JUVITOP 19/11/14

The slice spectral sequence

Previously on Juvitop:

$\exists\, \theta_j \in \pi_{2^{j+2}-2} \mathbb{S}$?

Strategy: We produced a C_8 spectrum Ω_O, $\Omega = \Omega_O^{hC_8}$ s.t.

① Hurewicz detects θ_j (if $\theta_j \,\exists$ $\quad H(\theta_j) \underset{\neq 0}{\in} \Omega^{2-2^{j+1}}(pt)$)

② Ω is 256 periodic

③ Ω has gap $\pi_i \Omega = 0$ for $i = 1,2,3$

ⓐ $\pi_n^{C_8} \Omega_O \xrightarrow{\simeq} \pi_n \Omega_O^{hC_8}$

ⓑ $\pi_n^{C_8} \Omega_O = 0$ for $n = -1, -2, -3$

Sp^G = G-spectra

$[X, Y]^G$ = mapping sets in $ho(Sp^G)$

$\underline{Sp}^G(X,Y)$ = mapping G-spectrum (all maps + action by conjugation)

$(\underline{\pi_n} X)(B) = [S^n \wedge B_+, X]^G$ Mackey functor

For every Mackey functor M there's an Eilenberg-MacLane spectrum HM

$$\pi_* HM = \begin{cases} M & * = 0 \\ 0 & * \neq 0 \end{cases}$$

$H_*^G(X, \underline{M}) = \pi_*^G (H\underline{M} \wedge X)$

Equivariant cohomology of G-CW-complexes

If $X^{(0)} \hookrightarrow X^{(1)} \hookrightarrow \dots \hookrightarrow X^{(n)}$ is a G-CW complex

$$X^{(i)}/X^{(i+1)} \simeq (X_i)_+ \wedge S^i \qquad X_i \text{: G-set}$$

We set $\quad C^n_{cw}(X, \underline{M}) = [X_n, HM]^G$

+ natural boundary maps

Fact: $H^*(X; \underline{M}) = H^*(C_{cw}(X, \underline{M}))$

Let us now look at $\underline{\mathbb{Z}}(B) = \mathrm{Hom}_G(B, \mathbb{Z})$ Mackey functor.

$$C^n_{cell}(X, \underline{\mathbb{Z}}) = \mathrm{Hom}(X_n, \underline{\mathbb{Z}}) = \underbrace{\mathbb{Z}[X_n]}_{\text{free abelian group on } X_n}{}^G = C^n_{cell}(X; \mathbb{Z})^G = C^n_{cell}(X/G; \mathbb{Z})$$

$$\Rightarrow \boxed{H^*_G(X; \underline{\mathbb{Z}}) = H^*(X/G; \mathbb{Z})}.$$

(2) Equivariant cell decomposition of representation spheres

G finite, let ρ_G be the regular representation, let $\rho_G - 1$ be the summand of of ρ_G split by the copy of the trivial rep, and $(\rho_G - 1)_{|\cdot|=1}$ be the unit sphere in it

Goal: $H^*_G(S^{\rho_G}; \underline{\mathbb{Z}})$ vanishes for $* = 1, 2, 3$

Note: Boundary of the standard simplex in $\mathbb{R}^G$ is homeomorphic to $(\rho_G - 1)_{|\cdot|=1}$.

Problem: The std. decomposition is not equivariant.

Solution: Barycentric subdivision is equivariant

$$(\rho_G - 1)_{|\cdot|=1} \cong |\text{poset of nonempty proper subsets of } G|$$

Important: We can build $S^{\rho_G - 1}$ out of G-cells of the form $(G/H)_+ \wedge S^k$ for H proper and $1 \leq k \leq \# G - 1$

Thm: If $\# G \neq 1, 3$, $n > 0$

$$H^i_G(S^{n\rho_G}; \underline{\mathbb{Z}}) = 0 \quad \text{for } i = 1, 2, 3$$

Proof: $\underbrace{S^{\rho_G - 1}}_{\text{unreduced suspension}} = S(\rho_G - 1)_{|\cdot|=1}$ Assume $n=1$ for simplicity

$S^{\rho_G - 1}$ is simply connected (suspension of $(\rho_G - 1)_{|\cdot|=1}$ connected)

$\Rightarrow S^{\rho_G - 1}/G$ is simply connected (if $\# G \geq 3$ we're covered because $(\rho_G - 1)^{|\cdot|=1}/G$ is connected, if $\# G = 2$ we compute explicitly)

$$H^i_G(S^{\rho_G};\underline{\mathbb{Z}}) = H^{i-1}_G(S^{\bar\rho_G-1};\mathbb{Z}) = H^{i-1}(S^{\bar\rho_G-1}/G;\mathbb{Z}) = 0 \quad \text{if } i=1,2.$$

For $i=3$, by universal coefficient thm

$$H^3_G(S^{\rho_G};\underline{\mathbb{Z}}) = H^2(S^{\bar\rho_G-1}/G;\mathbb{Z}) \hookrightarrow H^2(S^{\bar\rho_G-1}/G;\mathbb{Q})$$

