THE GOODWILLIE FILTRATION AND THE GENEALOGY OF UNSTABLE ELEMENTS

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1. Two notions of instability

Previously, we constructed the *transfinite Goodwillie spectral sequence (TGSS)*, whose signature is roughly

$$E^1_{t,J}(S^n) = \pi_t \mathbb{S}^{n-|J|+\|J\|} \Rightarrow \pi_t S^n.$$

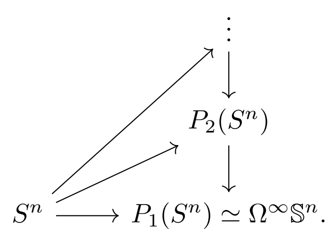
This spectral sequence computes unstable homotopy from stable data. Over the next couple of talks, we'll discuss the differentials in this spectral sequence, but for today, let's just focus on understanding the relationship between the E^1 page and the E^∞ page at a high level.

Before jumping in, recall that elements in the $E^1_{t,J}$ term of the TGSS are denoted $\alpha[J]$, where $\alpha \in \pi_t \mathbb{S}^{n-|J|+\|J\|}$ is a stable element and J is CU sequence with excess $\geq n$.

Let $\beta \in \pi_t S^n$. Where does it come from in the E_1 page? In other words,

Question 1.1 (Vague)**.** *How "stable" is β?*

One possible answer to this question is as follows. Consider the Goodwillie tower of the identity evaluated at S^n.



Given an element $\beta \in \pi_t S^n$, the further down the tower you can map it and get something nonzero, the "more stable" it is.

Definition 1.2. An element $\beta \in \pi_t S^n$ has *Goodwillie filtration* 2^k if its image in $\pi_t P_{2^k}(S^n)$ is nonzero, but its image in $\pi_t P_{2^{k-1}}(S^n)$ is null.

Equivalently, β is detected in the TGSS by elements of the form $\alpha[j_1, \ldots, j_k]$. This is because the 2^k-th layer of the Goodwillie tower is given by the homotopy of $L(k)_n$, and via the TAHSS they come from the homology of $L(k)_n$ which are indexed by CU sequences of length equal to k.

Example 1.3. β has Goodwillie filtration one iff it is *stable*.

There is also a different notion of instability arising from the EHP sequence. Recall that the EHP sequence gives us a long exact sequence

$$\cdots \to \pi_{t+2} S^{2n+1} \xrightarrow{P} \pi_t S^n \xrightarrow{E} \pi_{t+1} S^{n+1} \to \cdots.$$

Date: November 8, 2017.

Let $0 \neq \beta \in \pi_t S^n$ be an unstable element, i.e., it dies in $\pi_t \mathbb{S}^n$. Then there exists $r_1 \geq 0$ minimal such that $E^{r_1+1}(\beta) = 0$. By exactness, there exists $\alpha_1 \in \pi_{t+r_1+2} S^{2(n+r_1)+1}$ such that $P(\alpha_1) = E^{r_1}(\beta)$, so $\beta \in E^{-r_1} P(\alpha_1)$. In this situation, we shall say that "α_1 is a *child* of β".

If α_1 is still unstable, we can repeat this process and inductively get $\beta \in E^{-r_1} P \cdots E^{-r_l} P(\alpha_l)$. We say that "$\alpha_l$" is a l-th generation *descendant* of β". We also say that β is *unstable of degree at least* k if it has a k-th generation descendant.

