The Classical Adams Spectral Sequence

0) Motivation.

(a) Want to study $[Y, X]$ $\leftarrow$ (stable) htpy classes of maps.

Idea: Study the Hurewicz homomorphism:

$$\begin{aligned} [Y,X] &\longrightarrow \operatorname{Hom}(H^*(X), H^*(Y)). \\ f &\longmapsto f^* \end{aligned}$$

Fix a prime p. Then $H^*(X)$ is equipped with an $\mathcal{A}$-module structure. (Here $\mathcal{A} = \mathcal{A}_p$ is the mod p Steenrod algebra.)

We get:

$$[Y,X] \xrightarrow{d} \operatorname{Hom}_{\mathcal{A}}(H^*(X), H^*(Y)) \subset \operatorname{Hom}(H^*(X), H^*(Y)).$$

A priori, this gives no additional information.

But we can also look at $\{\operatorname{Ext}^n_{\mathcal{A}}\}$, which may carry more information because multiplication in $\mathcal{A}$ is complicated.

Example: $\operatorname{Ext}^1_{\mathcal{A}}$ measures whether SES of $\mathcal{A}$-modules split.

Given $f: S^{2n-1} \longrightarrow S^n$ $(n>1)$, form the mapping cone C_f.

Consider the exact sequence associated to the pair (C_f, S^n):

$$0 \longrightarrow H^*(S^{2n}) \longrightarrow H^*(C_f) \longrightarrow H^*(S^n) \longrightarrow 0.$$

Q: Does this split as $\mathcal{A}$-modules? (i.e.: is f stably essential?)

$\mathcal{A}$ acts trivially on $H^*(S^{2n}), H^*(S^n)$. So:

$Q \iff Q'$: Does $\mathcal{A}$ act trivially on C_f?
i.e.: is Sq^n trivial?

This is the Hopf invariant one problem.

Thm. (Adams '60) Sq^n is nontrivial only for $n=1, 2, 4, 8$.

$\hookleftarrow$ originally proven using ASS + secondary cohomology operations!

Upshot (of considering $\mathrm{Ext}^n_{\mathcal{A}}$): can apply tools of homological algebra to study geometric problems.
(e.g. Adams' vanishing thm, etc.)

Q: How much does $\{\mathrm{Ext}^n_{\mathcal{A}}\}$ see?

Surprisingly, enough for a reasonable first approximation.
(More precisely: enough to construct the E_2 page of a SS that converges to $\pi^Y_*(X)^\wedge_p$!).

(b) Recall Serre's method for computing homotopy groups.

Thm X $(n-1)$-connected, $n \geq 2$.

(a) $h: \pi_n(X) \xrightarrow{\sim} H_n(X)$.

(b) For abelian G, $\exists\, K(G,n)$ unique up to htpy equiv. s.t.

$$\pi_i(K(G,n)) = \begin{cases} G & i=n \\ 0 & i\neq n. \end{cases}$$

Let $G = \pi_n(X)$. $\exists f: X \to K(G,n)$ inducing isos on H_n, π_n.

Let $X' = \mathrm{fib}(f)$. Then using LES in htpy for fibrations, see:

$$\pi_i(X') = \begin{cases} \pi_i(X) & i \geq n+1 \\ 0 & i \leq n. \end{cases}$$

If we can compute $H_{n+1}(X')$, we will know $\pi_{n+1}(X)$, and we can use the Serre SS applied to $X' \to X \to K(G,n)$ to compute $H_*(X')$. (Computing the differentials in this step can become difficult.)

<u>Thus</u>, can inductively continue to compute the homotopy groups of X by "killing off the lowest nontrivial homotopy group".

$$\begin{tikzcd}
\vdots \arrow[d] & \\
X^2 \arrow[r] \arrow[d] & K(G_2, n_2) \\
X' \arrow[r] \arrow[d] & K(G_1, n_1) \\
X \arrow[r] & K(G, n)
\end{tikzcd}$$

<u>Moral</u>: Can use (co)homology to study $\pi_*(X)$.

<u>Unfortunately</u>: This is not very efficient.

Let's work mod p and in the stable range.

Adams' idea is to kill off <u>all</u> the homotopy that can be detected by mod p cohomology in one step.

How?

- Instead of mapping X to an Eilenberg-Mac Lane space, map X to a product of E-M spaces inducing a surjection on H^*.
- Codify all the information into one SS.

Statement of Main Theorem.

Thm (Adams). Let X, Y be connective spectra of finite type.

There exists a spectral sequence with

$$\begin{cases} E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(X), H^*(Y)) \\ d_r : E_r^{s,t} \longrightarrow E_r^{s+r,\, t+r-1} \end{cases}$$

converging to $[Y, X_p^\wedge]_{t-s}$.

