JUVITOP 29/10/14

Basics of equivariant htpy thry

Let G be a finite group.

Q: What is a G-spectrum?

Not: A spectrum w/ G-action ⚠

What it is is this + a lot of additional data.

Loosely speaking: Spaces → Spectra is obtained by inverting all spheres.

For G-spectra: start w/ G-spaces, invert all representation spheres S^ρ (one pt. compactification of ρ real G-representation).

Why? Without doing this the HHR detecting spectrum Ω doesn't even exist.

Orthogonal spectra

Def: An orthogonal spectrum E is an assignment

$$\left\{ \begin{matrix} \text{real orthogonal spaces} \\ \text{of finite dimension} \end{matrix} \right\} \longrightarrow \text{Spaces}$$

$$V \longmapsto E(V)$$

w/ an action of $O(V)$ on $E(V)$, such that for every isometric embedding $i: V \to W$, we get a $O(V) \times O(i(V)^\perp)$-equivariant map

$$E(V) \wedge S^{i(V)^\perp} \longrightarrow E(W)$$

which is "continuous in i"

Ex: Say exactly what you mean by this.

G-spectra are defined in a similar way

Def: An orthogonal G-spectrum

$$E: \{\text{orthogonal } G\text{-vector spaces}\} \longrightarrow (G\text{-Spaces})$$

and $O(V) \circlearrowright E(V)$ ⚠ $O(V) = \mathrm{Isom}(V,V)$, no reference to the G-action whatsoever

For every isometric inclusion $V \hookrightarrow W$ we have $E(V) \wedge S^{i(V)^\perp} \longrightarrow E(W)$

For all $g \in G$

$$\begin{tikzcd} E(V) \wedge S^{i(V)^\perp} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & E(W) \arrow[d] \\ E(V) \wedge S^{g(i(V))^\perp} \arrow[r] & E(W) \end{tikzcd}$$

⚠ i needs not to be G-equivariant!

If V is a G-subrepresentation, we're just saying that $E(V) \wedge S^{i(V)^\perp} \to E(W)$ is equivariant.

Q: What are the weak equivalences?

Let's answer first for G-spaces.

Let X be a G-space

Def: $\forall H < G \quad \pi_n^H(X) := \pi_n(X^H) = [S^n \wedge G/H_+, X]^G$ (htpy classes of eq. maps)

A map $X \to Y$ is a weak equivalence if $\pi_n^H(X) \xrightarrow{\sim} \pi_n^H(Y) \quad \forall H < G, \forall n$.

⚠ $X \to Y$ is equivariant & a w.e. of underlying spaces it does not have to be a w.e. of G-spaces. Ex: $EG \to *$

What about spectra?

Let E be a G-spectrum, $H < G$, $n \in \mathbb{Z}$

$$\pi_n^H(E) = \operatorname{colim}_V \pi_{n+V}^H(E(V)) = \operatorname{colim}_V [S^{n \oplus V} \wedge G/H_+, E(V)]^G$$

Def: A map $E \to F$ is a w.e. if $\forall H < G, \forall n \in \mathbb{Z}$

$$\pi_n^H(E) \xrightarrow{\sim} \pi_n^H(F)$$

Q: What structure do we find on $\pi_*^\bullet(E)$?

Mackey functors

Let X be a G-space. What acts on $\pi_*^H(X)$?

If $x \in X^H$ and $g \in N_G(H) \Rightarrow gx \in X^H$. The kernel of this action contains H-itself

WH = Weyl group $= N_G(H)/H$

WH acts on $\pi_n^H(E)$ also for any spectrum E.

$$\pi_n^H(E) = [\Sigma_+^{\infty+n}(G/H), E]^G$$

Suppose $K < H < G$ $\Rightarrow$ we have $G/K \to G/H \Rightarrow \Sigma_+^\infty G/K \to \Sigma_+^\infty G/H$

inducing $F_K^H : \pi_n^H(E) \to \pi_n^K(E)$ (<u>restriction map</u>)

We also have $\Sigma_+^\infty G/K \to \Sigma_+^\infty G/H$

<u>Ex</u>: $H = G$, $K = (e)$. We need a map $S^n \to \bigvee_{g \in G} S^n$ (pinch enough times) that is equivariant up to coherent htpy.

This gives rise to $V_K^H : \pi_n^K(X) \to \pi_n^H(X)$

Moreover $L, K < H$ we could form

$F_L^H V_K^H$ = double coset formula (see Mackey's thm)

$F_e^{C_2} V_e^{C_2} : \pi_n^e(E) \to \pi_n^e(E) \circlearrowleft C_2$

$= id + \tau$

<u>Def</u>: A Mackey functor M (for G) is
- $\forall H < G$ $M^H \circlearrowleft WH$ abelian grp
- $\forall K < H$ $F_K^H : M^H \to M^K$, $V_K : M^H \to M^K$ satisfying appropriate compatibilities

<u>Ex</u>: $\pi_n^\bullet(E)$

<u>Ex</u>: Let A be an abelian group w/ G-action. The fixed point Mackey functor M_A is
$\Rightarrow$ $H \mapsto A^H$

$F_K^H : A^H \to A^K$ inclusion of fixed pts

$V_K^H(a) = \sum_{g_i \in H/K} g_i a$

<u>Important principle</u>

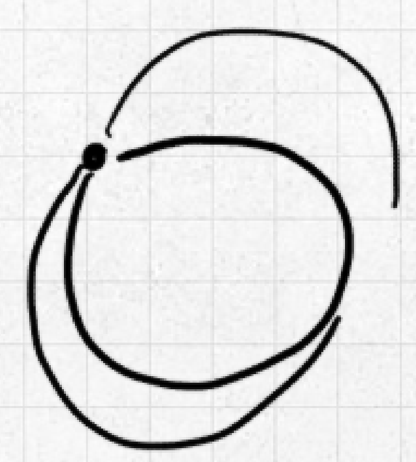
Mackey functors are the coefficients for G-stable htpy theory

<u>Ex</u>: For any Mackey functor M we can build an Eilenberg-MacLane spectrum HM

$\pi_0 HM = M$, $\pi_i HM = 0$ for $i \neq 0$.

Compute $H^*(S^2; M)$ where S^2 is acted by C_2 by reflection on the equator

Decompose S^2



The 0,1-cell are fixed,
the 2-cells are permuted

$$\partial D^2 \times C_2/e \longrightarrow S^1 \quad \text{attaching map}$$

$$C^n(X; M) = \bigoplus_i M(H_i)$$

set of n-cells $\coprod_i G/H_i$

$$0 \to M(C_2) \xrightarrow{0} M(C_2) \xrightarrow{F_e^{C_2}} M(e) \to 0 \to 0 \dots$$

$$H^0(S^2; M) = M(C_2)$$
$$H^1(S^2; M) = \ker F_e^{C_2}$$
$$H^2(S; M) = \operatorname{coker} F_e^{C_2}$$