Eventually, this process stops because we have produced an element α_k that is stable and survives to $\alpha \in \pi_*^s$. For some reason, we regard stable elements as infertile…

Pictorially, this looks like

$$(\dagger) \qquad \alpha \in \pi_*^s \xleftarrow{E^\infty} \pi_* S^{2j_1+1} \ni \alpha_k \xrightarrow{P} \pi_* S^{j_1} \xleftarrow{E^{r_k}} \pi_* S^{2j_2+1} \ni \alpha_{k-1} \xrightarrow{P} \cdots \xrightarrow{P} \pi_* S^{j_k} \xleftarrow{E^{r_1}} \pi_* S^n \ni \beta.$$

Such a sequence $(\alpha_1, \ldots, \alpha_k)$ is called a *lineage* of β, and we write $\beta \in \alpha \langle j_1, j_2, \ldots, j_k \rangle$. It is easy to see that the sequence $(j_1, \ldots, j_k)$ is CU of excess $\geq n$.

Examples 1.4. I'll give the lineage of a couple of elements, assuming we already know the relevant homotopy groups from some other method (e.g., Serre's method, Toda's method, …).

(a) We have $\pi_9 S^4 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ but $\pi_5^s = 0$, so let $x_1, x_2 \in \pi_9 S^4$ be the two generators. I claim that one is in $\eta^2 \langle 4 \rangle$ and the other is in $\eta \langle 5 \rangle$.

Consider part of an EHP sequence

$$\cdots \to \pi_{11} S^9 \xrightarrow{P} \pi_9 S^4 \xrightarrow{E} \pi_{10} S^5 \to \cdots.$$

This splits off a short exact sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0.$$

So one of the elements, say x_1, is in the image of P. But $\pi_{11} S^9$ is already in the stable range with generator η^2, so we have $x_1 = P(\eta^2)$, and $x_1 \in \eta^2 \langle 4 \rangle$.

Now consider $E(x_2) \neq 0$ in $\pi_{10} S^5$. We have

$$\cdots \to \cdots \pi_{12} S^{11} \xrightarrow{P} \pi_{10} S^5 \xrightarrow{E} \pi_{11} S^6 \to \cdots.$$

Since $\pi_{11} S^6 \cong \mathbb{Z}$, the map E must be zero, and therefore $E(x_2) \in P(\pi_{12} S^{11})$. The group $\pi_{12} S^{11}$ is in the stable range with generator η, so we have $x_2 \in E^{-1} P(\eta)$, and $x_2 \in \eta \langle 5 \rangle$.

(b) We have $\pi_8 S^2 \cong \mathbb{Z}/2$, so let $x \in \pi_6 S^2$ be the generator. I claim that $x \in \eta \langle 5, 2 \rangle$.

Consider part of an EHP sequence

$$\cdots \to \pi_{10} S^5 \xrightarrow{P} \pi_8 S^2 \xrightarrow{E} \pi_9 S^3 \to \cdots.$$

Since $\pi_9 S^3 = 0$, we see that $E(x) = 0$ and $x = P(y)$ for some $y \in \pi_{10} S^5 \cong \mathbb{Z}/2$. Now consider another EHP sequence

$$\cdots \to \pi_{12} S^{11} \xrightarrow{P} \pi_{10} S^5 \xrightarrow{E} \pi_{11} S^6 \to \cdots.$$

Because $\pi_{11} S^6 \cong \mathbb{Z}$, the homomorphism E is zero and therefore $y \in P(\pi_{12} S^{11})$. We are now in the stable range, and in fact $y = P(\eta)$, so x is descended from $\eta \in \pi_1^s$ with lineage $(5, 2)$.

Since we have these two notions of instability, a natural question to ask would be:

Question 1.5. *What is the relationship between the degree of instability and the Goodwillie filtration?*

To make this a little more precise, recall that we also have "Goodwillie names" for unstable elements. For example, $\eta^2[4]$ and $\eta[5]$ survive in the Goodwillie spectral sequence and detect elements in $\pi_9 S^4$. How are they related to $\eta^2 \langle 4 \rangle$ and $\eta \langle 5 \rangle$?