This gives a three-step process for computing stable htpy gps of spheres using the ASS.

① Compute E_2 page. $\begin{cases} \text{(i) Use cobar complex.} \\ \quad \text{Better: use minimal resolutions.} \\ \text{(ii) May SS} \\ \text{(iii) } \Lambda\text{-algebra.} \end{cases}$

② Compute differentials ← difficult!

③ Solve extension problem ← not too difficult.

1] Constuction of the ASS

(a) Preliminary facts about E-M spectra.

Spectra: Unless stated otherwise, <u>all</u> spectra we consider will be <u>connective</u> and of <u>finite type</u>.

↑ $\pi_n(X) = 0$ for all sufficiently negative n

↑ $H_*(X)$ is f.g. in each dimension.

Notation: $H = H\mathbb{F}_p$ is the <u>Eilenberg - Mac Lane spectrum</u>,

i.e. $H_n = K(\mathbb{F}_p, n)$

maps $H_n \to H_{n+1}$ are adjoint to the htpy equiv

$$K(\mathbb{F}_p, n) \xrightarrow{\sim} \Omega K(\mathbb{F}_p, n+1).$$

This is a E_∞ ring spectrum.

<u>Prop</u>^{n}. $\mathcal{A} \xrightarrow{\cong} H^*(H) = [H, H]_{-*}$.

Pf. ($p=2$ — other p similar)

fundamental class of H_n.

Using the Serre SS, $H^*(K(\mathbb{F}_2, n); \mathbb{F}_2) = \mathbb{F}_2[\{Sq^I \iota_n\}]$

where I is admissible and $e(I) < n$.

So the homomorphism

$$\begin{array}{ccc} \Sigma^r \mathcal{A} & \longrightarrow & \widetilde{H}^*(H_r) \\ \Sigma^r Sq^I & \longmapsto & Sq^I \iota_r \end{array}$$

is an isomorphism up to deg $\sim 2n$.

□.

Defn A generalized Eilenberg-Mac Lane spectrum K is a finite wedge sum of suspensions of E-M spectra.

Cor. For any spectrum Y, $d : [Y, K] \longrightarrow \mathrm{Hom}_{\mathcal{A}}(H^*(K), H^*(Y))$ is an isomorphism when K is a g. E-M spectrum.

Pf. Enough to prove for $K = \Sigma^n H$.

since $H^*(\Sigma^n H)$ is a free $\mathcal{A}$-module.

$$\begin{tikzcd}
{[Y, \Sigma^n H]} \arrow[r, "\sim"] & H^n(Y) & \mathrm{Hom}_{\mathcal{A}}(H^*(\Sigma^n H), H^*(Y)) \arrow[l, "\sim"'] \\
{[f]} \arrow[r, mapsto] & f^*(\iota_n) & f^* \arrow[l, mapsto]
\end{tikzcd}$$

□.

Cor. Y, K as above. Suppose $j : X \to K$ is a map of spectra s.t. $j^* : H^*(K) \to H^*(X)$ is onto. Then a map $f : Y \to X$ of spectra induces the zero homomorphism $f^* : H^*(X) \to H^*(Y)$ iff the composite $j \circ f : Y \to K$ is null homotopic.

Pf.

$$\begin{tikzcd}
{[Y, K]} \arrow[r, "\sim"] & \mathrm{Hom}_{\mathcal{A}}(H^*(K), H^*(Y)) & \mathrm{Hom}_{\mathcal{A}}(H^*(X), H^*(Y)) \arrow[l, hook'] \\
{[j \circ f]} \arrow[r, mapsto] & f^* \circ j^* & f^* \arrow[l, mapsto]
\end{tikzcd}$$

□.

(b) Adams resolutions.

<u>Defn</u> An <u>Adams resolution</u> for a spectrum X is a sequence of spectra:

$$\begin{tikzcd}
\vdots \arrow[d] & & \\
X^2 \arrow[r, "j_2"] \arrow[d, "i_1"'] & K^2 \arrow[r, "\partial_2"] & \Sigma X^3 \\
X^1 \arrow[r, "j_1"] \arrow[d, "i_0"'] & K^1 \arrow[r, "\partial_1"] & \Sigma X^2 \\
X^0 = X \arrow[r, "j_0"] & K^0 \arrow[r, "\partial_0"] & \Sigma X^1
\end{tikzcd}$$

such that:

(i) For each $s \geq 0$, $X^{s+1} \to X^s \to K^s \to \Sigma X^{s+1}$ is a cofiber sequence.

(ii) Each K^s is a g. E-M spectrum.

(iii) The induced map $j_s^*: H^*(K^s) \longrightarrow H^*(Y^s)$ is surjective.

