{E_r^{s,t}, d_r\}_{r \geq 1}$ computing $K(n)_*(K(\mathbf{Z}/p, 1))$ is isomorphic (from E_2 onwards) to the Bousfield-Kan spectral sequence associated to the simplicial space $X_\bullet = (\mathbf{Z}/p)^\bullet$ (in the manner described in the proof of Proposition A). This defines a filtration

$$0 = \text{gr}^{-1}B \subseteq \text{gr}^0B \subseteq \text{gr}^1B \subseteq \dots$$

of $B = K(n)_0(K(\mathbf{Z}/p, 1))$ with $\text{gr}^s B / \text{gr}^{s-1} B = E_\infty^{s, -s}$. The filtration is such that the map $\text{Prim}(B) \rightarrow \text{gr}^2 B / \text{gr}^1 B$ is an isomorphism (this is because, if we look at the Atiyah-Hirzebruch filtration, we find that $\text{Prim}(B)$ is generated as a κ -vector space by the dual of the element $x \in K(n)^0(K(\mathbf{Z}/p, 1))$ from the proof of Proposition 1.5).

Let $Y_\bullet = K(\mathbf{Q}_p/\mathbf{Z}_p, m-1)^\bullet$; this gives rise to the *dual* of the spectral sequence $\{E_r^{s,t}, d_r\}_{r \geq 1}$ of Proposition A. Let $F^s K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ denote the associated increasing filtration (which is dual to the one discussed in Proposition A). As above, the map

$$\begin{aligned} \text{Prim}(K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))) &\rightarrow F^2 K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \\ &\rightarrow F^2 K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) / F^1 K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \end{aligned}$$

is an isomorphism. One way to see this (this also provides another proof of the fact that the map $\text{Prim}(B) \rightarrow \text{gr}^2 B / \text{gr}^1 B$ is an isomorphism) is as follows. The filtration $\dots \subseteq I(1) \subseteq I(0) = K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) =: D$ of Proposition A (which, again, is *dual* to the filtration considered here) is a reindexing of the \mathfrak{m}_D -adic filtration, so that

$$I(s) = \begin{cases} \mathfrak{m}_D^k & \text{if } s = 2k, \\ \mathfrak{m}_D^k & \text{if } s = 2k + 1. \end{cases}$$

This shows that $I(2)/I(3) \cong \mathfrak{m}_D/\mathfrak{m}_D^2$. Dualizing, we find that

$$F^2 K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) / F^1 K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \cong (\mathfrak{m}_D/\mathfrak{m}_D^2)^\vee \cong \text{Prim}(K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))).$$

⁷Again, there is another typo here; the Hopf algebra denoted A is defined in [HL13] to be $K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m-1))$, but it should be $K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$, so that $\text{DM}(A) = \text{D}(Y) = \text{D}(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$.

Note that the spectral sequences associated to X_\bullet and Y_\bullet are exactly the Eilenberg-Moore spectral sequences associated to the fibrations $\mathbf{Z}/p \rightarrow * \rightarrow K(\mathbf{Z}/p, 1)$ and $K(\mathbf{Q}_p/\mathbf{Z}_p, m-1) \rightarrow * \rightarrow K(\mathbf{Q}_p/\mathbf{Z}_p, m)$.

The simplicial space Y_\bullet is the Čech nerve of the inclusion $* \rightarrow K(\mathbf{Q}_p/\mathbf{Z}_p, m)$. Moreover, X_\bullet is the Čech nerve of the inclusion $* \rightarrow K(\mathbf{Z}/p, 1)$. Since there is a commutative diagram

$$\begin{array}{ccc} * \times K(\mathbf{Z}/p, m-1) & \longrightarrow & * \\ \downarrow & & \downarrow \\ K(\mathbf{Z}/p, 1) \times K(\mathbf{Z}/p, m-1) & \xrightarrow{c} & K(\mathbf{Q}_p/\mathbf{Z}_p, m), \end{array}$$

