JUVITOP: NILPOTENCE AND THE NISHIDA RELATIONS

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Remark 0.1. To learn more about this circle of ideas, I highly recommend *On a nilpotence conjecture of J.P. May* by Akhil Mathew, Niko Naumann, and Justin Noel. Also, Haynes Miller wrote excellent and easily googled notes deriving the full Nishida relations.

1. Theorems of Nishida and Mahowald

Last week, Hood discussed the **Dyer-Lashof algebra**, which acts on the homotopy of any E_∞-ring in $\mathrm{H}\mathbb{F}_2$-modules. Like Hood, I will focus on the prime $p = 2$ for simplicity, though everything I say has an analogue for odd primes. Today I would like to explain the following theorem, which was one of the earliest and most striking applications of power operations:

Theorem 1.1 (Nishida, 1973)**.** *Suppose $x \in \pi_*(\mathbb{S})$ for $* > 0$. Then x is nilpotent.*

What we will actually prove today is:

Theorem 1.2. *Suppose $x \in \pi_*(\mathbb{S})$ is simple 2-torsion, meaning $2x = 0$. Then x is nilpotent.*

Remark 1.3. Suppose α is an element in $\pi_{2k+1}(\mathbb{S})$. Since the stable homotopy groups of spheres form a graded-commutative ring, $\alpha^2 = -\alpha^2$. By the previous theorem, α is nilpotent.

Remark 1.4. Suppose α is an element in $\pi_{2k}(\mathbb{S})$. A simple argument with the Kahn-Priddy theorem shows that, if $2^m\alpha = 0$, then 2^{m-1} kills some large power of α. Assembling the analogous remarks at every prime recovers the full strength of Nishida's result. (In fact, this style of argument can work even when $m = 1$, circumventing most of the hard work in Nishida's paper. However, the explicit k for which $\alpha^k = 0$ is much more accurately estimated using that hard work.)

Armed with modern understanding, it is conceptually enlightening to recast Nishida's result as a corollary of the following theorem of Mahowald:

Theorem 1.5 (Mahowald)**.** *The free $\mathbb{E}_2$-algebra with $2 = 0$ is $H\mathbb{F}_2$.*

Remark 1.6. Most of this talk will be devoted to the explanation and proof of Mahowald's theorem. A crucial ingredient is his use of the so-called 'Nishida relations': hence the connection to Nishida's original proof of the nilpotence result.

Corollary 1.6.1. *Let A denote an $\mathbb{E}_2$-algebra (e.g., $A = \mathbb{S}$), and suppose that $x \in \pi_n(A)$ is simple 2-torsion. Then $x : S^n \to A$ is nilpotent if and only if the composite $S^n \xrightarrow{x} A \wedge \mathbb{S} \to A \wedge H\mathbb{F}_2$ is nilpotent.*

Proof. We will show that $A[x^{-1}]$ is 0 if and only if $A[x^{-1}] \wedge \mathrm{H}\mathbb{F}_2$ is 0. Notice that $A[x^{-1}]$ is an $\mathbb{E}_2$-algebra in which $2 = 0$. By Mahowald's theorem, $A[x^{-1}]$ receives an $\mathbb{E}_2$-algebra map from $\mathrm{H}\mathbb{F}_2$, and so $A[x^{-1}]$ is an $\mathrm{H}\mathbb{F}_2$-module spectrum.

Since $\mathbb{F}_2$ is a field, every $\mathrm{H}\mathbb{F}_2$-module splits as a wedge of suspensions of $\mathrm{H}\mathbb{F}_2$. Furthermore, $\pi_*(\mathrm{H}\mathbb{F}_2 \wedge \mathrm{H}\mathbb{F}_2)$ is not trivial (it is the dual Steenrod algebra). It follows that, if X is any $\mathrm{H}\mathbb{F}_2$-module, then X is 0 if and only if $X \wedge \mathrm{H}\mathbb{F}_2$ is 0. $\square$

2. A bit of $\mathbb{E}_2$-algebra theory

For the rest of the talk, we will discuss the proof of Mahowald's theorem:

Theorem 2.1 (Mahowald)**.** *The free $\mathbb{E}_2$-algebra with $2 = 0$ is $H\mathbb{F}_2$.*

Definition 2.2. We begin by recalling (and/or introducing) the objects in question:

- If X is a spectrum, then the free $\mathbb{E}_2$-algebra (in spectra) on X is given by
$$F_{\mathbb{E}_2}(X) \simeq \mathbb{S} \vee X \vee \left(X^{\wedge 2} \wedge_{\Sigma_2} \Sigma^\infty_+ \mathrm{Config}_2(\mathbb{R}^2)\right) \vee \left(X^{\wedge 3} \wedge_{\Sigma_3} \Sigma^\infty_+ \mathrm{Config}_3(\mathbb{R}^2)\right) \vee ...$$
- If X is a connected space, then a theorem of Snaith gives an equivalence
$$F_{\mathbb{E}_2}(\Sigma^\infty X) \simeq \Sigma^\infty_+ \Omega^2 \Sigma^2 X.$$
- For any spectrum X, the free $\mathbb{E}_2$-$\mathrm{H}\mathbb{F}_2$-algebra on X is $\mathrm{H}\mathbb{F}_2 \wedge F_{\mathbb{E}_2}(X)$.

Definition 2.3. The free $\mathbb{E}_2$-algebra with $2 = 0$ is given by the pushout (in the category of $\mathbb{E}_2$-algebras) of the diagram

$$\begin{array}{ccc} F_{\mathbb{E}_2}(\mathbb{S}) & \xrightarrow{\overline{2}} & \mathbb{S} \simeq F_{\mathbb{E}_2}(0) \\ \downarrow{\scriptstyle \overline{0}} & & \\ \mathbb{S} & & \end{array}$$

Here, $\overline{m} : F_{\mathbb{E}_2}(\mathbb{S}) \to \mathbb{S}$ denotes the adjoint to $m : \mathbb{S} \to \mathbb{S}$. Notice that $\overline{0} : F_{\mathbb{E}_2}(S^0) \to S^0$ is just $F_{\mathbb{E}_2}(0 : \mathbb{S} \to 0)$, but $\overline{2}$ is not $F_{\mathbb{E}_2}$ applied to any morphism of spectra.

Remark 2.4. It is difficult to compute a colimit of $\mathbb{E}_2$-algebras. The general strategy is to note that the forgetful functor $U : \mathbb{E}_2$-**algebras** $\to$ **Spectra** preserves sifted colimits. One can hope to decompose an arbitrary colimit into a series of coproducts and sifted colimits, but it is not always easy to compute coproducts in $\mathbb{E}_2$-algebras or sifted colimits in spectra.

We will use R to denote the free E_2-algebra with $2 = 0$ for the remainder of the talk. Our eventual goal is to compute $\pi_*(R)$, and to see that this is a single $\mathbb{F}_2$ concentrated in degree 0. It is much easier, however, to compute $H_*(R; \mathbb{F}_2)$.

Proposition 2.5. *$H\mathbb{F}_2 \wedge R \simeq H\mathbb{F}_2 \wedge F_{\mathbb{E}_2}(S^1)$.*

Proof. In the category of $\mathrm{H}\mathbb{F}_2$-modules twice the identity is nullhomotopic. It follows that

$$H\mathbb{F}_2 \wedge \overline{2} : H\mathbb{F}_2 \wedge F_{\mathbb{E}_2}(\mathbb{S}) \longrightarrow H\mathbb{F}_2 \wedge \mathbb{S}.$$

is equivalent to

$$H\mathbb{F}_2 \wedge \overline{0} : H\mathbb{F}_2 \wedge F_{\mathbb{E}_2}(\mathbb{S}) \longrightarrow H\mathbb{F}_2 \wedge \mathbb{S}.$$

Since free functors preserve pushouts, the pushout diagram defining $\mathrm{H}\mathbb{F}_2 \wedge R$ degenerates into

$$\mathrm{H}\mathbb{F}_2 \wedge F_{\mathbb{E}_2} \left(\begin{array}{ccc} \mathbb{S} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & S^1 \end{array} \right).$$

$\square$

Remark 2.6. Mahowald noticed that R is a Thom spectrum. The above proposition can be seen as an instance of the Thom isomorphism theorem.

By the universal property of R there is a natural map

$$R \to H\mathbb{F}_2.$$

We will show that this map is an equivalence. Since $\pi_*(R)$ is a graded-commutative ring in which $2 = 0$, all the homotopy groups of R are $\mathbb{F}_2$-algebras. It follows that R is 2-complete, so to show that $R \to H\mathbb{F}_2$ is an equivalence it suffices to show that

$$H\mathbb{F}_2 \wedge F_{\mathbb{E}_2}(S^1) \simeq H\mathbb{F}_2 \wedge R \to H\mathbb{F}_2 \wedge H\mathbb{F}_2$$

is an equivalence.