<u>Lemma</u> Adams resolutions exist.

Pf. Suppose X^s has been constructed.

Let $K^s = H \wedge X^s$,

$$j_s = \eta \wedge 1 : \underbrace{S^0 \wedge X^s}_{X^s} \longrightarrow \underbrace{H \wedge X^s}_{K^s}$$

$\eta: S^0 \to H$ is the unit map.

<u>Why is K^s a g. E-M spectrum?</u>

Künneth $\Rightarrow$ $H^*(H \wedge E) \cong A \otimes_{\mathbb{F}_p} H^*(E)$

The el'ts $1 \otimes x$ give a map $H \wedge E \longrightarrow \bigvee_{H^*(E)} H$ which is an iso on homotopy groups and hence an equivalence by Whitehead's theorem.

Why is j_s^* a surjection?

The map $E \to H \wedge E$ induces the Steenrod action $\mathcal{A} \otimes H^*(E) \to H^*(E)$, which is surjective.

$$\left\{ \begin{array}{l} \text{Alternatively, more hands on way:} \\ \text{Pick generators } u_n \text{ for } H^*(X^s). \\ \text{Get a surjection } \bigoplus \mathcal{A}[u_n] \longrightarrow H^*(X^s) \\ \text{This gives a map } j_s : X^s \longrightarrow \bigvee_n \Sigma^{|u_n|} H . \end{array} \right.$$

Now let X^{s+1} be the homotopy fibre of j_s.

The LES in homotopy / homology show that $\pi_*(X^{s+1})$ is bdd below and $H_*(X^{s+1})$ has finite type.

Continue inductively. □

Rmk. (Canonical Adams resolution)

Let $\bar{H}$ be the cofiber of the unit map η. Get:

$$\Sigma^{-1}\bar{H} \to S^0 \to H \to \bar{H} .$$

η induces augmention $\varepsilon : \mathcal{A} \to \mathbb{F}_2$ in cohomology

$$Sq^i \longmapsto \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{else.} \end{cases}$$

Smashing with X^s gives

$$\underbrace{\Sigma^{-1}\bar{H} \wedge X^s}_{X^{s+1}} \longrightarrow X^s \longrightarrow \underbrace{H \wedge X^s}_{K^{s+1}} \longrightarrow \bar{H} \wedge X^s$$

The canonical Adams resolution is $\begin{cases} X^s = (\Sigma^{-1}\bar{H})^{\wedge s} \wedge X \\ K^s = H \wedge (\Sigma^{-1}\bar{H})^{\wedge s} \wedge X \end{cases}$

Upshot: This resolution is natural in X.

Prop. For any Adams resolution, let

$$P_s = H^*(\Sigma^s K^s), \quad \delta_s = (\Sigma^{s-1}\partial_{s-1})^* \circ (\Sigma^s j_s)^*.$$

$$\varepsilon = j_0^* : H^*(K^0) \to H^*(X).$$

Then the diagram

$$\cdots \to P_s \xrightarrow{\delta_s} P_{s-1} \to \cdots \to P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\varepsilon} H^*(X) \to 0$$

is a resolution of $H^*(X)$ by free $\mathcal{A}$-modules, each of which is bounded below and of finite type.

Pf. Look at LES in cohomology of cofibre sequence

$$\forall s \geq 0: \quad \cdots \to \Sigma^s X^{s+1} \to \Sigma^s X^s \xrightarrow{\Sigma^s j_s} \Sigma^s K^s \xrightarrow{\Sigma^s \partial_s} \Sigma^{s+1} X^{s+1} \to \Sigma^{s+1} X^s \to \cdots$$

Since j^* is onto, i^* is 0, and we get SES:

$$0 \to H^*(\Sigma^{s+1} X^{s+1}) \to H^*(\Sigma^s K^s) \to H^*(\Sigma^s X^s) \to 0.$$

Splice these SES together to get

$$\begin{tikzcd}
\cdots \arrow[r] & H^*(\Sigma^2 K^2) \arrow[rr, "\delta_2"] \arrow[dr] & & H^*(\Sigma K^1) \arrow[rr, "\delta_1"] \arrow[dr] & & H^*(K^0) \arrow[r, "\varepsilon"] & H^*(X) \\
\ddots \arrow[dr] & & H^*(\Sigma X^2) \arrow[ur] \arrow[dr] & & H^*(\Sigma X^1) \arrow[ur] \arrow[dr] & & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

By assumption, each of the $H^*(K^s)$ is a free $\mathcal{A}$-module.

Rmk. The homomorphisms δ_s, ε preserve the cohomological grading of $H^*(Y)$ and P_s, which is called the internal grading and is denoted by t.