Rationally $X_{hG} \simeq X/G$ and the Serre spectral sequence implies

$$H^2(S^{\bar\rho_G-1}_{hG};\mathbb{Q}) \simeq \left(H^2(S^{\bar\rho_G-1};\mathbb{Q})\right)^G$$

and $S^{\bar\rho_G-1} = S^{|G|-1}$. $\square$

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③ Slice connectivity

Write $\hat{S}(m,K) = G_+ \wedge_K S^{m\rho_K}$ for all $K<G$

Def: Slice cells are spaces (spectra!) of the form

- $\hat{S}(m,K)$ regular, of dimension $(|K|)^m$
- $\Sigma^{-1}\hat{S}(m,K)$ dimension $(|K|)^m-1$

Def: • A G-spectrum X is n-connective if $\underline{\pi}_i(X)=0 \quad \forall i<n$

Write $Sp^G_{\boxed{\geq n}}$ for the full subcategory of n-connective G-spectra

(smallest category containing $G/H_+ \wedge \mathbb{S}^n$ closed under hty colims & extensions)

$Sp^G_{\boxed{<n}} = (Sp^G_{\boxed{\geq n}})^\perp \quad \left(= \{X \mid \forall Y \in Sp^G_{\geq n} \quad \underline{Sp}^G(X,Y) = *\}\right)$

• The category $Sp^G_{\geq n}$ of slice n-connective (slice $(n-1)$-positive) spectra is the smallest category containing all slice cells of dim $\geq n$ and closed under colims & extensions.

④ Comparing connectivity

Lemma: $Sp^G_{\boxed{\geq 0}} = Sp^G_{\geq 0}$, $Sp^G_{\boxed{\geq -1}} = Sp^G_{\geq -1}$

Pf: The slice cells $\geq 0\,(-1)$ live in $Sp^G_{\geq 0(-1)}$, and S^0 lives in $Sp^G_{\geq -1}$. $\square$

Lemma: $X \in Sp^G_{\geq 0}$, $Y \in Sp^G_{\geq m} \Rightarrow X \wedge Y \in Sp^G_{\geq m}$.

Q: How connective are slice cells?

Ans: ① If $\hat{S}$ is a slice cell of dimension $n>0 \Rightarrow \hat{S} \in Sp^G_{\boxed{\geq \lfloor \frac{n}{\#G} \rfloor}}$

If $\hat{S}$ " " " $n<0 \Rightarrow \hat{S} \in Sp^G_{\boxed{\geq n}}$

Proof: $S^{\rho_G - 1}$ has a cell decomposition w/ cells of dim $0, \dots, \#G-1$

$\Rightarrow S^{\rho_G}$ " " $1, \dots, \#G$

$\Rightarrow m \geq 0$ $S^{m\rho_G}$ " " $m, \dots, m\#G$

$\Rightarrow S^{m\rho_G} \in Sp^G_{\boxed{\geq m}} = Sp^G_{\boxed{\geq \lfloor \frac{n}{\#G} \rfloor}}$

The rest is similar by inducing along subgroups. □

Corollary: For $n \geq 0$ $\quad Sp^G_{\geq n} \subseteq Sp^G_{\boxed{\geq \lfloor \frac{n}{\#G} \rfloor}}$

$n \leq 0$ $\quad Sp^G_{\geq n} \subseteq Sp^G_{\boxed{\geq n}}$

Using that all cells in the decomp. of $\rho_G - 1$ are isotropic (= nonfree) one shows:

Thm: For $n \geq 0$ $\quad Sp^G_{\boxed{\geq n}} \subseteq Sp^G_{\geq n}$

$n < 0$ $\quad Sp^G_{\boxed{\geq n}} \subseteq Sp^G_{\geq (n+1)(\#G-1)}$.

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⑤ Slice spectral sequence

One can show that there are unique functors $P_{n+1}: Sp^G \to Sp^G_{\geq n}$ + functorial fiber sequences

$P^n: Sp^G \to Sp^G_{\leq n}$

$$P_{n+1}X \to X \to P^n X$$

$\Rightarrow$ We can form a tower

$$X \to \dots \to P^n X \to P^{n-1} X \to \dots$$

The slice spectral sequence is the sequence associated to this tower

$$E_2^{s,t} = \underline{\pi}_{t-s}\left(\mathrm{fib}\left(P^t X \to P^{t-1} X\right)\right) \Rightarrow \underline{\pi}_{t-s} X$$

<u>Convergence thm:</u> If $n \geq 0$ $\left\lfloor \frac{n+1}{|G|} \right\rfloor > k \Rightarrow P_{n+1} X \in \mathrm{Sp}^G_{\geq n+1} \subseteq \mathrm{Sp}^G_{\boxed{\geq \lfloor \frac{n+1}{|G|} \rfloor}} \subseteq \mathrm{Sp}^G_{\boxed{\geq k+1}}$

$\Rightarrow \pi_k X \xrightarrow{\simeq} \pi_k P_n X$

$\Rightarrow X \xrightarrow{\simeq} \varprojlim P_n X$.

<u>Thm (Vanishing thm):</u> If X is a G-spectrum, $M = \mathrm{fib}(P^n X \to P^{n-1} X)$

$\Rightarrow \pi_k M = 0$ if k is outside $\left\lfloor \frac{n}{|G|} \right\rfloor \leq k \leq n$ for $n \geq 0$

$n \leq k \leq \left\lfloor \frac{n+1}{|G|} \right\rfloor$ for $n < 0$

This shows that E_2-term of the slice spectral sequence vanishes outside the marked region