we obtain a map of simplicial spaces $X_\bullet \times K(\mathbf{Z}/p, m-1) \rightarrow Y_\bullet$. This gives a map of spectral sequences, and hence, in particular, a map $\text{gr}^s B \otimes_\kappa B' \rightarrow F^s K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ of filtrations. The above considerations allow us to extend diagram (2) to a larger diagram

$$\begin{array}{ccc} \ker(V : M/p \rightarrow M/p) \times \text{DM}(B')/F \text{DM}(B') & \xrightarrow{\bar{\lambda}} & \ker(V : D(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \rightarrow D(K(\mathbf{Q}_p/\mathbf{Z}_p, m))) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{Prim}(B) \otimes_{\mathbf{m}_{B'}}/\mathbf{m}_{B'}^2 & \xrightarrow{\bar{\mu}} & \text{Prim}(K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{gr}^2 B/\text{gr}^1 B \otimes_\kappa \mathbf{m}_{B'} & \longrightarrow & F^2 K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m))/F^1 K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \\ \parallel & & \parallel \\ E_\infty^{2,-2} \otimes_\kappa \mathbf{m}_{B'} & \longrightarrow & E_\infty^{2,-2}. \end{array}$$

where all the vertical maps are again isomorphisms. It therefore suffices to show that the bottom horizontal map is surjective. Since $\{E_r^{s,t}, d_r\}_{r \geq 1}$ is the Atiyah-Hirzebruch spectral sequence computing $K(n)_*(K(\mathbf{Z}/p, 1))$, we know that $E_2^{2,-2} \cong H_2(B\mathbf{Z}/p; \pi_2 K(n))$. This consists of permanent cycles⁸, so it suffices to prove that there is a surjection $E_2^{2,-2} \otimes_\kappa \mathbf{m}_{B'} \rightarrow E_2^{2,-2}$ on the level of E_2 -pages.

Since $K(n)$ is 2-periodic, it suffices to prove that the map $E_2^{2,0} \otimes_\kappa \mathbf{m}_{B'} \rightarrow E_2^{2,0}$ is surjective. By identifying each of the terms, we find that this is a map

$$\phi : H_2(B\mathbf{Z}/p; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathbf{m}_{B'} \rightarrow \text{Tor}_2^{K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m-1))}(\kappa, \kappa).$$

⁸Let us provide a proof for the Atiyah-Hirzebruch spectral sequence computing the cohomology of $B\mathbf{Z}/p$. We have $H^*(B\mathbf{Z}/p; \mathbf{F}_p) \cong \mathbf{F}_p[x] \otimes E(y)$, with $\beta(y) = x$. We also have $P^i(x^k) = \binom{k+(p-1)i}{k} x^{k+(p-1)i}$. Recall that the Milnor Q_i 's are defined as $Q_0 = \beta$ and $Q_{i+1} = P^{p^i} Q_i - Q_i P^{p^i}$, and satisfy the property that $d_{2p^n-2}(x) = u^{p^n-1} Q_n(x)$ in the Atiyah-Hirzebruch spectral sequence for $K(n)$ -cohomology. We therefore have $Q_i(yx^k) = x^{k+p^i}$, and $Q_i(x^k) = 0$. The E_2 -page of the Atiyah-Hirzebruch spectral sequence is $\mathbf{F}_p[x, u^{\pm 1}] \otimes E(y)$. The first differential is $d_{2p^n-2}(y) = u^{p^n-1} x^{p^n}$ and $d_{2p^n-2}(x) = 0$; after this, we compute that the E_{2p^n-1} -page is $\mathbf{F}_p[x, u^{\pm 1}]/x^{p^n}$. There are no more differentials, due to sparsity. It follows that $E_{2p^n-1} = E_\infty$, and there is in fact no extension problem, so $K(n)_*(B\mathbf{Z}/p) \cong \mathbf{F}_p[x, u^{\pm 1}]/x^{p^n}$. Dualizing, we get the claimed result for the Atiyah-Hirzebruch spectral sequence for $K(n)_*(B\mathbf{Z}/p)$.