To answer this question, our main theorem today is

Theorem 1.6 ([Beh12, Thm. 3.5.1])**.** *Suppose we have an element $\alpha[j_1, \ldots, j_k] \in E^1_{*,J}(S^n)$ in the TGSS. Then*

$$\alpha[j_1, \ldots, j_k] \in E_*^{-l_k} P_* E_*^{-l_{k-1}} P_* \cdots E_*^{-l_1} P_* \alpha$$

for some $\alpha \in E^1_{*,\emptyset}(S^{2j_1+1})$, where $l_s = j_s - (2j_{s+1} + 1)$ for $1 \leq s < k$ and $l_k = j_k - n$.

Furthermore, if, for all s, the elements $\alpha[j_1, \ldots, j_{s-1}] \in E^1_{*,[j_1,\ldots,j_{s-1}]}(S^{2j_s+1})$ *are permanent cycles, converging to elements* $\alpha_s \in \pi_*(S^{2j_s+1})$*, then we have* $E^{l_s}(\alpha_{s+1}) = P(\alpha_s)$ *modulo elements of higher Goodwillie filtration.*

The first part of this theorem will be shown once we know how the P and E maps act on the TGSS. This was actually computed last time, so I'll recall the statement below. The second part of the theorem means that if all the intermediate steps in the zigzag (†) survive the spectral sequence, then we can lift this statement about the E^1 term of the TGSS to a statement on the level of the E^∞page about actual unstable homotopy groups, up to the associated graded.

In summary, "under favorable circumstances" – i.e., if the hypothesis of the second part of the theorem is satisfied and if descendants are unique, and so on – if $\beta \in \pi_* S^n$ is detected by $\alpha[j_1, \ldots, j_k]$, then

$$\beta \in E^{-l_k} P E^{-l_{k-1}} P \cdots E^{-l_1} P(\tilde{\alpha})$$

for $\tilde{\alpha}$ stabilizing to α. Moreover, the degree of instability of the element β is equal to k, and β has lineage $\beta \in \alpha \langle j_1, \ldots, j_k \rangle$.

This theorem is not too hard to show. We'll need the following facts, of which the harder ones were already discussed in a previous talk.

Lemma 1.7 ([Beh12, Cor. 3.4.2, 3.4.4])**.**

(a) The P *map induces a map of TGSS's:*

$$\begin{array}{ccc} E^1_{t+2,J}(S^{2n+1}) & \Longrightarrow & \pi_{t+2}(S^{2n+1}) \\ \downarrow P_* & & \downarrow P \\ E^1_{t,[J,n]}(S^n) & \Longrightarrow & \pi_t(S^n). \end{array}$$

The left map is described by

$$P_*(\alpha[J]) = \alpha[J, n].$$

In particular, the P *map takes an element of Goodwillie filtration* 2^{k-1} *to an element of Goodwillie filtration at least* 2^k*.*

(b) The E *map induces a map of TGSS's:*

$$\begin{array}{ccc} E^1_{t,J}(S^n) & \Longrightarrow & \pi_t(S^n) \\ \downarrow E_* & & \downarrow E \\ E^1_{t+1,J}(S^n) & \Longrightarrow & \pi_{t+1}(S^{n+1}). \end{array}$$

The left map is described by

$$E_*(\alpha[J]) = \begin{cases} \alpha[J], & e(J) \geq n+1 \\ 0, & \textit{otherwise.} \end{cases}$$

In particular, the E *map takes an element of Goodwillie filtration* 2^k *to an element of Goodwillie filtration at least* 2^k*.*

Proof sketch of lemma. To show that the P and E maps induce maps of TGSS's, we need to show that they induce maps of the transfinite towers that give rise to the spectral sequences. Since the TGSS is built from the TAHSS, a preliminary step is to show that P and E induce maps of TAHSS's and the towers that beget them. Unpackaging the definition of all these transfinite spectral sequences, this just boils to checking for maps between the Mitchell-Priddy spectra, which are constructed inductively using the cofiber sequences produced previously.

The description of the induced maps P_* and E_* follows from [Beh12, Prop. 2.2.6].

□

The theorem follows.

