JUVITOP 9/19: HOPF ALGEBRAS AND WITT VECTORS

Contents

Disclaimer: these notes include nothing original, and follow [HL] 1.1 and 1.2 very closely. My discussion is a bit less formal, and details and proofs are omitted, so the one advantage might be getting to read the main results more quickly. The odd spacing and color coding was intended to make it easier for me to follow the notes during my talk. If you're reading, the idea is roughly (1) bold is a guiding point or key idea that I wanted to be sure to say; (2) red is something that can be omitted in a rush.

§1. Quick recap from last week: context

Recall that the goal this semester is to follow Hopkins – Lurie to prove that, in the ∞-category of $K(n)$-local spectrum, limits and colimits indexed by π-finite Kan complexes coincide. In order to do this, Hopkins and Lurie construct a norm map from the colimit to the limit (generalizing the example from homotopy orbits to homotopy fixed points that we talked about last week), and show that it is an equivalence.

The latter part can be reduced to a calculation for diagrams constant at $K(n)$ and of shape $K(\mathbb{Z}/p\mathbb{Z}, m)$. Vaguely, proving the norm map is an equivalence is then reduced to an explicit calculation of a map between $K(n)$-homology groups, which can be analyzed using Dieudonne theory.

Our first goal today will be to construct a symmetric monoidal product $\boxtimes$ on the category of Hopf algebras over a fixed commutative ring k.

The main result of chapter 1 is then to prove a universal property of the Dieudonne module associated to the product of Hopf algebras (which determines the product since the functor DM to be defined next time is fully faithful and compatible with $\boxtimes$ in an appropriate sense):

1.1. Theorem (Goerss, Buchstaber–Lazarev; see [HL], 1.3). *Let H and H' be connected Hopf algebras over k. Then the Diuedonne module $DM(H \boxtimes H')$ is characterized by the following universal property: for any left D_k-module M, there is a bijective correspondence between D_k-module maps $DM(H \boxtimes H') \to M$ and $W(k)$-bilinear maps $\lambda\colon DM(H) \times DM(H') \to M$ such that*

$$V\lambda(x, y) = \lambda(Vx, Vy),$$

$$F\lambda(Vx, y) = \lambda(x, Fy),$$

$$F\lambda(x, Vy) = \lambda(Fx, y).$$

In order to do this, **we develop some results on Witt vectors, which is our second objective for today**.

Dexter will talk about Dieudonne modules and the proof of the main theorem of Chapter 1 (above) next week; today is about setting up algebraic foundations.

§2. Categories of Hopf algebras

The goal in this section is to construct a product $\boxtimes$ making the category $\mathbf{Hopf_k}$ of (commutative, cocommutative) Hopf algebras over k symmetric monoidal.

The strategy of [HL]1.1 is to construct a symmetric monoidal product on a larger category $\mathbf{BiAlg_k}$. Then one shows that the full subcategory $\mathbf{Hopf_k}$ closed under the product constructed.

First, some facts about the category $\mathbf{CoAlg_k}$ of coalgebra objects in k-modules:

2.1. Proposition ([HL]1.1.3). *$\mathbf{CoAlg_k}$ is locally presentable: it admits small colimits, and has a τ-compact generating set for some regular cardinal τ (meaning that for every element X in the generating set, the representable functor $\mathbf{CoAlg_k(X, -)}$ preserves τ-filtered colimits.*

This implies that $\mathbf{CoAlg_k}$ admits small limits and colimits.

2.2. Proposition ([HL] 1.1.9). *The Yoneda embedding*

$$h\colon \mathbf{CoAlg_k} \to \mathrm{Fun}(\mathbf{CoAlg_k^{op}}, \mathbf{Set})$$

admits a left adjoint L which commutes with finite products.

