$\boxed{\text{JUVITOP} \quad 03/11/23}$

Thm: h_j^2 is not a permanent cycle for $j>6$

$$\begin{tikzcd} E_2(MU) \arrow[r, "i"] \arrow[d] \arrow[dr, Rightarrow] & H^*(C_8; \pi_* L) \arrow[d, Rightarrow] \\ E_2(H) \Rightarrow \pi_* \mathbb{S} \arrow[r] & \pi_*(L^{hC_8}) \end{tikzcd}$$

ANSS

ASS

htpy fixed pt SS

(1) Detection theorem

In the appropriate dimensions $\ker i \subseteq \ker \mu$ and $\pi_{odd}(L) = 0$

If h_j^2 is a permanent cycle detecting θ_j, then there's $x \in E_2(MU)$ detecting θ_j

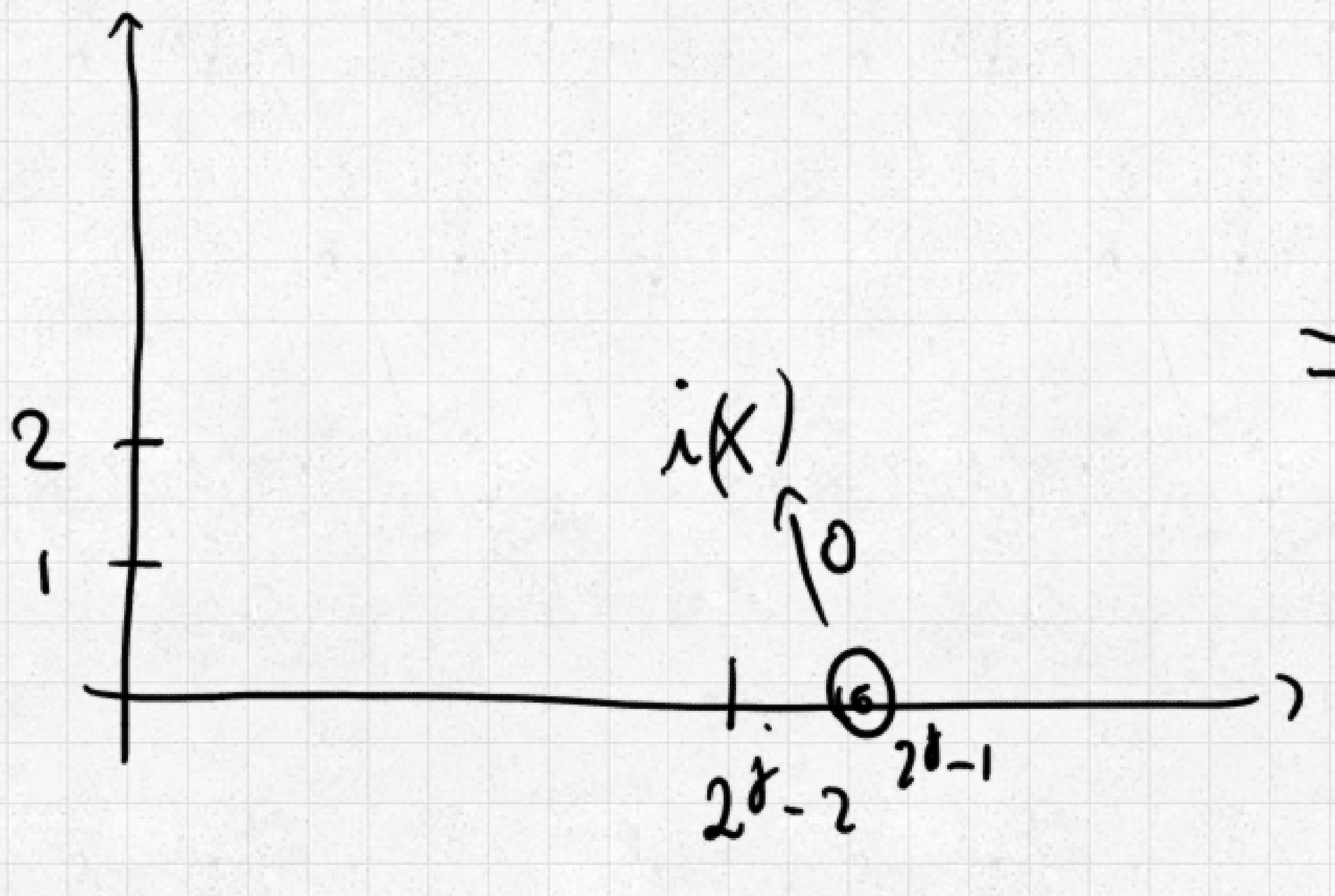
$\Rightarrow i(x) \neq 0$ permanent cycle (and it can't be hit by a differential since $\pi_{odd} L = 0$)