(c) The exact couple and the E_2 page.

Fix a finite spectrum Y (think $Y = S^0$).

Apply $\pi_t^Y = [\Sigma^t Y, -]$ to the Adams resolution, and get an unrolled exact couple of Adams type.

$$\begin{tikzcd}
\vdots & \cdots \\
\pi_*^Y(X^2) \arrow[u] \arrow[r] & \pi_*^Y(K^2) \arrow[ul] \\
\pi_*^Y(X^1) \arrow[u] \arrow[r] & \pi_*^Y(K^1) \arrow[ul] \\
\pi_*^Y(X^0) \arrow[u] \arrow[r] & \pi_*^Y(K^0) \arrow[ul]
\end{tikzcd}$$

Q: Why Y finite?

A: Want $\pi_t^Y(Z)$ to be f.g.

<u>Exact couple</u>

$$\begin{tikzcd}
\pi_*^Y(X^{s+1}) \arrow[rr, "(i_s)_*"] & & \pi_*^Y(X^s) \arrow[dl, "(j_s)_*"] \\
& \pi_*^Y(K^s) \arrow[ul, "(\partial_s)_*"] &
\end{tikzcd}$$

Splicing together these LES, get the E_1 page of the ASS:

$$\begin{cases} E_1^{s,t} = \pi_{t-s}^Y(K^s) \\ d_1^{s,t} = (j \circ \partial)_* : \pi_{t-s}^Y(K^s) \to \pi_{t-s-1}^Y(K^{s+1}) \end{cases}$$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & \pi^Y_{t-s+1} X^s \arrow[r] \arrow[d] & \pi^Y_{t-s+1} K^s \arrow[r] \arrow[d] & \pi^Y_{t-s} X^{s+1} \arrow[r] \arrow[d] & \pi^Y_{t-s} K^{s+1} \arrow[r] \arrow[d] & \pi^Y_{t-s-1} X^{s+2} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi^Y_{t-s+1} X^{s-1} \arrow[r] \arrow[d] & \pi^Y_{t-s+1} K^{s-1} \arrow[r] \arrow[d] \arrow[rr, bend right, "d_1"'] & \pi^Y_{t-s} X^s \arrow[r] \arrow[d] & \pi^Y_{t-s} K^s \arrow[r] \arrow[d] & \pi^Y_{t-s-1} X^{s+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi^Y_{t-s+1} X^{s-2} \arrow[r] \arrow[d] & \pi^Y_{t-s+1} K^{s-2} \arrow[r] \arrow[d] & \pi^Y_{t-s} X^{s-1} \arrow[r] \arrow[d] & \pi^Y_{t-s} K^{s-1} \arrow[r] \arrow[d] & \pi^Y_{t-s-1} X^s \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

We expect this SS to abut to $\pi^Y_*(X)$, filtered by image $(\iota^s_* : \pi_* X^s \to \pi_* X)$.

<u>Prop</u> The E_2 term of the ASS is

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X), H^*(Y)).$$

(independent of the choice of Adams resolution

Pf. The Adams E_1 term and d_1-differential is the complex

$$\cdots \leftarrow \pi^Y_*(\Sigma^2 K^2) \leftarrow \pi^Y_*(\Sigma K^1) \leftarrow \pi^Y_*(K^0) \leftarrow 0 .$$

Since the $\{\Sigma^s K^s\}$ are g. E-M spectra,

$$d : \pi^Y_*(\Sigma^2 K^2) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(H^*(\Sigma^s K^s), H^*(Y))$$

So by definition, the cohomology groups of the complex are $\mathrm{Ext}^s_{\mathcal{A}}(H^*(X), H^*(Y))$

What about the internal grading?

$$E_1^{s,t} = \pi_{t-s}(K^s) = [\Sigma^{t-s} Y, K^s] \cong \mathrm{Hom}_{\mathcal{A}}(H^*(\Sigma^s K^s), \Sigma^t H^*(Y))$$

$\rightsquigarrow$ $\mathcal{A}$-module homomorphisms $H^*(\Sigma^s K^s) \to H^*(Y)$ that lowers degrees by t.

We denote this by $\mathrm{Hom}_{\mathcal{A}}^{t}(H^*(\Sigma^s K^s), H^*(Y))$.

So $E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X), H^*(Y))$.

(d) The Adams filtration and convergence.

<u>Defn</u> Given an Adams resolution of X, the <u>Adams filtration</u> of $\pi_*^Y(X)$ is $\cdots \subset F^{s+1} \subset F^s \subset \cdots \subset F^0 = \pi_*^Y(X)$.

where $F^s = \text{image}\,(\pi_*^Y(X^s) \to \pi_*^Y(X))$.