As mentioned previously, the goal is to study bialgebras which contain Hopf algebras as a subcategory. These are obtained from $\mathbf{CoAlg_k}$ by taking monoid objects: we define $\mathbf{BiAlg_k}$ be the set of commutative monoid objects in $\mathbf{CoAlg_k}$. More generally, given any category $\mathcal{C}$ with finite products, we get a natural association $\mathcal{C} \mapsto \mathrm{CMon}(\mathcal{C})$. Abusing notation slightly, let CMon be the category of objects in **Set**.

Applying CMon to the adjunction $L \dashv h$, we get:

$$\mathrm{CMon}(\mathrm{Fun}(\mathbf{CoAlg_k^{op}}, \mathbf{Set})) \underset{\mathrm{CMon}(L)}{\overset{\mathrm{CMon}(h)}{\rightleftarrows}} \mathbf{BiAlg_k}$$

Note that since both h and L preserve products, they take monoid objects in one category to monoid objects in the other. Morever a morphism in either category is a monoid map if and only if its transpose is, and hence $\mathrm{CMon}(L) \dashv \mathrm{CMon}(h)$.

How does this help us put a symmetric monoidal structure on $\mathbf{BiAlg_k}$? Observe first the left hand side of the above equation is equivalent to $\mathrm{Fun}(\mathbf{CoAlg_k}, \mathrm{CMon})$. Following [HL], we'll use the following notation for the previous adjunction rewritten:

$$\mathrm{Fun}(\mathbf{CoAlg_k^{op}}, \mathrm{CMon}) \underset{L}{\overset{h}{\rightleftarrows}} \mathbf{BiAlg_k}$$

where $L \dashv h$.

First, we will describe a symmetric monoidal structure on $\mathrm{Fun}(\mathbf{CoAlg_k}, \mathrm{CMon})$ which arises from one on CMon object-wise.

CMon **has a symmetric monoidal structure given by "tensor product of monoids"**: the tensor product of two monoids M, M' is the universal monoid amongst those receiving a bilinear map from $M \times M'$. Here, bilinearity means that the map λ is additive in each component and takes an element with either component zero to the zero element of the target:

$$\lambda(x + x', y) = \lambda(x, y) + \lambda(x', y), \lambda(x, y + y') = \lambda(x, y) + \lambda(x, y') \quad (1)$$

$$\lambda(0, y) = 0 = \lambda(x, 0) \quad (2)$$

Then the goal is to pass the product on $\mathrm{Fun}(\mathbf{CoAlg_k}, \mathrm{CMon})$ along the adjunction $h \vdash L$ in order to endow $\mathbf{BiAlg_k}$ with a symmetric monoidal structure. A general fact:

2.3. Lemma. *Let $R \vdash L$ with $R\colon \mathcal{C} \to \mathcal{D}$ and $(\mathcal{C}, \otimes)$ symmetric monoidal. Suppose that for any isomorphism $h\colon C \to C'$ and any object C'' in $\mathcal{C}$, the induced map $L(C \otimes C') \to L(C)$ is an isomorphism. Then $\mathcal{D}$ has a symmetric monoidal structure given by $(D, D') \mapsto L(R(D) \otimes (D'))$ which makes L symmetric monoidal. This structure is unique up to a canonical isomorphism.*

We need only check that $L \dashv h$ as defined above satisfies the lemma to obtain:

2.4. Theorem. $\mathbf{BiAlg_k}$ *has a symmetric monoidal product* $\boxtimes\colon \mathbf{BiAlg_k} \times \mathbf{BiAlg_k} \to \mathbf{BiAlg_k}$ *making L symmetric monoidal. $\boxtimes$ is unique as such, up to canonical isomorphism.*

Proof. Let L be the left adjoint to yoneda, and let $\alpha\colon F \to F'$ be a morphism in $\mathcal{C} := \mathrm{Fun}(\mathbf{CoAlg_k^{op}}, \mathrm{CMon})$.

Suppose that the induced map $\beta\colon L(F) \to L(F')$ is an isomoprhism of bialgebras (and hence of coalgebras, too), and let G be arbitrary in $\mathcal{C}$.