3. Dyer-Lashof Operations and Free $\mathbb{E}_n$-algebras

In light of the above, our goal should be to compute

$$H_*(F_{\mathbb{E}_2}(S^1); \mathbb{F}_2) \cong H_*(\Omega^2 S^3; \mathbb{F}_2).$$

In general, one might want to understand the homology of loop spaces of spheres. This computation is highly related to what Hood discussed last week.

Construction 3.1. Suppose A is an E_n-algebra and $x \in H_i(A; \mathbb{F}_2)$ for $i > 0$. Then one obtains another element $Qx \in H_j(A; \mathbb{F}_2)$ for every $Q \in H_j(\Omega^n S^{n+i}; \mathbb{F}_2)$.

Proof. We may represent x by a map $S^i \to \mathrm{H}\mathbb{F}_2 \wedge A$. Since $\mathrm{H}\mathbb{F}_2 \wedge A$ is an $\mathbb{E}_n$-algebra in $\mathrm{H}\mathbb{F}_2$-modules, we can use the free-forgetful adjunction to produce a map

$$\mathrm{H}\mathbb{F}_2 \wedge F_{\mathbb{E}_n}(S^i) \to \mathrm{H}\mathbb{F}_2 \wedge A.$$

Finally, we can compose with

$$Q : S^j \to \mathrm{H}\mathbb{F}_2 \wedge \Sigma^\infty_+ \Omega^n S^{n+i} \simeq \mathrm{H}\mathbb{F}_2 \wedge F_{\mathbb{E}_n}(S^i).$$

□

In other words, power operations for $\mathbb{E}_n$-algebras in $\mathrm{H}\mathbb{F}_2$-modules are controlled by the homologies of loop spaces of spheres. To compute these homologies, one can proceed inductively using Eilenberg-Moore spectral sequences. All of the relevant spectral sequences degenerate nicely on the E_2-page.

Remark 3.2. To prove Mahowald's theorem, one does not even need to know that the spectral sequences computing $H_*(\Omega^2 S^3)$ degenerate. The spectral sequences are only needed to establish an upper bound on the size of this homology.

A summary of the calculation, which should look familiar following Hood's talk, is contained in the following:

Theorem 3.3. *If A is an $\mathbb{E}_n$-algebra in $H\mathbb{F}_2$-modules, then there are operations $Q^r : \pi_*(A) \to \pi_{*+r}(A)$ for each $r \geq 0$ satisfying*

1. *For $i > 0$ and $x_i \in \pi_i(A)$, $Q^r(x_i) = 0$ when $r < i$ or $r > i + n - 1$*
2. *$Q^i(x_i) = x_i^2$.*
3. *The Q^r satisfy the Adem relations and Cartan formula.*

As n tends to ∞ we recover the action of the Dyer-Lashof algebra that Hood spoke about last week.

We can draw a picture of $H_*(F_{\mathbb{E}_2}(S^1); \mathbb{F}_2) \cong H_*(\Omega^2 S^3; \mathbb{F}_2) \cong H_*(R; \mathbb{F}_2)$. [DRAW PICTURE]

4. The Nishida relations and a proof of Mahowald's Theorem

To recap, there is a natural map

$$H_*(\Omega^2 S^3; \mathbb{F}_2) \cong H_*(R; \mathbb{F}_2) \to H_*(\mathrm{H}\mathbb{F}_2; \mathbb{F}_2),$$

which we are trying to show is an equivalence. The calculations summarized in the previous section show that both graded vector spaces have the same ranks, but it remains to understand why the map has no kernel.