That is, we filter $\pi_*^Y(X)$ by how far back an el't pulls back in the Adams tower.

We say that $f \in \pi_*^Y(X)$ has <u>filtration degree s</u> if $f \in F^s$ but $f \notin F^{s+1}$.

<u>Lemma</u> The Adams filtration is independent of the choice of Adams resolution.

Pf) In fact, $f \in \pi_t^Y(X)$ has Adams filtration $\geq s$ iff it factors as

$$\Sigma^t Y \to U_s \to U_{s-1} \to \cdots \to U_0 = X$$

where each of the maps $U_i \to U_{i-1}$ induces zero in cohomology.

($\Rightarrow$) If f has Adams filtration $\geq s$, then it can be factored

$$\Sigma^t Y \to X^s \to X^{s-1} \to \cdots \to X^0 = X$$

and each of the maps $X^i \to X^{i-1}$ is trivial in cohomology (since j_i^* is onto).

($\Leftarrow$) Conversely, we shall construct lifts inductively.

$$\begin{tikzcd}
\Sigma^t Y \arrow[r] & U_s \arrow[r, dashed] \arrow[d] & X^s \arrow[r] \arrow[d] & K^s \\
& \vdots \arrow[d] & \vdots \arrow[d] & \\
& U_1 \arrow[r, dashed] \arrow[d] & X^1 \arrow[r] \arrow[d] & K^1 \\
& U_0 \arrow[r, equal] & X \arrow[r] & K^0
\end{tikzcd}$$

Suppose we have $U_{i-1} \to X^{i-1}$.

1. It lifts to $U_i \to X^i$ iff $U_i \to U_{i-1} \to X^{i-1} \to K^{i-1}$ is null.
   (b/c $[U_i, X^i] \to [U_i, X^{i-1}] \to [U_i, K^{i-1}]$ is exact)
2. K^{i-1} is a g. E-M spectrum, so it suffices to check $U_i \to U_{i-1} \to X^{i-1}$ induces zero in cohomology, which it does.

□.

Convergence.

Thm. X, Y connective spectra of finite type.
Then $E_2^{s,t} = \mathrm{Ext}_{A}^{s,t}(H^*(X), H^*(Y)) \Rightarrow \pi^Y_{t-s}(X)^\wedge_p$ ← if Y finite: p-component of $\pi^Y_{t-s}(X$

Sketch of a high-level proof.

Idea: · Milnor showed that the obstruction to conditional convergence (see Boardman '99) is $\mathrm{holim}\, X^s = \mathrm{hofib}(\prod_s X^s \xrightarrow{1-i} \prod_s X^s)$.
· p-completion annihilates this obstruction.

1. Smash Adams resolution $\{X^s\}$ with Moore spectra S/p^e, for $e \geq 1$. Get a tower of Adams resolutions $\{X^s/p^e\}$ of X/p^e.
2. Pass to homotopy limit over e, get $\{(X^s)^\wedge_p\}$, show $\mathrm{holim}\,(X_s)^\wedge_p \simeq *$.
3. Show $E_1^{s,t} = \pi^Y_{t-s} \xrightarrow{\sim} \pi_{t-s}((K^s)^\wedge_p)$, which converges conditionally to $\pi_*(X)^\wedge_p$
4. For strong convergence, check Boardman's criterion $RE_\infty = 0$.
   (Satisfied when Y is connective of finite type, or if SS collapses at finite stage).

A low-tech proof.

We shall show:

(a) For fixed s,t, $E_r^{s,t}$ is independent of r for sufficiently large r, and the stable groups $E_\infty^{s,t}$ are isomorphic to $F^{s,t}/F^{s+1,t+1}$, where $F^{s,t} = \text{image}\,(\pi_{t-s}^Y(X^s) \to \pi_{t-s}^Y(X))$.

(b) $\bigcap_n F^{s+n,t+n}$ is the subgroup of $\pi_{t-s}^Y(X)$ consisting of torsion el'ts of order prime to p.

(a) Look at the r^{th} derived couple

$$\begin{tikzcd}
& & & & E_r^{\cdot\cdot} \\
& & E_r^{s,t} \arrow[r] & A_r^{\cdot\cdot} \arrow[d, "i_r"] & \\
& A_r^{\cdot\cdot} \arrow[d] & & A_r^{\cdot\cdot} \arrow[uur] & \\
& A_r^{\cdot\cdot} \arrow[uur] & & & \\
\cdot \arrow[uur, dashed] & & & &
\end{tikzcd}$$

We claim that i_r is injective for sufficiently large r.

For non-p torsion and non-torsion elements, this follows since $E_r^{s,t}$ is a $\mathbb{F}_p$ vector space.