We must show that β induces a bijection

$$\theta\colon\ \mathrm{Fun}_{\mathbf{BiAlg_k}}(L(F' \otimes G), H) \to \mathrm{Fun}_{\mathbf{BiAlg_k}}(L(F \otimes G), H),$$

for H arbitrary in $\mathbf{BiAlg_k}$.

Transposing, this corresponds to

$$\theta'\colon\ \mathrm{Fun}_{\mathcal{C}}(F' \otimes G, h_H) \to \mathrm{Fun}_{\mathcal{C}}(F \otimes G, h_H).$$

The domain and codomain of θ' are identified with bilinear maps $F' \times G \to h_H$ and $F \times G \to h_H$, respectively. Since L commutes with finite products, both sides are identified with the same subset of

$$\mathrm{Fun}_{\mathbf{CoAlg_k}}(L(F') \otimes_k L(G), H) \simeq \mathrm{Fun}_{\mathbf{CoAlg_k}}(L(F) \otimes_k L(G), H).$$

□

Now let's try to understand what $\boxtimes$ does.

As we'll see, it's very different from the tensor product of underlying modules. Consider $H' \boxtimes H \to H''$ in $\mathbf{BiAlg_k}$. By definition, this is a map in $\mathbf{BiAlg_k}$ from $L(h_{H'} \otimes h_H)$ to H''. Transposing by the adjunction, we obtain

$$h_{H'} \otimes h_H \Rightarrow h_{H'}$$

which is defined via the properties of $\otimes$ on $\mathrm{Fun}(\mathbf{CoAlg_k^{op}}, \mathrm{CMon})$. By yoneda, this translates into a map $H \otimes_k H' \to H''$ satisfying concrete relations.

Conclusion: $H' \boxtimes H$ is a quotient of the symmetric algebra on $H \otimes_k H$.

2.5. Remark. *All the maps in the diagram (which is commutative up to canonical isomorphism)*

$$\begin{array}{ccc} \mathbf{BiAlg_k} & \xrightarrow{h} & \mathrm{Fun}(\mathbf{CoAlg_k^{op}}, \mathrm{CMon}) \\ \downarrow & & \downarrow \alpha \\ \mathbf{CoAlg_k} & \xrightarrow{h} & \mathrm{Fun}(\mathbf{CoAlg_k^{op}}, \mathbf{Set}) \end{array}$$

admit left adjoints. The adjoint the the forgetful functor $\mathbf{BiAlg_k} \to \mathbf{CoAlg_k}$ *is given by* $C \mapsto \mathrm{Sym}^*(C)$*, while the adjoint to* α *is given by pointwise compositive with the free monoid functor* $\mathbb{Z}_{\geq 0}[-]$*. Therefore the diagram of left adjoints also commutes up to isomorphism. Since* $\mathbb{Z}_{\geq 0}[-]$ *is symmetric monoidal (where* **Set** *is considered with the cartesian product), we get that for* $C, D \in \mathbf{CoAlg_k}$*,*

$$\boxed{\mathrm{Sym}^*(C \otimes_k D) \simeq \mathrm{Sym}^* C \boxtimes \mathrm{Sym}^* D.}$$

There's also a "reduced" version of this, where $\mathbf{CoAlg_k}$ is replaced with augmented coalgebras, **Set** is replaced with $\mathbf{Set}_*$ – see end of 1.1. It is (apparently) useful for computations involving Hopf algebras with augmentations. The upshot:

$$\boxed{\mathrm{Sym}^*_{red}(C) \boxtimes \mathrm{Sym}^*_{red}(D) \simeq \mathrm{Sym}^*_{red}(C \wedge D).}$$

This is computationally useful:

2.6. Example. $k[x]$ *with* $x \mapsto 1 \otimes x + x \otimes 1$*. Note that* $k[x] \wedge k[x] \simeq k[x]$ *(induced by* $x \mapsto x \otimes x$*. Moreover,* $k[x] = \mathrm{Sym}^*_{red}(C)$ *where* C *is the subcoalgebra of* $k[x]$ *generated by* 1 *and* x*. Thus deduce:* $k[x] \boxtimes k[x] \simeq k[x]$*.*