Remark 1.8. The analogue of this lemma for the TAHSS can be used to give an inductive procedure for computing the TAHSS's. The main observations are:

- E^{n-1} induces a map of TAHSS

$$E^{n-1}_* : \{E^{\alpha}_{*,*}(L(k))\} \to \{E^{\alpha}_{*,*}(L(k)_n)\}$$

given by *truncation*.
- P induces an *injective* map of TAHSS

$$P_* : \{E^{\alpha}_{*,*}(L(k-1)_{2m+1})\} \to \{E^{\alpha}_{*,*}(L(k)_m)\}.$$

For example, suppose we completely understand the TAHSS for $L(k-1)$ including its differentials. By truncation, we also completely understand the TAHSS for $L(k-1)_{2m+1}$. But this embeds into the TAHSS for $L(k)$: namely,

$$\begin{array}{ccc} E^1_{t,*}(L(k-1)_{2m+1}) & \Longrightarrow & \pi_t(L(k-1)_{2m+1} \\ \downarrow & & \downarrow \\ E^1_{t,*}(L(k)_m) & \Longrightarrow & \pi_t(L(k)_m). \end{array}$$

This gives us a bunch of differentials in the TAHSS for $L(k)$ via the correspondence

$$d^{L(k-1)}(\alpha[J]) = \alpha'[J'] \qquad \Leftrightarrow \qquad d^{L(k)}(\alpha[J,m]) = \alpha'[J',m].$$

Then it remains to compute the remaining differentials in order to understand the TAHSS for $L(k)$.

2. Some counterexamples

Theorem 1.6 gives a relationship between the lineage of an unstable element and the element that detects it in the E^1 term of the TGSS. One might wonder if something stronger holds: perhaps every "circumstance" is "favorable". For example,

Question 2.1. *Is it true that every element of Goodwillie filtration 2^k has degree of instability equal to k?*

This fails in an uninteresting way:

Example 2.2. There is a unique element $\overline{\alpha_{6/3}[4]} \in \pi_{18}S^4$ that is detected in the TGSS for S^4 by $\alpha_{6/3}[4]$ with the property that $E(\overline{\alpha_{6/3}[4]}) = 0$. The element $\alpha_{6/3}[4]$ has Goodwillie filtration 2^1. Moreover, $\overline{\alpha_{6/3}[4]} = P(\alpha_{6/3})$, so it has lineage $\alpha_{6/3}\langle 4\rangle$, and the two notions agree.

However, there is another element $\overline{1[11,5]}$ in $\pi_{18}S^4$ that is the unique element detected by $1[11,5]$. Consider the sum $x = \overline{\alpha_{6/3}[4]} + \overline{1[11,5]}$. The element x is still detected by $\alpha_{6/3}[4]$, so it still has Goodwillie filtration 2^1. However, $E(x) = E(\overline{1[11,5]}) = P(P(1))$ for $1 \in \pi_{23}S^{23}$:

$$\pi_{23}S^{23} \xrightarrow{P} \pi_{21}S^{11} \xrightarrow{P} \pi_{19}S^5.$$

So x has lineage $1\langle 11,5\rangle$, i.e., it has degree of instability 2.

We can "repair" this example by choosing $x = \overline{\alpha_{6/3}[4]}$ instead, i.e., by throwing out all the terms of higher Goodwillie filtration. So a better question is:

Question 2.3. *Is every element x of Goodwillie filtration 2^k equivalent to another element x' modulo Goodwillie filtration 2^{k+1} such that the degree of instability of x' is equal to k?*

This turns out to still be false, but for a much more interesting reason. It is related to the "bad differential" of [Beh12, Sec. 5.4], a differential in the first 20 stems of the TEHPSS that cannot be lifted from one in the TGSS by the standard method.