For p-torsion, follows from (b), since $A_r^{\cdot\cdot}$ consists of el'ts that pull back at least $r-1$ units vertically.

Then also, for large r, $E_r^{s,t}$ is the coker $(A_r^{\cdot\cdot} \to A_r^{\cdot\cdot})$, and this map is just the inclusion of $F^{s+1,t+1}$ into $F^{s,t}$.

(b) Claim: If $f \in \pi_t^Y(X)$ and f is divisible by p, then $f \in F^n$.

Induction on n. Clear if $n=0$.

For $n=1$,

$$\begin{tikzcd} & X^1 \arrow[r] \arrow[d] & K^1 \\ \Sigma^t Y \arrow[ur, dotted, "?"] \arrow[r, "f"'] & X \arrow[r] & K^0 \end{tikzcd}$$

the lift exists if f is zero in mod p cohomology, which it is.

For the inductive step, write $f = pg$ where g is divisible by p^{n-1}. So g factors through X^{n-1}. Same argument lets me lift f to $\Sigma^t Y \to X^n \to X$.

__Claim__. If f is not divisible by p^n for some n, then $f \notin F^s$ for some s.

Consider the pullback of the path-fibration $PX \to \Sigma X$ along $\Sigma(p^r): \Sigma X \to \Sigma X$

$$\begin{tikzcd} \Omega\Sigma X \arrow[r, "m"] & W \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & PX \arrow[d] \\ & \Sigma X \arrow[r, "\Sigma(p^r)"'] & \Sigma X \end{tikzcd}$$

We get an exact sequence

$$\pi_*^Y(X) \xrightarrow{p^r} \pi_*^Y(X) \xrightarrow{m_*} \pi_*^Y(W) \xrightarrow{\pi_*} \pi_*^Y(\Sigma X) \xrightarrow{p^r} \pi_*^Y(\Sigma X$$

So: $0 \to \operatorname{coker} p^r \to \pi_*^Y(W) \to \ker p^r \to 0$

$$\Rightarrow p^{2r} \cdot \pi_*^Y(W) = 0.$$

Let $G = \pi_s^Y(W)$ be the first nonzero homotopy group of W

Let $K = \Sigma^s K(G \otimes \mathbb{Z}/p)$, $g: W \to K$ s.t.

$$\begin{tikzcd} g_*: & \pi_s^Y(W) \arrow[r] \arrow[d, equal] & \pi_s^Y(K) \arrow[d, equal] \\ \text{is} & G \arrow[r] & G \otimes \mathbb{Z}/p \end{tikzcd}$$

Construct the pullback

$$\begin{tikzcd} \Omega K \arrow[r] & W_1 \arrow[r] \arrow[d, "q_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & PK \arrow[d] \\ & W \arrow[r, "g"'] & K \end{tikzcd}$$

The LES in homotopy

$$\begin{tikzcd} 0 = \pi_{s+1}^Y(K) \arrow[r] & \pi_s^Y(W_1) \arrow[r] & \pi_s^Y(W) \arrow[r] & \pi_s^Y K \\ & & G \arrow[r] & G \otimes \mathbb{Z}/p \end{tikzcd}$$

shows $\pi_s(W_1) \cong p \cdot \pi_s(W)$.

Also, $p^{2n} \cdot \pi_*(W_1) = 0$.

Iterate this to increase connectivity of W_k, until $\pi_t^Y(W_k) = 0$.

Consider the Adams resolution.

$$\begin{tikzcd}
 & X^k \arrow[r, "m_k"] \arrow[d, dotted] & W_k \arrow[r] \arrow[d, dotted] & K_k. \\
 & X^1 \arrow[r] \arrow[d] & W_1 \arrow[r] \arrow[d] & K_1 \\
\Sigma^t Y \arrow[r, "f"'] \arrow[uur, dotted, "\tilde{f}", "?"' near end] & X \arrow[r, "m"'] & W \arrow[r] & K
\end{tikzcd}$$

The same lifting argument from before lets us construct a chain map.

If $f \in \bigcap_s F^s$, then f lifts to

$\tilde{f} : \Sigma^t Y \to X^k$.

Then $m_k \circ \tilde{f} = 0$

$\Rightarrow m \circ f = 0$

$\Rightarrow f \in \ker m_*$

$\Rightarrow f$ is divisible by p^r. $\Rightarrow\!\Leftarrow$.

$\square$.

In summary, this proves $\bigcap_s F^s = p^\infty \pi_*^Y(X)$.