Recall that, if X is any spectrum, $H_*(X; \mathbb{F}_2)$ receives a degree-lowering action on the right by the Steenrod algebra. If X is furthermore an E_n-algebra, then $H_*(X)$ is also acted on by Dyer-Lashof operations. The map $H_*(R) \to H_*(\mathrm{H}\mathbb{F}_2)$ must preserve both Steenrod operations and Dyer-Lashof operations. This gives two (highly related) strategies to finish the proof:

(1) Compute the Steenrod algebra action on $H_*(R;\mathbb{F}_2)$
(2) Compute the Dyer-Lashof algebra action on $H_*(H\mathbb{F}_2;\mathbb{F}_2)$

Both strategies lead to a proof. Pursuing the first strategy in its full generality leads to the following very useful relations:

Theorem 4.1 (Nishida relations)**.** *For A an $\mathbb{E}_n$-algebra, $t \geq k$, and $y \in H_*(A)$,*

$$(Q^t y)Sq^k = \sum_i \binom{t-k}{k-2i} Q^{t-k+i}(ySq^i).$$

Proof. See, e.g., Haynes Miller's notes. $\square$

Proof of Mahowald's Theorem. Since R is the free $\mathbb{E}_2$-algebra with $2 = 0$, it receives a map from the free spectrum with $2 = 0$, which is the mod 2 Moore spectrum. This means that the bottom two classes in $H_*(R;\mathbb{F}_2)$ are connected by a Sq^1. The Nishida relations then completely determine the Steenrod algebra action on $H_*(R;\mathbb{F}_2)$. The map $H_*(R;\mathbb{F}_2) \to H_*(\mathrm{H}\mathbb{F}_2)$ must preserve Steenrod operations, and it follows that the map is an isomorphism. $\square$

5. Dyer-Lashof operations on $H_*(\mathrm{H}\mathbb{F}_2;\mathbb{F}_2)$

Recall that $H_*(H\mathbb{F}_2) = \mathbb{F}_2[\xi_1, \xi_2, ...]$ where the degree of ξ_i is $2^i - 1$. The following result is enough to prove Mahowald's theorem:

Proposition 5.1. *For each $i \geq 1$, $Q^{2^i}(\xi_i) = \xi_{i+1}$*

Proof. Consider the map

$$x : \mathbb{RP}^\infty \to \Sigma H\mathbb{F}_2$$

which picks out the generator in cohomology. In homology, x_* sends $b_{2^i} \in H_{2^i}(\mathbb{RP}^\infty;\mathbb{F}_2)$ to ξ_i. There is a sequence of C_2-equivariant maps

$$\mathbb{RP}^\infty \xrightarrow{\Delta} \mathbb{RP}^\infty \times \mathbb{RP}^\infty \xrightarrow{(x,x)} \left(S^1 \wedge \mathrm{H}\mathbb{F}_2\right) \wedge \left(S^1 \wedge \mathrm{H}\mathbb{F}_2\right) \simeq S^{1+\sigma} \wedge \mathrm{H}\mathbb{F}_2 \wedge \mathrm{H}\mathbb{F}_2$$

Taking homotopy orbits, we get maps

$$\begin{aligned} \mathbb{RP}^\infty \times \mathbb{RP}^\infty &\xrightarrow{\Delta_{hC_2}} (\mathbb{RP}^\infty \times \mathbb{RP}^\infty)_{hC_2} \simeq BD_8 \\ &\to (S^{1+\sigma} \wedge \mathrm{H}\mathbb{F}_2 \wedge \mathrm{H}\mathbb{F}_2)_{hC_2} \\ &\simeq \Sigma^\infty \Sigma \mathbb{RP}^\infty \wedge (\mathrm{H}\mathbb{F}_2 \wedge \mathrm{H}\mathbb{F}_2)_{hC_2} \\ &\xrightarrow{m} \Sigma^\infty \Sigma \mathbb{RP}^\infty \wedge \mathrm{H}\mathbb{F}_2, \\ &\xrightarrow{x} \Sigma^2 \mathrm{H}\mathbb{F}_2 \wedge \mathrm{H}\mathbb{F}_2, \\ &\xrightarrow{m} \Sigma^2 \mathrm{H}\mathbb{F}_2 \end{aligned}$$

This fits into a commutative triangle

$$\begin{tikzcd} \mathbb{RP}^\infty \times \mathbb{RP}^\infty \arrow[r, "\Delta_{hC_2}"] \arrow[dr, "{(x,x)}"'] & BD_8 \arrow[d] \\ & \Sigma^2 \mathrm{H}\mathbb{F}_2 \end{tikzcd}$$

In homology, the top map in the above triangle will take $b_1 \otimes b_{2^{i+1}}$ to $b_1 \otimes b_{2^i} \otimes b_{2^i}$. By what Hood spoke about last time, this maps down to $Q^{2^i}(\xi_i)$. On the other hand, the third map in the triangle takes $b_1 \otimes b_{2^{i+1}}$ to ξ_{i+1}. $\square$