SUVITOP 12/11/14

Fixed points

Let X be an orthogonal G-spectrum

Def: The naive fixed point spectrum is $(X^G)_n = X_n^G$

⚠ This notion is not homotopically correct unless X is a G-Ω-spectrum

Def: The fixed point spectrum is

$$(F^G X)_n = \mathrm{Map}_G(S^{n\bar\rho_G}, X(n\rho_G))$$

ρ_G = regular repr. $\bar\rho_G \oplus 1 = \rho_G$

This is homotopically invariant & gives a model for the derived functor of X^G

Structure maps

$$\mathrm{Map}_G(S^{n\bar\rho_G}, X(n\rho_G)) \wedge S^1 \xrightarrow{\sim} \mathrm{Map}_G(S^{n\bar\rho_G}, X(n\rho_G)\wedge S^1)$$

$$\xrightarrow{\wedge S^{\bar\rho_G}} \mathrm{Map}_G(S^{(n+1)\bar\rho_G}, X(n\rho_G)\wedge S^{\rho_G}) \longrightarrow \mathrm{Map}(S^{(n+1)\bar\rho_G}, X((n+1)\rho_G))$$

Properties: I) $\pi_*^G X = \pi_* F^G X$

Pf: $[S^{n\rho_G}, X(n\rho_G)]^G = [S^n, \mathrm{Map}(S^{n\bar\rho_G}, X(n\rho_G))]^G$ & take the limit

II) F^G commutes w/ htpy limits & colimits

III) $F^G(X)\wedge F^G(Y) \to F^G(X\wedge Y)$

Failures:
- $F^G\Sigma^\infty X \neq \Sigma^\infty(X^G)$
- If $i : Sp \to Sp^G$ inclusion of spectra w/ trivial action $F^G_i \neq id$

Q: How to compute F^G? π_*^G?

Answer I) Approximate by homotopy fixed pts $X^{hG} = \operatorname{holim}_{BG} X$

$$X^{hG} = F^G(X^{EG_+})$$

$$\pi_*^G X \to \pi_*^G(X^{EG_+}) = \pi_* X^{hG}$$

and we have a spectral sequence $H^*(G; \pi_* X) \Rightarrow \pi_*(X^{hG})$.

Answer II) Approximate by geometric fixed points

Def: The geometric fixed point spectrum

$$\Phi^G(X) := X(n\rho_G)^G$$

$$X(n\rho_G)^G \wedge S^1 \simeq (X(n\rho_G)\wedge S^{\rho_G})^G \longrightarrow X((n+1)\rho_G)^G$$

We have an evaluation map $F^G X \to \Phi^G X$

$$f \longmapsto f(0) \quad 0 \in S^{n\bar\rho_G}$$

Formula for Φ^G in terms of F^G.

$E\mathcal{P}$ = G-space given by $(E\mathcal{P})^H = \begin{cases} * & H \neq G \\ \emptyset & H = G \end{cases}$

$E\mathcal{P}_+ \to S^0 \to \widetilde{E\mathcal{P}}$ cofiber sequence

$$\left.\begin{array}{l} (\widetilde{E\mathcal{P}} \wedge A)^G = A^G \\ (\widetilde{E\mathcal{P}} \wedge A)^H = * \quad H \neq G \end{array}\right\} \forall\, G\text{-space } A$$

Lemma: $\mathrm{Map}_G(B, \widetilde{E\mathcal{P}} \wedge A) \xrightarrow{\sim} \mathrm{Map}(B^G, A^G)$ is an equivalence

$\forall A$ G-space

B G-cw-complex

$\Rightarrow \forall$ G-spectrum E $\quad F^G(\widetilde{E\mathcal{P}} \wedge E) \xrightarrow{\sim} \Phi^G(\widetilde{E\mathcal{P}} \wedge E) \simeq \Phi^G(E)$

because it is a levelwise equivalence.

$\Rightarrow \Phi^G$ is homotopical and we have a cofiber seq.

$$F^G(E\mathcal{P}_+ \wedge E) \longrightarrow F^G E \longrightarrow \Phi^G E$$

Properties of Φ^G:

i) $\Phi^G(\Sigma^\infty A) = \Sigma^\infty A^G$

ii) Φ^G commutes w/ homotopy colimits

iii) Φ^G is strong symm. monoidal

$$\Phi^G(X) \wedge \Phi^G(Y) \xrightarrow{\sim} \Phi^G(X \wedge Y)$$

iv) If X is a spectrum $X \simeq \Phi^G(iX)$

v) $\Phi^H(X) = \Phi^G(N_H^G X)$

Thm (tom Dieck splitting):

$$F^G \Sigma^\infty A = \prod_{\substack{H \in \text{conj} \\ \text{classes of subgp.}}} \Sigma^\infty (E(WH)_+ \wedge_{WH} A^H) \qquad WH = NH/H$$

$$\pi_*^G (\Sigma^\infty A) \simeq \prod \pi_*^{WH} (EWH_+ \wedge_{WH} A^H)$$

Ex $A = C_2$ we get the Adams isomorphism $F^G(EC_{2+} \wedge \Sigma^\infty A) \simeq \Sigma^\infty A_{hC_2}$

Ex: $A = S^0$ $\pi_*^G \mathbb{S} = A(G) \leftarrow$ group completion of finite G-sets under $\amalg$

Prop: $\forall f: X \to Y$ map of G-spectra, TFAE

1. f is a π_*-isomorphism
2. $\forall H < G$ $f: F^H X \xrightarrow{\sim} F^H Y$
3. $\forall H < G$ $f: \Phi^H X \xrightarrow{\sim} \Phi^H Y$

Pf: ① $\Leftrightarrow$ ② by definition

② $\Rightarrow$ ③ $\qquad \widetilde{EP} \wedge X \to \widetilde{EP} \wedge Y$ is still a π_*-isom.