(2) Gap theorem $\pi_{-2} L = 0$

Periodicity theorem $\pi_k L^h \simeq \pi_{k+256} L^h$

Fixed pt $L \longrightarrow L^h$ is an equivalence

$\Rightarrow \pi_{2^j-2}(L^h) = 0 \ \forall j \geq 8$



$\Rightarrow$ no differential can hit $i(x)$

Recall: γ tautological bundle on BU $\Rightarrow MU = BU^\gamma$

We can equip MU w/ a C_2 action making $MU_{\mathbb{R}}$ (using complex conjugation)

But we want a C_8-space!

If X is an H-object we can induce up it to $\coprod_{|G/H|} X$ = G-object

For G-spectra we can use $\wedge$ to "multiplicatively induce" it $N_H^G(X)$

So we can form
$$MU^{((4))} = N_{C_2}^{C_4}(MU) \quad (C_\infty\text{-spectrum})$$
and $L = D^{-1}(MU^{((4))})$ (suitable localization)

Homotopy fixed point spectral sequence

$EG = B(*, G, G)$

$(EG)_n = G^n \times G$

$\Rightarrow$ We can form a cosimplicial spectrum $C^n(G, X) = \mathrm{Map}^G((EG)_n, X)$

$\mathrm{Tot}\left(C^n(G,X)\right) = X^{hG}$

This cosimplicial spectrum induces a SS w/ $E_2 = H^*(G; \pi_* X)$

Formal group laws

<u>Def</u>: A FGL / R ring is a power series $F \in R[[x,y]]$

- $F(x,y) = F(y,x)$
- $F(x, F(y,z)) = F(F(x,y), z)$
- $F(x,y) = x + y + \text{stuff}$
- $F(0,x) = x = F(x,0)$

<u>Quillen</u>: $(\text{Maps } MU_* \to R) \overset{1:1}{\longleftrightarrow} (FGL/R)$

<u>Strict isomorphisms of FGL</u>: Power series $\varphi = x + b_1 x^2 + \dots$ w/ $\varphi^{-1} F(\varphi x, \varphi y) = G(x,y)$

Power series like that are represented by $\mathbb{Z}[b_1, \dots]$

$\Rightarrow$ A map of Hopf algebras $(MU_*, MU_*MU) \longrightarrow (R; ?)$ is a FGL on R w/ a strict isom.

We want to consider the Hopf algebroid $R^G \rightrightarrows R$

$(g \mapsto r) \longleftarrow r$

$(g \mapsto gr) \longleftarrow r$

So we want maps $(MU_*, MU_*MU) \longrightarrow (R, R^G)$

Let $A = \mathbb{Z}_2[\zeta_8]$ ⟵ primitive 8th root of unity

$R = A[u^{\pm 1}]$ $\quad |u| = 2$

$G = C_8$ and the map is given by

$$F = u^{-1}\ell^{-1}\big(\ell(ux) + \ell(uy)\big) \qquad \ell(x) = \sum_{i=0}^{\infty} \frac{x^{2^i}}{\zeta - 1}$$

