JUVITOP 22/10/16

Ravenel's work at odd primes

A^* = dual Steenrod algebra

$p=2$ $\quad A^* = \mathbb{F}_2[\xi_1, \xi_2, \dots]$ $\qquad |\xi_n| = 2^n - 1$

p odd $\quad A^* = E[\tau_0, \tau_1, \dots] \otimes P[\xi_1, \xi_2, \dots]$ $\qquad |\xi_n| = 2p^n - 2$ $\quad |\tau_n| = 2p^n - 1$

$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ $\qquad |v_n| = 2p^n - 2$

$BP_*BP = BP_*[t_1, t_2, \dots]$ $\qquad |t_n| = 2p^n - 2$

At $p=2$ $\quad \exists\, h_j = [\xi_1^{2^j}] \in E_2^{1,2^j}(\mathbb{S}, H\mathbb{F}_2)$

Kervaire invariant problem $\Leftrightarrow$ Is h_j^2 a permanent cycle?

$p>2$ $\quad \exists\, h_j = [\xi_1^{p^j}] \in E_2^{1,(2p-2)p^j}(\mathbb{S}, H\mathbb{F}_p)$

$$h_j^2 = h_j \cdot h_j = -h_j^2 \Rightarrow \boxed{h_j^2 = 0}$$

We can define $b_i = \underbrace{\langle h_i, \dots, h_i \rangle}_{p\text{-times}}$ iterated Massey product

$$b_i = \sum_{j=1}^{p-1} \frac{1}{p}\binom{p}{j} [\xi_1^{jp^i} \,|\, \xi_1^{(p-j)p^i}] \in E^{2,(2p-2)p^{j+1}}(\mathbb{S}, H\mathbb{F}_p)$$

For $p=2$ $\quad b_i = h_i^2$.

<u>FACTS:</u>
- b_0 is a permanent cycle, detecting a class $\pi_{2p^2-2p-2}\mathbb{S}$ (which is the first nontrivial element in coker J)
- Toda proved $d_{2p-1} b_1 = h_0 b_0^p$

b_2? At $p=3$ b_2 is a permanent cycle.
$p \geq 5$?

<u>Thm</u> (Ravenel) For $p \geq 5$, $i \geq 1$ b_i is <u>not</u> a permanent cycle

At $p=3$ it is an open problem.

Proof strategy: The classes we care about live in

$$b_i \in E_2^{2,(2p-2)p^{i+1}}(\mathbb{S}, H\mathbb{F}_p)$$

Suppose b_i is a permanent cycle $\Rightarrow$ it detects some htpy class of Adams filtration 2
(b_i cannot be hit by a diff $\Rightarrow$ if it is a perm cycle it is nontrivial)

We have a map of SS

$$E_2^{2,(2p-2)p^{i+1}}(\mathbb{S}, H\mathbb{F}_p) \longleftarrow E_2^{2,(2p-2)p^{i+1}}(\mathbb{S}, BP)$$

$\Rightarrow$ it must lie in Novikov filtration ≤ 2. Moreover by sparseness of the ANSS
$\Rightarrow$ Novikov filtration 2 $\Rightarrow$ b_i has a lift which is a permanent cycle

We have β_{p^i/p^i} which is a lift of b_i. This is defined by the cobar cycle

$$\sum_{j=1}^{p^{i+1}-1} \frac{1}{p}\binom{p^{i+1}}{j} [t_1^j | t_1^{p^{i+1}-j}]$$

We have SES of BP_*-comod.

$$0 \to BP_* \xrightarrow{p} BP_* \to BP_*/p \to 0 \qquad \delta_0, \delta_1 \text{ conn homom.}$$

$$0 \to BP_*/p \xrightarrow{v_1^{p^i}} BP_*/p \to BP_*/(p, v_1^{p^i}) \to 0$$

$$\beta_{p^i/p^i} = \delta_0 \delta_1 (v_2^{p^i})$$

Now
$$E_2^{2,(2p-2)p^i}(\mathbb{S}, BP) = \mathbb{F}_p\langle \beta_{p^i/p^i}\rangle \oplus \underbrace{\bigoplus \mathbb{F}_p\langle \beta_{\dots/\dots}\rangle}_{\text{goes to 0 in } E_2(\mathbb{S}, H\mathbb{F}_p)}$$

$\Rightarrow$ lots of lifts.

$$E_2^{2,(2p-2)p^{i+1}}(\mathbb{S}, BP) = \mathrm{Cotor}^{2,(2p-2)p^{i+1}}_{BP_*BP}(BP_*, BP_*) \longrightarrow \mathrm{Cotor}^{2,(2p-2)p^{i+1}}_{BP_*BP}(BP_*, BP_*/I_3)$$

$I_3 = (p, v_1, v_2)$

Any element in $\bigoplus \mathbb{F}_p\langle \beta_{\dots/\dots}\rangle$ is sent to 0

We want to show that $d^{2p-2} x \neq 0$ if x is a lift of b_i

$d^{2p-1}x \in \mathrm{Cotor}^{2p+1,(2p-2)(p^{i+1}+1)}(BP_*, BP_*) \longrightarrow \mathrm{Cotor}^{2p+1,\cdots}(BP_*, BP_*/I)$

if $x \in \bigoplus \mathbb{F}_p\langle\beta_1\rangle \Rightarrow x \mapsto 0$ and if $p \geq 3$ this comes from a map of SS given by $\mathbb{S} \to \mathbb{S}/(p, v_1, v_2) \Rightarrow d^{p+1}x = 0$

So we just need to show $d^{p+1}(\beta_{p^i/p^i}) \neq 0$.