However, $H_2(B\mathbf{Z}/p; \mathbf{F}_p)$ is free over \mathbf{F}_p on one generator (namely, the generator dual to the element of $H^2(B\mathbf{Z}/p; \mathbf{F}_p)$ classifying the Bockstein $B\mathbf{Z}/p \rightarrow K(\mathbf{F}_p, 2)$, which is induced by the short exact sequence expressing \mathbf{Z}/p^2 as an extension of two copies of \mathbf{Z}/p). It follows that the map ϕ is simply a map $\mathfrak{m}_{B'} \rightarrow \mathrm{Tor}_2^{K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m-1))}(\kappa, \kappa)$.

In the previous lecture, we constructed an isomorphism $\psi : \mathrm{Ext}_{A[p]}^1(\kappa, \kappa) \rightarrow \mathrm{Ext}_A^2(\kappa, \kappa)$ (which was used to great effect in Proposition A). Dualizing this, we obtain an isomorphism $\mathrm{Tor}_2^A(\kappa, \kappa) \rightarrow \mathrm{Tor}_1^{A[p]}(\kappa, \kappa)$. Now, $B' = K(n)_0(K(\mathbf{Z}/p, m-1))$ is the kernel of the map $[p]$ on $K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m-1))$, so we have an isomorphism

$$\mathrm{Tor}_2^{K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m-1))}(\kappa, \kappa) \xrightarrow{\sim} \mathrm{Tor}_1^{B'}(\kappa, \kappa).$$

Since there is a canonical isomorphism $\mathrm{Ext}_{A[p]}^1 \cong \mathfrak{m}_{A[p]}/\mathfrak{m}_{A[p]}^2$ (by a cobar complex computation, which we discussed last time), we find that there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{m}_{B'} & \xrightarrow{\psi} & \mathrm{Tor}_2^{K(n)_0(K(\mathbf{Q}_p/\mathbf{Z}_p, m-1))}(\kappa, \kappa) \\ \downarrow & & \downarrow \sim \\ \mathfrak{m}_{B'}/\mathfrak{m}_{B'}^2 & \xrightarrow{\sim} & \mathrm{Tor}_1^{B'}(\kappa, \kappa). \end{array}$$

It follows that the map ψ is surjective, as desired.

4. The proof of Proposition C

We now discuss the proof of Proposition C. We will, however, omit a bunch of details, since this is meant to be a seminar talk, and going through this proof in its entirety will no doubt be unentertaining.

Let $G' = K(\mathbf{Z}_p/p^k, m-1)$, $G = K(\mathbf{Q}_p/p^k\mathbf{Z}_p, m-1)$, and $G'' = K(\mathbf{Q}_p/\mathbf{Z}_p, m-1)$, so that there is a fiber sequence $G' \rightarrow G \xrightarrow{p^k} G''$ of topological abelian groups. Let $A' = K(n)_0G'$, and let $A = K(n)_0G \cong K(n)_0G''$ be the associated Hopf algebras, so there is a short exact sequence $\kappa \rightarrow A' \rightarrow A \xrightarrow{[p^k]} A \rightarrow \kappa$ of Hopf algebras over κ .

Let us identify $K(\mathbf{Z}/p^k, m)$ with BG' , so that we have the bar/Bousfield-Kan spectral sequence converging to $K(n)_0K(\mathbf{Z}/p^k, m)$ whose E_2 -page is given by $E_2^{s,t} = \mathrm{Ext}_{A'}^s \otimes_{\kappa} \pi_t K(n)$. Define $R = K(n)_0K(\mathbf{Q}_p/\mathbf{Z}_p, m)$, so that the proof of Proposition A (see the discussion in the proof of Proposition B) gives a canonical isomorphism $\mathfrak{m}_R^s/\mathfrak{m}_R^{s+1} \cong \mathrm{Ext}_A^{2s} \otimes_{\kappa} \pi_{2s} K(n)$, where we recall that \mathfrak{m}_R is the augmentation ideal of R . Recall also that Andy constructed a map $\psi : \mathrm{Ext}_{A'}^1 \rightarrow \mathrm{Ext}_A^2$.

