The Goodwillie Tower of the Identity

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October 11, 2017

These notes are a transcript of what I said in my talk.

1 Overview

We know that $D_nX \simeq \partial_n \wedge_{\Sigma_n} X^{\wedge n}$ for a spectrum ∂_n which is given as an Ω-spectrum by $\Omega^{i\overline{\rho}}(D_{1,\ldots,1}\chi_n Id)(S^i,\ldots,S^i)$ where $\overline{\rho}$ is the reduced regular representation of Σ_n, $D_{1,\ldots,1}$ is the multilinearization or first derivative in each variable, and χ_n is the cross-effect. Because χ_n is a symmetric functor, ∂_n has a very naive Σ_n-action. It also has a Σ_n equivariant filtration, from the definition of the cross-effects functor as an iterated limit.

In Johnson's thesis [3], she built a map

$$(\chi_n Id)(X_1,\ldots,X_n) \to \text{Some simpler filtered } \Sigma_n\text{-functor}$$

that preserves the filtration and has the correct asymptotic connectivity in terms of the connectivity of the X_i to be a $D_{1,\ldots,1}$ equivalence. She does this by writing down an explicit model for the iterated homtopy fiber and restricting to a small subset of the data of the iterated fiber. This expresses $\partial_n \simeq \mathbb{D}\,P_n$

Arone and Mahowald analyze the filtration on P_n – it comes from a Čech cover with all intersections contractible or empty. The combinatorics of this cover correspond to the partition complex which is defined in terms of the poset of partitions of a set of size n ordered by refinement. They use the spectral sequence corresponding to this filtration and the combinatorics of partitions to compute $H_*D_*S^d$.

Chains of partition that start at the finest partition and end at the coarsest partition can be identified with rooted trees with labeled leaves where the length of every path from the root to a leaf is the same. Ching defined the bar construction for operads in terms of some rooted trees with labeled leaves and labels on the internal vertices coming from the operad. Having no labels on the internal vertices corresponds to only having one element in each term in the symmetric sequence for the operad, so $P_n \simeq \mathrm{B}(\mathrm{Comm})(n)$ and $\partial_* \simeq \mathbb{D}\,\mathrm{B}(\mathrm{Comm})$ is the Koszul dual to Comm, which is a shifted Lie algebra.

Arone and Dwyer identified the ∂_n with certain suspensions of the spectra $L(n)$, which Robert will discuss next week. They do not use this paper or the computation of $H_*L(n)$ so an alternate approach would be to compute $H_*L(n)$ directly and transfer that over using Arone and Dwyer.

According to Ching, D_*X is the free ∂_*Id-algebra on X. Because of this, H_*D_*X should be generated by $H_*X \cong H_*D_1X$ under the operations on the homology of ∂_*Id-algebras. According to Omar's thesis [2] (which only actually deals with the prime 2), these are given by:

- A degree -1 Lie bracket $H_iD_mX \otimes y \in H_jD_nX \to H_{i+j-1}D_{m+n}X$
- Operations $\beta^\epsilon \widetilde{Q}^j$ with degree $j-\epsilon-1$ that take $\beta^\epsilon \widetilde{Q}^j : H_iD_mX \to H_{i+j-\epsilon-1}D_{pm}X$

The $\beta^\epsilon \widetilde{Q}^j$ satisfy Adem-style relations which reduce $(\beta^\epsilon \widetilde{Q}^i)(\beta^{\epsilon'} \widetilde{Q}^{i'})$ if this would be admissible in the Dyer Lashof algebra, so that the action is generated by "completely unadmissible" monomials in the $\beta^\epsilon Q^i$. All brackets of terms involving DL operations vanish. (This issue is not settled at odd primes, so this is a bit misleading. Behrens is only interested in the prime 2 so it doesn't really matter.) We're interested in $H_*D_nS^d$. If d is odd, then all of the brackets vanish and $H_*D_*S^d$ is generated by a single element under the $\widetilde{Q}^I$, and the relations are given by an excess criterion. In particular, the only nonvanishing derivatives are p-powers. If d is even, then there is one nontrivial bracket, and all of the elements of $H_*D_*S^d$ are generated by u and $[u,u]$ under the $\widetilde{Q}^I$ again subject to excess criteria. In this case, the nonvanishing derivatives are in degrees p^k and $2p^k$. I'm not going to make it back to this perspective in this talk, but Rasmus will talk about this story in two weeks.

2 Arone and Mahowald's calculuation of $H_*(D_nS^d)$

We're only going to cover the case of an odd sphere.