We can map this to $E_{\mathbb{F}_{p^{p-1}}}[h] \otimes P_{\mathbb{F}_{p^{p-1}}}[b]$ (<u>main point</u>)

sending $d^{2p-1}\beta_{p^i/p^i} = h_0 \beta^p_{p^{i-1}/p^{i-1}}$ + elements killed by $\beta_{1/1}^{a_i-1}$

if $a_i = \frac{p(p^i-1)}{p-1}$

Choose $c \in \mathbb{F}_{p^{p-1}}^\times$, our map will send this to

$$c^{p^{i+1}}(hb^p) + \underbrace{\text{elements in the kernel of } (c^p b)^{a_i-1}}_{\text{must all be } 0!}.$$

$\Rightarrow c^{p^{i+1}}(hb^p) \neq 0$

So if we are able to build this map we're good.

The detection map is the composition of 6 maps

for $n = p-1$

① $\mathrm{Cotor}_{BP_*}(BP_*, BP_*/I_3) \longrightarrow \mathrm{Cotor}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n)$

② $\mathrm{Cotor}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n) \xrightarrow{\sim} \mathrm{Cotor}_{K(n)_*K(n)}(K(n)_*, K(n)_*)$

③ $\mathrm{Cotor}_{K(n)_*K(n)}(K(n)_*, K(n)_*) \longrightarrow \mathrm{Cotor}_{K(n)_*K(n)\otimes_{K(n)_*}\mathbb{F}_{p^n}}(\mathbb{F}_{p^n}, \mathbb{F}_{p^n})$

④ $K(n)_*K(n) \otimes_{K(n)_*} \mathbb{F}_{p^n} \xrightarrow{\sim} \mathrm{Hom}_{cts}(\mathbb{F}_{p^n}[S_n], \mathbb{F}_{p^n})$

⑤ $\mathrm{Hom}_{cts}(\mathbb{F}_{p^n}[S_n], \mathbb{F}_{p^n}) \longrightarrow \mathrm{Hom}(\mathbb{F}_p[\mathbb{Z}/p], \mathbb{F}_p) = A$

And $\mathrm{Cotor}_A(\mathbb{F}_{p^n}, \mathbb{F}_{p^n}) = E[h] \otimes P[b]$ and we need to figure out what happens to $h_0, \beta_{p^i/p^i}$

$K(n)_*$ is a BP_*-module

$\overset{\shortparallel}{\mathbb{F}_p[v_n^{\pm 1}]}$

$$K(n)_* K(n) \overset{\text{def}}{=} \underbrace{K(n)_* \otimes_{BP_*} BP_* BP}_{K(n)_*[t_1, \ldots]} \otimes_{BP_*} K(n)_*$$

$\boxed{!}$ $K(n)_* K(n)$ is not the $K(n)$-homology of $K(n)$

$$\Rightarrow K(n)_* K(n) = K(n)_*[t_1, \ldots] \big/ (v_n t_i^{p^n} - v_n^{p^i} t_i)$$

We have $K(n)_* \longrightarrow \mathbb{F}_{p^n}$ (non graded $K(n)_*$-mod.)
$v_n \longmapsto 1$

$$\Rightarrow \mathbb{F}_{p^n} \otimes_{K(n)_*} K(n)_* K(n) = \mathbb{F}_{p^n}[t_1, \ldots] \big/ (t_i^{p^n} - t_i)$$

Let $W\mathbb{F}_{p^n}$ be the Witt vectors $\Rightarrow W\mathbb{F}_{p^n}/p \simeq \mathbb{F}_{p^n}$.

$\Rightarrow$ Let $\tau : \mathbb{F}_{p^n} \longrightarrow W\mathbb{F}_{p^n}$ be the Teichmuller map (given by Hensel's lemma)

and let $\sigma : W\mathbb{F}_{p^n} \circlearrowleft$ be the Frobenius.

$E_n = W\mathbb{F}_{p^n}\langle\langle T \rangle\rangle \big/ (T^n - p)$ power series, T non commuting w/ elets in $W\mathbb{F}_{p^n}$.

$Tw = w^\sigma T$

$S_n = \{ f(T) \in E_n^\times \mid f(T) \equiv 1 \mod T \}$ (<u>Morava stabilizer group</u>)

$f(T) = 1 + \sum e_i T^i$ where $e_i^{p^n} = e_i$ $\quad (e_i = \tau(\ldots))$

$$\mathbb{F}_{p^n}[t_1, t_2, \ldots] \big/ (t_i^{p^n} - t_i) \longrightarrow \mathrm{Hom}_{cts}(\mathbb{F}_{p^n}[S_n], \mathbb{F}_{p^n}) = \mathrm{Map}_{cts}(S_n, \mathbb{F}_{p^n})$$
$$t_i \longmapsto \left(1 + \sum e_i T^i \mapsto \bar{e}_i \in \mathbb{F}_{p^n}\right)$$

coordinate ring of S_n as a $\mathbb{F}_{p^n}$-group scheme

In fact this is an iso of Hopf-algebra

$\mathbb{Z}/p =$ primitive p-th roots of unity $\subseteq S_n \subseteq E_n^\times$

We need to see where t_1 goes, and it goes to $ct \in A = \mathbb{F}_p[t]/(t^p - t)$

In fact this map is just the map induced in cohomology by

$$B\mathbb{Z}/p \hookrightarrow M_{FG} \text{ of the Honda FG.}$$