Fix a nonzero (necessarily invertible) element $u \in \pi_2 K(n)$. Then, we have a map

$$\Phi : \mathfrak{m}_R^s/\mathfrak{m}_R^{s+1} \cong \mathrm{Ext}_A^{2s} \otimes_{\kappa} \pi_{2s} K(n) \rightarrow \mathrm{Ext}_{A'}^{2s} \otimes_{\kappa} \pi_{2s} K(n) = E_2^{2s, 2s} \xrightarrow{\cdot u^{-1}} E_2^{2s, 2(s-1)}.$$

We claim that Proposition C is a consequence of the following technical lemma.

Lemma 4.1. *Let $x \in E_2^{1,0} = \mathrm{Ext}_{A'}^1 \otimes_{\kappa} \pi_0 K(n) = \mathrm{Ext}_{A'}^1$, and suppose $y \in \mathfrak{m}_R$ is a lift of the element $\psi(x) \otimes u \in \mathrm{Ext}_A^2 \otimes_{\kappa} \pi_2 K(n) \cong \mathfrak{m}_R/\mathfrak{m}_R^2$. Assume that the map $[p^k] : R \rightarrow R$ takes y to an element $y' \in \mathfrak{m}_R^s$, and let $x' = \Phi(y')$. Then both x and x' survive to E_{2s-1} . Moreover, $d_{2s-1}(x_{2s-1}) = u^{-1}x'_{2s-1}$.*

Let us first show how to deduce Proposition C from Lemma 4.1. There are two cases, namely when $m = n$ and $m \neq n$. We'll only discuss the case when $m = n$; in this case, we know from Proposition B that $\mathrm{Spf} R$ is a 1-dimensional p -divisible group of height 1 over κ . The argument is similar in the case when $m \neq n$, but more complicated, since $\mathrm{Spf} R$ will no longer be 1-dimensional or of height 1. We will indicate the changes in the case when $m \neq n$ while we go through the proof when $m = n$, but the reader should keep in mind that we are sweeping a ton of details under the rug.

There is only one 1-dimensional p -divisible group of height 1 (namely, $\widehat{\mathbf{G}}_m$) over an algebraically closed field, so by replacing κ with its algebraic closure if necessary we can assume that $\mathrm{Spf} R$ is $\widehat{\mathbf{G}}_m$. This means that there is an isomorphism $R \cong \kappa[[y]]$ (which is equivalent to choosing a coordinate) such that $[p^k](y) = y^{p^k}$ (so this is the choice of a p -typical coordinate). In particular, $y \in \mathfrak{m}_R$. The image \bar{y} of y in $\mathfrak{m}_R/\mathfrak{m}_R^2 = \mathrm{Ext}_A^2 \otimes_{\kappa} \pi_2 K(n)$ is of the form $\psi(x) \otimes u$, for some unique element $x \in E_2^{1,0} = \mathrm{Ext}_{A'}^1$. By Lemma 4.1, the element x survives to E_{2p^k-1} , and $d_{2p^k-1}(x) = u^{-1}\bar{y}^{p^k}$.