2.7. Example. k *as a* k*-coalgebra is the unit with respect to* $\otimes_k$ *on* $\mathbf{CoAlg_k}$*. Since* Sym^* *is symmetric monoidal, it follows that* $\mathrm{Sym}^* k \simeq k[x]$ *is the unit for* $\boxtimes$ *on* $\mathbf{BiAlg_k}$*.*

A Hopf algebra is an abelian group object in the category of bialgebras. Equivalently, Hopkins – Lurie's definition:

2.8. Lemma–Definition. *Note that* $\mathbb{Z}_{\geq 0} \hookrightarrow \mathbb{Z}$ *induces an in* CMon $\mathbb{Z} \simeq \mathbb{Z}_{\geq 0} \otimes Z \to \mathbb{Z} \otimes \mathbb{Z}$*. Hence* $\mathbb{Z}$ *is idempotent as a monoid, and the category* $\mathrm{Mod}_{\mathbb{Z}}(\mathrm{CMon})$ *is a full subcategory of* CMon*. Moreover, admitting a* $\mathbb{Z}$*-module structure in* CMon *is equivalent to being an abelian group. Thus,* Ab *inherits a symmetric monoidal structure from* CMon *with the same product, but with different monoidal unit.*

$H \in \mathbf{BiAlg_k}$ *is a* **Hopf algebra** *if the functor* $h_H \colon \mathbf{CoAlg_k^{op}} \to$ CMon *factors through the full subcategory* Ab $\subset$ CMon*. We denote the full subcategory of* $\mathbf{BiAlg_k}$ *spanned by such* H *by* $\mathbf{Hopf_k}$*.*

Our next goal is to show that Hopf algebras inherit a symmetric monoidal structure from $\boxtimes$. To do this, we try to understand Hopf algebras.

We can characterize Hopf algebras handily: let $\underline{\mathbb{Z}}$ be the constant functor $\mathbf{CoAlg}_{\mathbf{k}}^{\mathrm{op}} \to \mathrm{Ab}$ with value $\mathbb{Z}$, and L as before be the left adjoint to yoneda.

Then $L(\underline{\mathbb{Z}})$ can be identified with the ring of Laurent polynomial $k[\mathbb{Z}] = k[t^{\pm 1}]$, with comultiplication $t \mapsto t \otimes t$. L is symmetric monoidal, so $k[\mathbb{Z}]$ is idempotent in $\mathbf{BiAlg}_{\mathbf{k}}$ wrt $\boxtimes$. Moreover, $k[\mathbb{Z}]$ is a Hopf algebra: the set of coalgebra maps from any coalgebra to $k[\mathbb{Z}]$ are multiplicative invertible: the inverse of a given map $X \to k[\mathbb{Z}]$ is given by postcomposing with a bialgebra automorphism of $k[\mathbb{Z}]$.

By definition, we have that for any $H \in \mathbf{BiAlg}_{\mathbf{k}}$: $h_{H \boxtimes k[\mathbb{Z}]} \simeq h_H \otimes h_{k[\mathbb{Z}]}$. Since the latter takes values in Ab, we see that $H \boxtimes k[\mathbb{Z}]$ is a Hopf algebra for any $H \in \mathbf{BiAlg}_{\mathbf{k}}$. If H is already a Hopf algebra, then $H \boxtimes k[\mathbb{Z}] \simeq L(h_H) \simeq L(h_H \otimes \underline{\mathbb{Z}}) \simeq H \boxtimes k[\mathbb{Z}]$.

This shows that the subcategory of Hopf algebras can be identified with the category of modules over the idempotent object $k[\mathbb{Z}]$ in $\mathbf{BiAlg}_{\mathbf{k}}$, and, as such, has symmetric monoidal structure given by $\boxtimes$ restricted to $\mathbf{Hopf}_{\mathbf{k}}$ with monoidal unit $k[\mathbb{Z}]$.