Definition 2.1. *Let* $\mathrm{Part}(n)$ *be the poset of equivalence relations on* $[n]$*, partially ordered by refinement. This has a maximum and minimum element, so the corresponding nerve is contractible. Let* $\mathrm{Part}(n)^\pm$ *be the poset of partitions with the minimum and maximum deleted, respectively. Then*

$$P_n = \frac{N(\mathrm{Part}(n))}{N(\mathrm{Part}(n)^+) \cup N(\mathrm{Part}(n)^-)}.$$

The n*-simplices of* P_n *are chains* $0 = \lambda_0 \leq \cdots \leq \lambda_n = 1$.

We're going to take it as a fact (pretty much proven by Johnson but stated explicitly in Arone-Mahowald) that $\partial_n = \mathbb{D}\,P_n$. This implies that $D_n(X) = \partial_n \wedge_{\Sigma_n} \Sigma^\infty X^{\wedge n} \simeq \mathrm{Map}_*(P_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}$. The skeletal filtration on P_n gives fibrations

$$\mathrm{Map}_*(\Sigma^i P_n(k)) \to \mathrm{Map}_*(P_n^{\leq k}, \Sigma^\infty X^{\wedge n})_{h\Sigma_n} \to \mathrm{Map}_*(P_n^{\leq k-1}, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}$$

where by $P_n(k)$ I mean the discrete set of k simplices in P_n. This gives us a spectral sequence

$$E_1^{s,t} = H_{t-s}(\mathrm{Map}_*(\Sigma^s P_n(s), \Sigma^\infty X^{\wedge n})) = H_t(\mathrm{Map}_*(P_n(s), \Sigma^\infty X^{\wedge n})) \Rightarrow H_* D_n(X)$$

When we compute the E_2 page of this, we'll find out that it is concentrated in one column, so the spectral sequence degenerates at E_2. Let $C^i = E^{i,*}$ be the chain complex of columns of the E_1 page.

Let's analyze this spectral sequence now. There is an equivalence

$$\mathrm{Map}_*(P_n(s), \Sigma^\infty X^{\wedge n})_{h\Sigma_n} \simeq \bigvee_\Lambda \Sigma^\infty X^{\wedge n}_{h\Sigma_\Lambda}$$

where the wedge is over orbits of $P_n(s)$ under the action of Σ_n and Σ_Λ is the stabilizer of the orbit Λ. The differentials in the E_1 page are given by restricting along face maps. Because of the way we formed P_n as a quotient, the outer boundary maps in E_1 are zero. The inner boundary maps are given under this isomorphism as transfers – if $\Lambda' = (\lambda_0 \leq \cdots \leq \lambda_{s+1})$ is a refinement of $\Lambda = (\lambda_0 \leq \cdots \leq \widehat{\lambda_i} \leq \cdots \leq \lambda_{s+1})$, then $\Sigma_{\Lambda'} \subseteq \Sigma_\Lambda$ and the boundary map d_i is given by the sum of the transfers from $H_*(\Sigma^\infty X^{\wedge n}_{h\Sigma_\Lambda}) \to H_*(\Sigma^\infty X^{\wedge n}_{h\Sigma_{\Lambda'}})$ where Λ' varies over such refinements that add an extra partition in the ith place.

Now note that $H_*(X^p_{h\Sigma_p})$ is generated by the union of the image of the orbit map from $H_*(X^p)$ and the image of the diagonal map from $H_*(B\Sigma_{p+} \wedge X)$. The first type of elements are products of elements in the homology of X and the second type are external Dyer Lashof elements. This is true more generally.

Definition 2.2. *A chain of partitions Λ of $n = p^k$ is pure if Σ_Λ is a wreath product: $\Sigma_{p^{l_0}} \wr \cdots \wr \Sigma_{p^{l_i}}$. An element of $H_*(X^{\wedge n}_{h\Sigma_\lambda})$ is pure if it is given by applying Dyer Lashof operations to elements of $H_*(X)$.*

The idea is that pure elements of $H_*(D_n Id(X))$ correspond to compositions of operations $\widetilde{Q}^j$ applied to the fundamental class and that the impure elements correspond to expressions that contain at least one lie bracket. We're going to show:

Proposition 2.3 ([1, Proposition 3.9]). *The set of pure elements forms a subcomplex $P^* \to C^*$.*