Our goal is to now understand the E_{∞} -page of our bar spectral sequence. To do so, we will construct another spectral sequence $\{E'_r{}^{s,t}, d'_r\}$ which collapses, and an isomorphism of spectral sequences $\{E'_r{}^{s,t}, d'_r\} \rightarrow \{E_r{}^{s,t}, d_r\}$. The spectral sequence $\{E'_r{}^{s,t}, d'_r\}$ is defined as follows. Let $E_r{}^{*,*} = (\pi_* K(n))[Y]\{1, X\}$ (the free $(\pi_* K(n))[Y]$ -module on generators $1, X$) for $r \leq 2p^k - 1$, where $|Y| = (s, t) = (2, 2)$ and $|X| = (s, t) = (1, 0)$. The only nontrivial differential is d'_{2p^k-1} , given by

$$d'_{2p^k-1}(1) = 0, \quad d'_{2p^k-1}(X) = -u^{-1}Y^{p^k}.$$

This defines $E_r{}^{*,*}$; it is isomorphic to $((\pi_* K(n))[Y]/Y^{p^k})\{X\}$.

Lemma 4.1 gives a map $\{E'_r{}^{s,t}, d'_r\} \rightarrow \{E_r{}^{s,t}, d_r\}$ of spectral sequences, which sends $Y^i \mapsto \bar{y}^i$ and $XY^j \mapsto x\bar{y}^j$. It also shows that this map is an isomorphism of $(2p^k - 1)$ -pages, so it is an isomorphism of spectral sequence. Since $\{E'_r{}^{s,t}, d'_r\}$ collapsed (by construction), we conclude that

$$E_{\infty}{}^{*,*} \cong E'_{\infty}{}^{*,*} = E'_{2p^k}{}^{*,*} \cong (\pi_* K(n))[\bar{y}]/\bar{y}^{p^k}.$$

In the case when $m \neq n$, the situation is not so simple. Instead, there is a finite collection $\{E'_r{}^{s,t}(I), d_r\}$ of collapsing spectral sequence, indexed by the set S of all subsets of $\{0, \dots, n-1\}$ of size m which contain $n-1$. By carefully defining elements y_I analogous to the element y defined above, one obtains an isomorphism

$$\bigotimes_S \{E'_r{}^{s,t}(I), d_r\} \rightarrow \{E_r{}^{s,t}, d_r\},$$

which gives an isomorphism

$$E_{\infty}{}^{s,t} \cong \bigotimes_S (\pi_* K(n))[\bar{y}_I]/(\bar{y}_I^{e_I^{(k)}}),$$

where each \bar{y}_I has bidegree $(2, 2)$ and $e_I^{(k)}$ is some number defined from I and k which we will not define here.

Returning to the case when $m = n$, we find that the spectral sequence is therefore strongly convergent, so there is a decreasing complete filtration $F^s K(n)^i(K(\mathbf{Z}/p^k, m)) \subseteq$

$K(n)^i(K(\mathbf{Z}/p^k, m))$ such that

$$F^s K(n)^i(K(\mathbf{Z}/p^k, m))/F^{s+1} K(n)^i(K(\mathbf{Z}/p^k, m)) \cong E_\infty^{s, s-i}.$$

The description of the E_∞ -page above shows that:

- (1) there is an isomorphism $\mathrm{gr}^* K(n)^0(K(\mathbf{Z}/p^k, m)) \cong \kappa[\bar{y}]/\bar{y}^{p^k}$ of κ -algebras, and
- (2) $K(\mathbf{Z}/p^k, m)$ is even since $E_\infty^{s, t} = 0$ unless s and t are both even.

In the case when $m \neq n$, we instead have an isomorphism

$$\mathrm{gr}^* K(n)^0(K(\mathbf{Z}/p^k, m)) \cong \bigotimes_S \kappa[\bar{y}_I]/\bar{y}_I^{p^{e_I}{}^{(k)}}$$

of κ -algebras. We also find that $K(\mathbf{Z}/p^k, m)$ is even.