§3. Witt Vectors and associated Hopf Algebras

In order to prove 1.1 next week, we need some algebraic background.

For R a commutative ring, define $W_{\mathrm{Big}}(R)$ to be the subset of $R[[t]]$ of elements with constant term 1. These form a group under multiplication: **the group of big Witt vectors of R.** We also define Wt_{Big} to be the polynomial ring $\mathbb{Z}[c_1, c_2, \ldots]$ on countably many variables. Then:

$$W_{Big}(R) \simeq \mathrm{Fun}_{\mathrm{Ring}}(Wt_{Big}, R),$$

via

$$f \mapsto 1 + f(c_1)t + f(c_2)t^2 + \ldots.$$

Since $R \to W_{Big}(R)$ takes values in abelian groups, we can regard Wt_{Big} as a Hopf algebra over $\mathbb{Z}$.

(Note: this is kind from what was done before, i.e. we start with algebra structure and use a covariant functor to produce a coalgebra structure. Not an issue.)

The comultiplication $Wt_{Big} \to Wt_{Big} \otimes Wt_{Big}$ that induces the group structure on $W_{Big}(R)$ for each R is given by

$$c_n \mapsto \sum_{i+j=n} c_i \otimes c_j :$$

we can just check this does the trick and use uniqueness.

The goal of the rest of this section is to study this Hopf algebra and some sub-Hopfalgebras.

Note also that Wt_{Big} is the cohomology ring of BU with integer coefficients. Each c_n can be identified with a virtual Chern class of the tautological bundle on BU.

3.1. Lemma–Definition. $1 + c_1 t + \cdots = \prod_{n>0}(1 - a_n t^n)$, *for* $a_n \in R$. *Each* c_m *can be written as an integer polynomial in* a_i*'s, and vice versa. If we take* $R = Wt_{Big}$*, we get a sequence of elements* $a_n \in Wt_{Big}$*, where* a_n *is an integer polynomial in* c_i*'s and is called the* n**-th Witt component**. *These determine an isomorphism* $\mathbb{Z}[a_1, a_2, \dots] \simeq Wt_{Big}$.

If $f(t) \in Wt_{Big}[[t]]$ is the element $1 + c_1 t + c_2 t^2 + ..$ **where** c_i**'s are the polynomial variables of** Wt_{Big}, then:

$$\boxed{td \log f(t) = tf'(t)/f(t) = w_1 t + w_2 t^2 + \dots}$$

for some $w_i \in Wt_{Big}$. We call w_n the n-th ghost component. Moreover, each w_n is **primitive**: $\Delta(w_n) = w_n \otimes 1 + 1 \otimes w_n$.

Over the rationalized ring $\mathrm{Wt}_{\mathrm{Big}}$, which we denote $\mathrm{Wt}^{\mathbb{Q}}_{\mathrm{Big}}$, we have that

$$\log(f(t)) = \sum_{n \geq 0} w_n / nt^n,$$

so that

$$f(t) = \exp(\sum w_n / nt^n),$$

and each c_n can be written as a rational polynomial in w_i's. In particular, $\mathrm{Wt}^{\mathbb{Q}}_{\mathrm{Big}}$ is a polynomial ring on the ghost components:

$$\mathrm{Wt}^{\mathbb{Q}}_{\mathrm{Big}} \simeq \mathbb{Q}[w_1, w_2, \dots].$$

Also note that, we actually have formulas for w_n in terms of a_m as defined before,

$$\sum_{n>0} w_n/nt^n = \sum_{m>0} \log(1-a_m t^m) = \sum_{m>0,d>0} a_m^d t^{md}/d = \sum_{n>0}\sum_{d|n} a_{n/d}^d t^n/d.$$