(e) Naturality and multiplicative structure

<u>Thm</u> Let $f : X \to X'$ be a map of connective spectra of finite type. Then there is a map $f_* : \{E_r(X), d_r\} \to \{E_r(X'), d_r\}$ of Adams spectral sequences, given at the E_2 level by the homomorphism

$$(f^*)^* : \mathrm{Ext}_A^{s,t}(H^*(X), H^*(Y)) \longrightarrow \mathrm{Ext}_A^{s,t}(H^*(X'), H^*(Y))$$

induced by the A-module homomorphism $f^* : H^*(X') \to H^*(X)$.

(Proof is by a comparison theorem of Adams resolutions: i.e., $\exists$ a chain map, unique up to chain homotopy, lifting the map f.)

Defn Let $\{'E_r\}$, $\{''E_r\}$, $\{E_r\}$ be three SS's.
A pairing of SS's is a sequence of homomorphisms

$$\phi_r : 'E_r^{*,*} \otimes ''E_r^{*,*} \longrightarrow E_r^{*,*} \quad \text{for } r \geq 1$$

satisfying the Leibniz rule

$$d_r(\phi_r(x \otimes y)) = \phi_r(d_r(x) \otimes y) + (-1)^{|x|}\phi_r(x \otimes d_r(y))$$

and s.t. $\phi_{r+1}([x] \otimes [y]) = [\phi_r(x \otimes y)]$

$\uparrow$ homology class of x in $'E_{r+1}^{*,*}$.

Rmk. A spectral sequence pairing $\{\phi_r\}$ induces a pairing
$\phi_\infty : 'E_\infty^{*,*} \otimes ''E_\infty^{*,*} \longrightarrow E_\infty^{*,*}$.

Defn. An algebra spectral sequence is a spectral sequence $\{E_r\}$ with a SS pairing $\{\phi_r : E_r \otimes E_r \to E_r\}$ that is associative and unital.

It is commutative if the pairing satisfies

$$\phi_r(y \otimes x) = (-1)^{|x||y|}\phi(x \otimes y).$$

Interestingly, there are two possible pairings one can define on the Adams spectral sequence.

Pairing	E_2 page	converges to...
Tensor product	Tensor product	Smash product pairing
Composition	Yoneda pairing	Composition pairing.

Thm (Adams '58) (Tensor product pairing)

There is a natural pairing $E_r^{s,t}(X) \otimes E_r^{s,t}(Y) \longrightarrow E_r^{s+u,t+v}(X \wedge Y)$ of Adams SS, given at the E_2 term by the tensor product pairing

$$\mathrm{Ext}_A^{s,t}(H^*(X), \mathbb{F}_2) \otimes \mathrm{Ext}_A^{u,v}(H^*(Y), \mathbb{F}_2) \to \mathrm{Ext}_A^{s+u,t+v}(H^*(X \wedge Y), \mathbb{F}_2)$$

and converging to the smash product pairing

$$\pi_{t-s}(X_2^\wedge) \otimes \pi_{v-u}(Y_2^\wedge) \longrightarrow \pi_{t-s+v-u}((X \wedge Y)_2^\wedge)$$

Thm (Moss '68) (Composition pairing)

There is a pairing of SS $E_r^{*,*}(Y,Z) \otimes E_r^{*,*}(X,Y) \to E_r^{*,*}(X,Z)$ which agrees for $r=2$ with the Yoneda pairing

$$\mathrm{Ext}_A^{*,*}(H^*(Z), H^*(Y)) \otimes \mathrm{Ext}_A^{*,*}(H^*(Y), H^*(X)) \longrightarrow \mathrm{Ext}_A^{*,*}(H^*(Z), H^*(X))$$

and which converges to the composition pairing

$$[Y, Z_p^\wedge]_* \otimes [X, Y_p^\wedge]_* \longrightarrow [X, Z_p^\wedge]_*.$$

This pairing is associative and unital.

Rmk When all the spectra are S^0, both pairings coincide.

2] Examples.

(a) $\mathrm{Ext}_{\mathcal{A}_2}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ — via minimal resolutions.

Defn A <u>minimal resolution</u> of $H^*(X)$ is a free resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to H^*(X) \to 0$$

where we choose the minimum number of generators in each degree.

Lemma For a min'l resolution, all the boundary maps in the dual complex

$$\cdots \leftarrow \mathrm{Hom}_{\mathcal{A}}(P_2, \mathbb{F}_2) \leftarrow \mathrm{Hom}_{\mathcal{A}}(P_1, \mathbb{F}_2) \leftarrow \mathrm{Hom}_{\mathcal{A}}(P_0, \mathbb{F}_2) \leftarrow 0$$

are zero; hence $\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X); \mathbb{F}_2) = \mathrm{Hom}_{\mathcal{A}}^{t}(P_s, \mathbb{F}_2)$.