Multiplication by p^k is nullhomotopic on $K(\mathbf{Z}/p^k, m)$, so the map $R \cong \kappa[[y]] = K(n)^0(K(\mathbf{Q}_p/\mathbf{Z}_p, m)) \rightarrow K(n)^0(K(\mathbf{Z}/p^k, m))$ kills $[p^k](y) = y^{p^k}$, so we have a map $R/y^{p^k} \rightarrow K(n)^0(K(\mathbf{Z}/p^k, m))$ of κ -algebras. It suffices to check that this map is an isomorphism on associated graded, but this follows from observation (1) above. The same proof works when $m \neq n$ to conclude that the map $\ker([p^k] : R \rightarrow R) \rightarrow K(n)^0(K(\mathbf{Z}/p^k, m))$ is an isomorphism. This concludes the proof of Proposition C, modulo Lemma 4.1.

We now address the proof of Lemma 4.1. In order to do so, we will need to go back to the definition of the bar spectral sequence. Recall that if $X : N(\mathbf{Z})^{op} \rightarrow \mathrm{Sp}$ is a filtered spectrum, we can define $X(\infty) = \lim X$. The r th page of the spectral sequence $\{E_r^{s, t}, d_r\}$ converging to $\pi_* X(\infty)$ can be defined as follows. Let $X(n, m)$ be the fiber of the map $X(n) \rightarrow X(m)$ for $n \geq m$. Then $E_r^{s, t}$ is the image of the map $\pi_{t-s} X(s+r-1, s-1) \rightarrow \pi_{t-s} X(s, s-r)$. If $X(j) = 0$ for $j < 0$, then we can therefore identify $E_r^{s, t}$ with the image of the map $\pi_{t-s} X(s+r-1, s-1) \rightarrow \pi_{t-s} X(s)$ for $r > s$. The filtration on $\pi_n X(\infty)$ is given by $F^s \pi_n X(\infty) = \ker(\pi_n X(\infty) \rightarrow \pi_n X(s))$, so that $F^s \pi_n X(\infty)/F^{s+1} \pi_n X(\infty) \cong E_\infty^{s, n+s}$.

Let us define three filtered spectra as follows. Let $G = K(\mathbf{Q}_p/\mathbf{Z}_p, m-1)$, $G'' = K(\mathbf{Q}_p/p^k \mathbf{Z}_p, m-1)$, and $G' = K(\mathbf{Z}/p^k, m-1)$. Let Y_\bullet be the simplicial space defined by the bar construction of G , and similarly for Y'_\bullet and Y''_\bullet . Set $X(s) = \mathrm{Tot}^s K(n)^{Y_\bullet}$ and $X'(s) = \mathrm{Tot}^s K(n)^{Y'_\bullet}$. Let W denote the constant filtered spectrum with value $K(n)$, so there is a commutative diagram of filtered spectra

$$\begin{array}{ccc} X'' & \longrightarrow & X \\ \downarrow & & \downarrow \\ W & \longrightarrow & X'. \end{array}$$

Note that $X(\infty) = K(n)^{K(\mathbf{Q}_p/\mathbf{Z}_p, m-1)}$, and similarly for $X'(\infty)$ and $X''(\infty)$.

We found in the proof of Proposition A that there are isomorphisms

$$\mathrm{Ext}_A^2 \otimes_\kappa \pi_0 K(n) \cong E_\infty^{2, 0}(X) = \mathrm{im}(\pi_{-2} X(\infty, 0) \rightarrow \pi_{-2} X(2, 0)).$$

Let $x \in E_2^{1, 0} = E_2^{1, 0}(X')$ be as in Lemma 4.1, so we have an element in $\pi_{-2} X(2, 0)$ corresponding to the element $\psi(x) \in \mathrm{Ext}_A^2$. We will denote this element by $\psi(x)$ as well. Similarly, we have

$$E_2^{1, 0}(X') = \mathrm{im}(\pi_{-1} X'(2, 0) \rightarrow \pi_{-1} X'(1, 0)),$$

so we get an element $x \in \pi_{-1}X'(1, 0)$. We claim that Lemma 4.1 is a consequence of the following assertion.