3.2. Lemma–Definition. *For $S \subset \mathbb{Z}_{>0}$ closed under multiplication, let Wt_S be the subalgebra of $\mathrm{Wt}_{\mathrm{Big}}$ generated by a_n for $n \in S$, and let $Wt_S^{\mathbb{Q}}$ be the tensor with $\mathbb{Q}$. Then Wit_S inherits the strucutre of a Hopf algebra from Wt_{Big}.*

Proof. The following steps lead to the conclusion:

- $Wt_S = Wt_S^{\mathbb{Q}} \cap \mathrm{Wt}_{\mathrm{Big}}$.
- Wt_S is a polynomial algebra over $\mathbb{Q}$ on w_n for $n \in S$.
- Wt_S is preserved by the antipode mape $w_n \mapsto -w_n$.
- For $n \in S$, $\Delta(w_n) = 1 \otimes w_n + w_n \otimes 1 \in Wt_S \otimes Wt_S \subset \mathrm{Wt}_{\mathrm{Big}} \otimes \mathrm{Wt}_{\mathrm{Big}}$, so that comultiplication on $\mathrm{Wt}_{\mathrm{Big}}$ carries Wt_S to $Wt_S \otimes Wt_S$ and Wt_S has the structure of a Hopf algebra.

□

Example: **ring of p-typical Witt vectors**, $\mathbb{Z}[a_1, a_p, a_{p^2}, \dots] \subset \mathrm{Wt}_{\mathrm{Big}}$.

The goal for the remainder of the section is to study the structure of $\mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}$ and $Wt_S \boxtimes Wt_S$ (with $S \subset \mathbb{Z}_{>0}$ as before), and how different $\boxtimes$-products of these are related.

The next four results (and their generalizations) will be referenced in [HL] 1.3, which we'll talk about next week. An idea of some of the proofs is given after the statements.

3.3. Proposition ([HL] 1.2.13). *Let S and T be finite subsets of $\mathbb{Z}_{>0}$ which are divisibility-closed. Then $Wt_S \boxtimes Wt_T$ is smooth over $\mathbb{Z}$.*

This generalizes to finite collections $S_1, \dots, S_n$. I.e. [HL] 1.2.14.

3.4. Theorem ([HL] 1.2.16). *Let S be closed under divisibility. Then the canonical map $Wt_S \boxtimes Wt_S \to \mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}$ is injective, and identifies $Wt_S \boxtimes Wt_S$ with $Wt_S^{\mathbb{Q}} \boxtimes Wt_S^{\mathbb{Q}} \cap \mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}} \subset \mathrm{Wt}_{\mathrm{Big}}^{\mathbb{Q}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}^{\mathbb{Q}}$.*

3.5. Theorem ([HL] 1.2.20). . *There exists a unique Hopf algebra map $\iota\colon \mathrm{Wt}_{\mathrm{Big}} \to \mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}$ such that $\iota(w_n) = \frac{w_n \boxtimes w_n}{n}$.*

Similarly, we also have the following:

3.6. Corollary ([HL] 1.2.21). *Let S be a divisibility-closed set. Then there exists a unique Hopf algebra map $\iota_S\colon Wt_S \to Wt_S \boxtimes Wt_S$ such that $\iota_S(w_n) = \frac{w_n \boxtimes w_n}{n}$.*

3.7. Theorem ([HL] Scholium 1.2.15). *Let $S' \subset S \subset \mathbb{Z}_{>0}$ and $S' \subset S \subset \mathbb{Z}_{>0}$. Then the inclusion maps*

$$Wt_{S'} \hookrightarrow Wt_S$$

and

$$Wt_{T'} \hookrightarrow W_t$$

induce a faithfully flat map

$$\phi\colon Wt_{S'} \boxtimes Wt_{T'} \to Wt_S \boxtimes Wt_T.$$

Proof notes. This proof is more or less a stand-alone algebraic result, although it does use some observations from the proofs of other results.

Step 1: reduce to finite sets S, T using direct limit argument. Factor ϕ to assume $T = T'$.