Proof (Proof sketch). Let A_k be the elementary abelian group of order p^k. There is a map $A_k \hookrightarrow \Sigma_{p^k}$ given by the regular representation of A_k. The group Σ_Λ contains a conjugate of A_k if and only if Λ is pure. Assume that Λ is pure and for convenience that Σ_Λ contains A_k. We need to show that the differential applied to pure elements is pure. The map $H_*(BA_{k+} \wedge X) \to H_*(X^{\wedge p^k}_{h\Sigma_\Lambda})$ sends an element

$$Q_{i_1} \otimes \cdots \otimes Q_{i_k} \otimes u \mapsto Q_{i_1} \cdots Q^{i_?} \wr \cdots \wr Q^{i_?} \cdots Q^{i_k} u$$

where the positions of the wreath products is the same as the positions of the wreath products in Σ_Λ. In particular, this map surjects onto the pure part and is zero onto the rest. The differential is given by certain transfer maps, and the elements that we are concerned with are in the image of restriction from A_k, so we use the double coset formula:

$$\mathrm{Tr}_K^G \mathrm{Res}_H^G = \sum_{Hx_iK} \mathrm{Res}^K_{H \cap K^{x_i}}(\mathrm{Tr}^H_{H \cap K^{x_i}}(c_{x_i}(u)))$$

where $K^{x_i} = x_i K x_i^{-1}$ and c_{x_i} is conjugation by x_i, and the sum is over double cosets. The useful fact here is that transfers from elementary abelian groups to proper subgroups are trivial, so this implies that if Λ' is a refinement of Λ such that Σ'_Λ is not conjugate to a group containing A_k then $\mathrm{Tr}_{\Sigma_{Lambda'}}^{\Sigma_\Lambda}(u) = 0$ for u pure. If Σ'_Λ is conjugate to a group containing A_k then the double coset formula gives us an expression of $\mathrm{Tr}_{\Sigma_{\Lambda'}}^{\Sigma_\Lambda}(u)$ as a sum of elements in the image of $\mathrm{Res}_{A_k}^{\Sigma_{\Lambda'}}$, so it's also pure.

Proposition 2.4 ([1, Lemma 3.11]). *I^* is acyclic for $D_* Id(S^{2d+1})$.*

Heuristically the idea is that I^* is the complex consisting of elements of $H_*(D_* Id(X))$ that involve at least one Lie bracket. The Lie bracket vanishes on the odd sphere, so if X is the odd sphere, the complex I^* should be acyclic.

Corollary 2.5 ([1, Theorem 3.13]). *If n is not a power of p then $D_n S^{2d+1} \simeq 0$.*

Corollary 2.6. *The E_2 term of the spectral sequence is given by $H^*(P^*)$.*

Definition 2.7 ([1, Definition 3.15]). *Let u be a generator of $H_{2d+1}(S^{2d+1})$. For fixed k let CU_* be the free graded $\mathbb{F}_p$ module on the following generators:*

$$\{\beta^{\epsilon_1} Q^{s_1} \cdots \beta^{\epsilon_k} Q^{s_k} u \mid |s_k \geq d,\ s_i > p s_{i+1} - \epsilon_{i+1}\} \qquad \text{if } p > 2$$
$$\{\beta^{\epsilon_1} Q^{s_1} \cdots \beta^{\epsilon_k} Q^{s_k} u \mid |s_k \geq 2d+1,\ s_i > 2 s_{i+1}\} \qquad \text{if } p = 2$$

Here CU stands for completely unadmissable, because these are sequences such that every pair of consecutive operations would satisfy an Adem relation in the Dyer Lashof algebra.

Theorem 2.8 ([1, Theorem 3.16]). *Let $n = p^k$. The cohomology of P^* is concentrated in degree k and there are isomorphisms of Steenrod modules*

$$H^k(P^*) \cong CU_* \cong \Sigma^k D_{p^k}(S^{2d+1})$$

Proof (Vague idea of proof). The top degree term of P^* looks like the pure part of $H_*(X^{\wedge n}_{h(\Sigma_p \wr \cdots \wr \Sigma_p)}$. This includes all external Dyer Lashof words of length k, regardless of admissibility. On the other hand, the bottom degree group of P^* is the pure part of $H_*(X^{\wedge n}_{h\Sigma_n})$ which contains only admissible Dyer Lashof words of length k. A counting argument shows that $H^k(P^*)$ contains a copy of CU_*, so the trick is to show that the differentials have as large as possible images.

References

[1] Greg Arone and Mark Mahowald. The goodwillie tower of the identity functor and the unstable periodic homotopy of spheres. *Inventiones mathematicae*, 135(3):743–788, 1999.

[2] Omar Antolín Camarena. The mod 2 homology of free spectral lie algebras. *arXiv preprint arXiv:1611.08771*, 2016.

[3] Brenda Johnson. The derivatives of homotopy theory. *Transactions of the American Mathematical Society*, 347(4):1295–1321, 1995.