(*) There is an element $z \in \pi_{-2} \text{fib}(X''(2, 0) \rightarrow X(2, 0))$ whose image in $\pi_{-2}X(2, 0)$ coincides with $\psi(x)$. Moreover, the image under the composite $\pi_{-2} \text{fib}(X''(2, 0) \rightarrow X(2, 0)) \rightarrow \pi_{-2} \text{fib}(W(2, 0) = * \rightarrow X'(2, 0)) \cong \pi_{-1}X'(2, 0) \rightarrow \pi_{-1}X'(1, 0)$ is x .

Let us prove Lemma 4.1 assuming (*). Since $E_r^{s,t}$ can be identified with the image of the map $\pi_{t-s}X'(s+r-1, s-1) \rightarrow \pi_{t-s}X'(s)$ for $r > s$, we need to show that the element

$$x \in E_2^{1,0}(X') = \text{im}(\pi_{-1}X'(2, 0) \rightarrow \pi_{-1}X'(1))$$

lies in the image of the map $\pi_{-1}X'(2s-1, 0) \rightarrow \pi_{-1}X'(1)$. We shall construct such an element in $\pi_{-1}X'(2s-1, 0)$.

Let $y \in \mathfrak{m}_R = \pi_0X''(\infty, 0)$ represent $u\psi(x)$. Then using the composite

$$X''(\infty, 0) \rightarrow X''(2, 0) \rightarrow X(2, 0),$$

we find that the map

$$\pi_{-2} \text{fib}(X''(\infty, 0) \rightarrow X(2, 0)) \rightarrow \pi_{-2}X''(\infty, 0) \times_{\pi_{-2}X''(2, 0)} \pi_{-2} \text{fib}(X''(2, 0) \rightarrow X(2, 0))$$

is surjective. In particular, there is an element $\bar{z} \in \pi_{-2} \text{fib}(X''(\infty, 0) \rightarrow X(2, 0))$ such that its image in $\pi_{-2} \text{fib}(X''(2, 0) \rightarrow X(2, 0))$ is the element z from (*) and its image in $\pi_{-2}X''(\infty, 0)$ is $u^{-1}y = \psi(x)$ (or, more precisely, the element in $\pi_{-2}X''(\infty, 0)$ mapping to $\psi(x) \in \pi_{-2}X''(2, 0)$).

The image of $u^{-1}y$ under the map $\pi_{-2}X''(\infty, 0) \rightarrow \pi_{-2}X(\infty, 0)$ lifts to an element $\bar{y} \in \pi_{-2}X(\infty, 2s-1)$, since we assumed that $[p^k](y) \in \mathfrak{m}_R^s$. There is a map

$$\text{fib}(X''(\infty, 0) \rightarrow X(2, 0)) \rightarrow \text{fib}(X(\infty, 0) \rightarrow X(2, 0)) = X(\infty, 2)$$

which sends \bar{z} to an element $\bar{z}' \in \pi_{-2}X(\infty, 2)$. The image of \bar{y} under the map $X(\infty, 2s-1) \rightarrow X(\infty, 2)$ gives another element $\bar{y}' \in \pi_{-2}X(\infty, 2)$. By construction, the image of \bar{y} in $\pi_{-2}X(\infty, 0)$ is $u^{-1}y$, and the discussion from the previous paragraph guarantees that the image of \bar{z} in $\pi_{-2}X(\infty, 0)$ is also $u^{-1}y$.