Step 2: Work by induction on size of $S - S'$; reduce to case of $S = S' \cup \{n\}$ with n maximal. We get a cofiber sequence:

$$Wt_{S'} \to Wt_S \to Wt_1,$$

where the second map is V_n, the **Verschiebung map** defined as follows:

$$V_n(c_m) = c_{m/n},\ n|m,$$

$$V_n(c_m) = 0,\ else.$$

This map will come up later.

Step 3:
Apply $\boxtimes$ to get a cofiber sequence

$$Wt_S \boxtimes W_T \to Wt_{S'} \boxtimes W_T \to Wt_1 \boxtimes Wt_T,$$

where the first map is ϕ.

After tensoring with $\mathbb{Q}$, can explicitly check that φ is flat. Use fibre-by-fibre flatness criterion to reduce to checking each ϕ_p is flat, and check this.

□

To prove [HL] 1.2.13 and 1.2.20, we need some general results about constructions on Hopf algebras:

3.8. Lemma ([HL] 1.2.8). *Let $H_1, \ldots, H_n$ be a finite collection of bialgebras over $\mathbb{Z}$ which are free when regarded as $\mathbb{Z}$-modules. If each H_i is finitely generated as a commutative ring, then $H_1 \boxtimes \cdots \boxtimes H_n$ is finitely generated as a commutative ring.*

3.9. Proposition ([HL]1.2.10). *Let H be a Hopf algebra which is finitely generated over $\mathbb{Z}$. Assume that for each prime p, the affine scheme* $\operatorname{spec} H/pH$ *is connected. Then the following are equivalent:*

(1) *The Hopf algebra H is smooth over $\mathbb{Z}$.*
(2) *The module of indecomposeables $Q(H)$ is free.*
(3) *For every prime p,*

$$\dim_{\mathbb{Q}}(Q(H) \otimes_{\mathbb{Z}} \mathbb{Q}) \geq \dim_{F_p}(Q(H) \otimes_{\mathbb{Z}} F_p).$$

Given these, we have:

Proof for [HL] *1.2.13.* Sketch:

- Using 1.2.8, we conclude that $Wt_S \boxtimes W_t$ is finitely generated as a $\mathbb{Z}$-algebra.
- Next, we show that the rationalization is a polynomial algebra and hence smooth (can be explicitly computed using observations about "commuting" Sym^*_{red} with $\boxtimes$ from 1.1.26).
- Finally, from 1.2.10 is suffices to show $specH/pH$ is connected and of dimension st. This is proven by induction on st.

□

Proof for [HL] *1.2.20.* Sketch:

- Prove over $\mathbb{Q}$, using that $\mathrm{Wt}^{\mathbb{Q}}_{\mathrm{Big}}$ is a polynomial ring on the ghost components w_n. From this, we get a unique ring homomorphism: we need to show it's a map of bialgebras.
- Show that ι restricted to $\mathrm{Wt}_{\mathrm{Big}}$ inside $\mathrm{Wt}^{\mathbb{Q}}_{\mathrm{Big}}$ factors through $\mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}$. (This is pretty non-trivial.)

□

Another results that comes up:

3.10. Remark ([HL] 1.2.23). *Let $S, T \subset \mathbb{Z}_{>0}$ be closed under divisibility. Let $n > 0$, and suppose $nm \in S \implies m \in T$. Then:*

$$\begin{array}{ccc} Wt_S & \xrightarrow{\iota_S} & Wt_S \boxtimes Wt_S \\ \downarrow{\scriptstyle V_n} & & \downarrow{\scriptstyle \iota_S \boxtimes \iota_S} \\ Wt_T & \xrightarrow{\iota_T} & Wt_T \boxtimes W_T \end{array}$$

By the way that we constructed ι, it's enough to show this over $\mathbb{Q}$!

References

[HL] M.J. Hopkins and J. Lurie. *Ambidexterity in the $K(n)$-local homotopy category.* Preprint, see Lurie's website.