③ $\Rightarrow$ ② induction on $|G|$. $|G| = 1$ it's obvious.

Suppose now then that $\Phi^H X \xrightarrow{\sim} \Phi^H Y$ $\forall H < G$

Then $EP_+ \wedge X \to EP_+ \wedge Y$ is a π_*-isomorphism

Since $EP_+ \to S^0$ is an H-equivalence $\Rightarrow$ $F^H X \xrightarrow{\sim} F^H Y$ $\forall H \neq G$

$Z = \text{fiber}(X \to Y)$ $\Rightarrow \pi_*^H Z = 0$ $\forall H \neq G$

Then $(EP_+ \wedge Z(V))^H = *$ $\forall H$ ✓ (since $(EP_+)^G = *$)).

$$\begin{tikzcd}
F^G(EP_+ \wedge X) \arrow[r] \arrow[d, "?"] & F^G X \arrow[r] \arrow[d] & \Phi^G(X) \arrow[d, "?"] \\
F^G(EP_+ \wedge Y) \arrow[r] & F^G Y \arrow[r] & \Phi^G Y
\end{tikzcd}$$

✓ □

Prop: For a G-spectrum X, TFAE

(i) $\forall H < G$ nontrivial $\Phi^H X = *$

(ii) $EG_+ \wedge X \to X$ equivalence ($\Leftrightarrow \widetilde{EG} \wedge X = 0$)

This gives a method for showing when $F^G X \xrightarrow{\sim} F^G(X^{EG_+})$. Namely when $\Phi^H(X)=0$ $\forall H$ nontrivial.

Pf:

$$\begin{tikzcd} EG_+\wedge X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \widetilde{EG}\wedge X \simeq * \arrow[d] \\ EG_+\wedge X^{EG_+} \arrow[r, "\sim"] & X^{EG_+} \arrow[r] & \widetilde{EG}\wedge X^{EG_+} \simeq * \end{tikzcd}$$

And $EG_+\wedge X \to EG_+\wedge X^{EG_+}$ is an eq. because when you apply Φ^H I've freed up the action $\Rightarrow$ it is an eq on Φ^H for $H\neq e$ and on Φ^e because $EG_+\sim S^0$ there. □

This proposition is applied to show that the HHR detection spectrum Ω_O

$$F^{C_8}(\Omega_O)\simeq \Omega_O^{hC_8}$$

The strategy is $\Phi^H(\Omega_O)=0$.

Let $a_V\in \pi^G_{-V}\mathbb{S}=\pi^G_0 S^V$ given by $S^0\hookrightarrow S^V$

$S^{\infty\bar\rho_G}=\underrightarrow{\mathrm{colim}}_n\, S^{n\bar\rho_G}$ is a model for $\widetilde{EP}$

In fact $\bar\rho_G$ has no nontrivial G-fixed pts but $(\bar\rho_G)^H\neq 0$ $\forall H<G$.

$$\widetilde{EP}\wedge X=\underrightarrow{\mathrm{colim}}_n\, S^{n\bar\rho_G}\wedge X$$

$$\Phi^G(X)=\underrightarrow{\mathrm{colim}}_n\, F^G(S^{n\bar\rho_G}\wedge X)$$

and $S^{n\bar\rho_G}\wedge X\to S^{(n+1)\bar\rho_G}\wedge X$ is given by $a_{\bar\rho_G}$

$$\pi_\star\Phi^G(X)=(\pi^G_\star X)[a^{-1}_{\bar\rho_G}]$$

RO(G)-graded htpy groups

$G=C_2$, $\bar\rho_G=\sigma$ and let's consider real K-theory KR

$a_\sigma\in\pi^{C_2}_{-\sigma}(\mathbb{S})$

Assume $F^{C_2}(KR) = (KR)^{hC_2} \simeq KO$

and the underlying spectrum is KU

Moreover we have Bott periodicity $KR \simeq \Sigma^{\rho_{C_2}} KR$

$1 \in \pi_0^{C_2} KR$

$a_\sigma \in \pi_{-\sigma}^{C_2} KR \simeq \pi_1 KO = \mathbb{Z}/2 \langle \eta \rangle \qquad a_\sigma = \eta$

$\Phi^{C_2}(KR) = \pi_\star^{C_2} KR[\eta^{-1}] = 0$ because η is nilpotent

$\Rightarrow \Phi^{C_2}(KR) = 0$