Proposition 2.4. *Every element of $\pi_{24}S^5$ that is detected in the TGSS by $\alpha_{8/5}[5]$ has degree of instability equal to 2.*

Proof. Let $\overline{\alpha_{8/5}[5]} \in \pi_{24}S^5$ be the unique unstable element detected by $\alpha_{8/5}[5]$. We wish to show it has a grandchild. In order to compute its lineage, we'll need to rely on calculations in sections 5 and 6 of [Beh12]. We have

$$\begin{aligned} &E(\overline{\alpha_{8/5}[5]}) = \overline{\eta^2[13,6]} \neq 0 & \\ &E(\overline{\eta^2[13,6]}) = 0 & \Rightarrow \overline{\eta^2[13,6]} = P(\overline{\eta^2[13]}) \\ &E(\overline{\eta^2[13]}) = 0 & \Rightarrow \overline{\eta^2[13]} = P(\eta^2). \end{aligned}$$

So we have $\overline{\alpha_{8/5}[5]} \in E^{-1}PP\eta^2$ and $\overline{\alpha_{8/5}[5]} \in \eta^2 \langle 13, 6 \rangle$. This is an element in $\pi_{24}S^5$ with Goodwillie filtration 2^1 but degree of instability 2.

In fact, every element of $\pi_{24}S^5$ that is detected by $\alpha_{8/5}[5]$ has degree of instability 2. This is because all the other generators of $\pi_{24}S^5$ are stably nontrivial, i.e., of *lower* Goodwillie filtration. □

Question 2.3 has a positive answer in some cases. It is tautological if $k = 0$, and it is also true when $k = 1$ in the metastable range.

Proposition 2.5. *Suppose that* $0 \neq \beta \in \pi_{t+m}S^m$, $t \leq 3n - 2$, *is detected by* $\alpha[n]$ *in the TGSS for* S^m *for some* $\alpha \in \pi^s_{t-n+1}$. *Then* $\beta \in \alpha \langle n \rangle$.

Example 2.6. The elements in $\pi_9 S^4$ from example 1.4(a) satisfy the hypotheses of this proposition.

Proof. We want to compute the lineage of β. We claim that $E^{n-m}(\beta) \neq 0$. The element $E^{n-m}(\beta)$ is detected by $\alpha[n]$ in the TGSS for S^n by lemma 1.7. If $\alpha[n] \in E^1(S^n)$ is hit by a TGSS differential, then $\alpha[n] \in E^1(S^m)$ must be hit by a corresponding TGSS differential. But it is not, and we conclude that $\alpha[n] \in E^1(S^n)$ survives the spectral sequence and $E^{n-m}\beta \neq 0$ in $\pi_{t+n}S^n$.

On the other hand, $t \leq 3n-2$ imply $2n+1 \geq (t+n+2) - (2n+1) + 2$, so we're in the stable range and $\pi^s_{t-n+1} \cong \pi_{t+n+2}(S^{2n+1})$. Under this isomorphism, the stable element α corresponds to an unstable element $\alpha \in \pi_{t+n+2}S^{2n+1}$, detected by $\alpha[\emptyset]$ in the TGSS for S^{2n+1}. In the TGSS for S^n, we have $P_*(\alpha[\emptyset]) = \alpha[n]$ again by (1.7).

Therefore, $E^{n-m}\beta$ and $P(\alpha)$ agree modulo elements of Goodwillie filtration ≥ 2. But there are no nonzero elements of $\pi_{t+n}(S^n)$ of Goodwillie filtration ≥ 2. That is to say, the group $E^1_{t,J}(S^n) = \pi_{t+n}S^{n-|J|+\|J\|}$ is zero if $|J| \geq 2$, because $t + n \leq 4n - 2$ and $n - |J| + \|J\| \geq n - 2 + (2n+1) = 4n - 1$. So $E^{n-m}\beta = P(\alpha)$. □

References

[Beh12] Behrens, Mark. "The Goodwillie tower and the EHP sequence." *Mem. Amer. Math. Soc.* 218 (2012), no. 1026.