Let $\theta(t) = \zeta^{-1} u^{-1} [\zeta](ut)$

θ induces an action of C_8 on (R, F) by strict isomorphisms

$(MU_*^{(n)}, MU_*^{(n)}MU) \longrightarrow (R, R^G)$ classify FGL w/ an action of G by strict iso such that if $\gamma^n F = -F(-x,-y)$

$$\gamma^{2(n-1)}\theta \circ \dots \circ \theta = [-1]$$

So finally we get a map of Hopf algebroids $(MU_*^{(4)}, MU_*^{(4)}MU) \longrightarrow (R, R^G)$

So we have a map of SS

$$\begin{tikzcd}
\bar{\imath} : E_2(MU) \arrow[r] \arrow[dd, Rightarrow] & E_2(MU^{(4)}) \arrow[r] \arrow[d, phantom, "\wr"] & H^*(C_8, R) \arrow[dd, Rightarrow] \\
& H^*(C_8; MU_*^{(4)}) \arrow[d, Rightarrow] & \\
\pi_* \mathbb{S} \arrow[r] & \pi_*(MU_*^{(4)})^{C_8} & ?
\end{tikzcd}$$

We want to show $\ker \mu \supseteq \ker \bar{\imath}$.

Slice cells

Let $S_{>m}$ be the category of m-connected spectra $\Rightarrow P^m$ (the m-th Postnikov piece) is the localization wrt $S_{>m}$

Let's say that E is $S_{>m}$-null if $\mathrm{Map}(E, F) = *$ $\forall F \in S_{>m}$

- $\forall X$ $P^m X$ is $S_{>m}$-null
- $\forall X$ $\forall Y$ $S_{>m}$ null $\quad \mathrm{Map}(P^m X, Y) \simeq \mathrm{Map}(X, Y)$

The idea is to replace $S_{>m}$.

We replace $\{S^m\}_{m>0}$ with regular slices

- $\hat{S}(k\rho_H) = G_+ \wedge_H S^{k\rho_H}$ ← regular repres. of H
- $\Sigma^{-1}\hat{S}(k\rho_H)$

$\Big\}$ slice cells

So we have a slice filtration

$$\begin{tikzcd} P^0 & {}^G P^1 X \arrow[l] & {}^G P^2 X \arrow[l] & \cdots \arrow[l] \\ \arrow[u] & {}^G P^1_1 X \arrow[u] & {}^G P^2_2 X \arrow[u] & \cdots \end{tikzcd}$$

Out of this tower of filtrations we get a SS, called slice spectral sequence

$$E_1^{s,t} = \pi_{t-s}\, {}^G P^t_t \Rightarrow \pi_{t-s} X$$

- ${}^G P^{-1}_{-1} X = H\pi_{-1} X$
- ${}^G P^0_0 X = HM$

If X is n-slice $\underline{\pi}_k X = 0$ outside of the range $\left\lfloor \frac{n}{|G|} \right\rfloor \le k < n$ if $n \geq 0$

$n \le k \le \left\lfloor \frac{n+1}{|G|} \right\rfloor$ $n<0$

Def: X is pure if $P^t_t X = H\underline{\mathbb{Z}} \wedge$ (wedge of t-dim slice cells)

Purity theorem: $MU^{((4))}$ is pure.

$$S^{\ell\rho_8} \xrightarrow{D} MU^{((4))}$$

$$MU^{((4))} \to S^{-\ell\rho_8} \wedge MU \to \cdots$$

Slice lemma: For every regular G-isotropic $\hat{S}$

$$\underbrace{H^G_{-1,-2,-3}(\hat{S}, \underline{\mathbb{Z}})}_{\pi^{C_8}_*(H\underline{\mathbb{Z}} \wedge \hat{S})} = 0$$