We will now establish that they have the same image in $\pi_{-2}X(\infty, 1)$. It follows from the long exact sequence

$$\cdots \rightarrow \pi_{-1}X(2, 0) \rightarrow \pi_{-2} \text{fib}(X(\infty, 0) \rightarrow X(2, 0)) = \pi_{-2}X(\infty, 2) \rightarrow \pi_{-2}X(\infty, 0) \rightarrow \cdots$$

associated to the fiber sequence

$$X(\infty, 2) \rightarrow X(\infty, 0) \rightarrow X(2, 0)$$

that $\bar{y}' - \bar{z}'$ is in the image of the map $\pi_{-1}X(2, 0) \rightarrow \pi_{-2}X(\infty, 2)$. There is a commutative diagram

$$\begin{array}{ccc} \pi_{-1}X(2, 0) & \longrightarrow & \pi_{-2}X(\infty, 2) \\ \downarrow & & \downarrow \\ \pi_{-1}X(1, 0) & \longrightarrow & \pi_{-2}X(\infty, 1) \\ \downarrow & & \downarrow \\ * = \pi_{-1}X(0, 0) & \longrightarrow & \pi_{-2}X(\infty, 0), \end{array}$$

so to show that \bar{y} and \bar{z} have the same image in $\pi_{-2}X(\infty, 1)$, it suffices to show that the map $\pi_{-1}X(2, 0) \rightarrow \pi_{-1}X(1, 0)$ is zero. But the image of this map is $E_2^{1,0}(X)$, which we saw in Proposition A vanishes.

We now use the above observation to construct an element $\bar{x} \in \pi_{-1}X'(2s-1, 0)$ whose image under the map $\pi_{-1}X'(2s-1, 0) \rightarrow \pi_{-1}X'(1)$ corresponds to an element of $E_{2s-1}^{1,0}$ which maps to $x \in E_2^{1,0}$. Consider the following pullback diagram

$$\begin{array}{ccc} F & \longrightarrow & \text{fib}(X(\infty, 0) \rightarrow X(2s-1)) \simeq X(\infty, 2s-1) \\ \downarrow & & \downarrow \\ \text{fib}(X''(\infty, 0) \rightarrow X(1, 0)) & \longrightarrow & \text{fib}(X(\infty, 0) \rightarrow X(1, 0)) = X(\infty, 1). \end{array}$$

Since we established above that $\bar{y} \in \pi_{-2}X(\infty, 2s-1)$ and $\bar{z} \in \pi_{-2}\text{fib}(X''(\infty, 0) \rightarrow X(1, 0))$ have the same image in $\pi_{-2}X(\infty, 1)$, we obtain an element $w \in \pi_{-2}F$. There is a map

$$\pi_{-2}F \cong \pi_{-2}(X''(\infty) \times_{X(\infty)} X(\infty, 2s-1)) \rightarrow \pi_{-2}(W(\infty) \times_{X'(\infty)} X'(\infty, 2s-1)) \cong \pi_{-1}X'(2s-1, 0),$$

so we have an element $\bar{x} \in \pi_{-1}X'(2s-1, 0)$. This element satisfies the desired property, so we find that $x \in E_2^{1,0}(X')$ survives to the $(2s-1)$ st page.

In order to finish the proof of Lemma 4.1, we also need a description of the element $d_{2s-1}(x)$. But this is the image of the element \bar{x} under either of the composites in the following commuting diagram:

$$\begin{array}{ccc} \pi_{-1}X'(2s-1, 0) & \longrightarrow & \pi_{-2}\text{fib}(X'(2s, 0) \rightarrow X'(2s-1, 0)) \cong \pi_{-2}X'(2s, 2s-1) \\ \downarrow \cong & & \uparrow \\ \pi_{-2}(X''(\infty) \times_{X(\infty)} X(\infty, 2s-1)) & \xrightarrow{p^k} \longrightarrow & \pi_{-2}X(\infty, 2s-1) \longrightarrow \pi_{-2}X(2s, 2s-1) \end{array}$$

From the description as the longer composite, we find that $d_{2s-1}(x)$ is indeed as claimed in Lemma 4.1.

In order to truly finish the proof of Lemma 4.1, we will also need to prove statement (*). Unfortunately, we will not do this here; the notes are long enough as is, and I will definitely be unable to cover all of this material in two talks anyway.

References

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