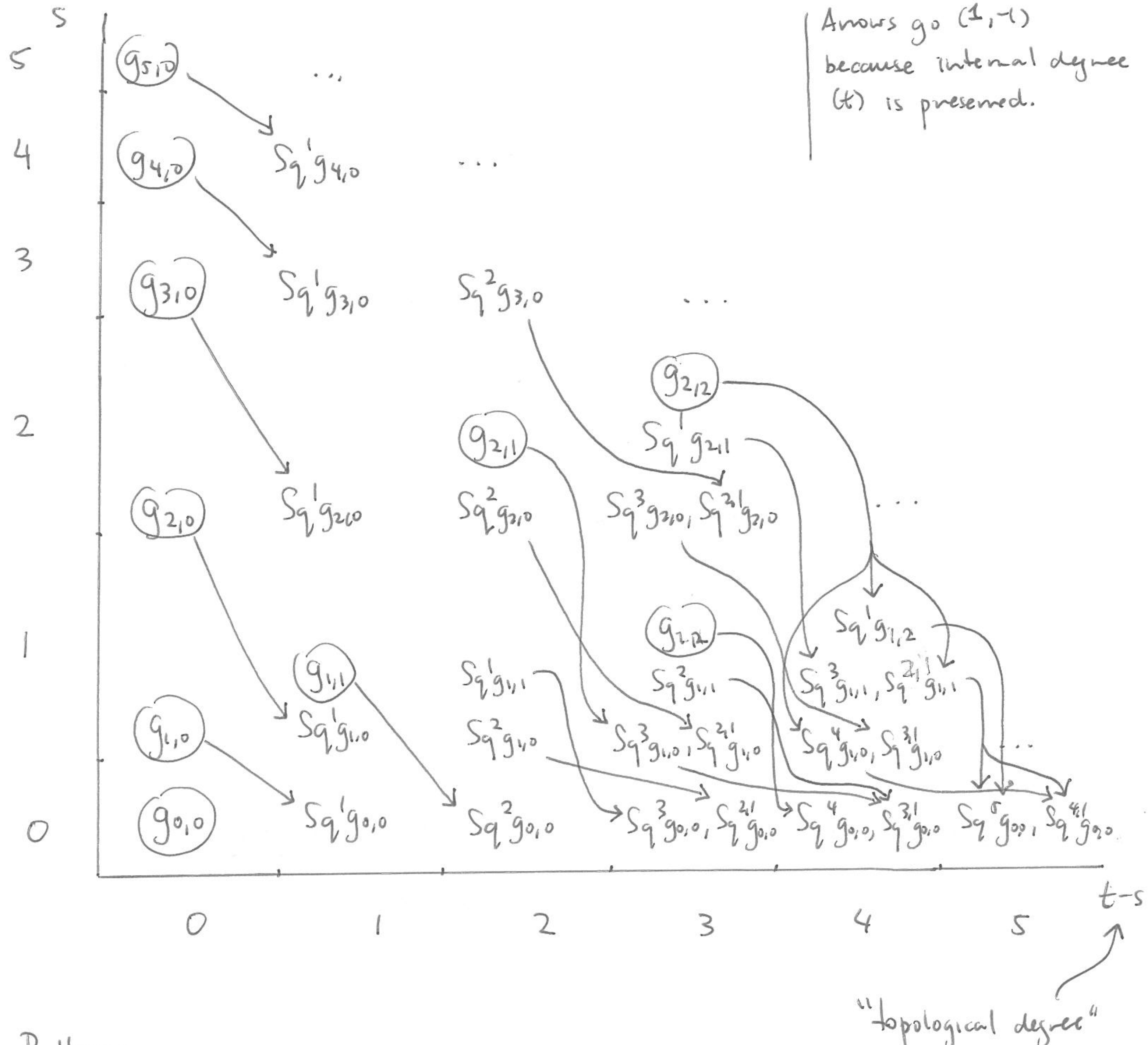
We now construct by hand a min'l resolution of $\mathbb{F}_2$ by free $\mathcal{A}$-modules

Recall:
- The admissible monomials form a vector space basis for $\mathcal{A}_2$.
- Adem relations.

For degrees ≤ 5:

Admissible monomials	Adem relations	
Sq^1	$Sq^{1,1} = 0$	$Sq^{2,3} = Sq^5 + Sq^{4,1}$
Sq^2	$Sq^{1,2} = Sq^3$	$Sq^{3,2} = 0$.
$Sq^3, Sq^{2,1}$	$Sq^{1,3} = 0$	
$Sq^4, Sq^{3,1}$	$Sq^{2,2} = Sq^{3,1}$	
$Sq^5, Sq^{4,1}$	$Sq^{1,4} = Sq^5$	

The E_2 page for $t \leq 5$:



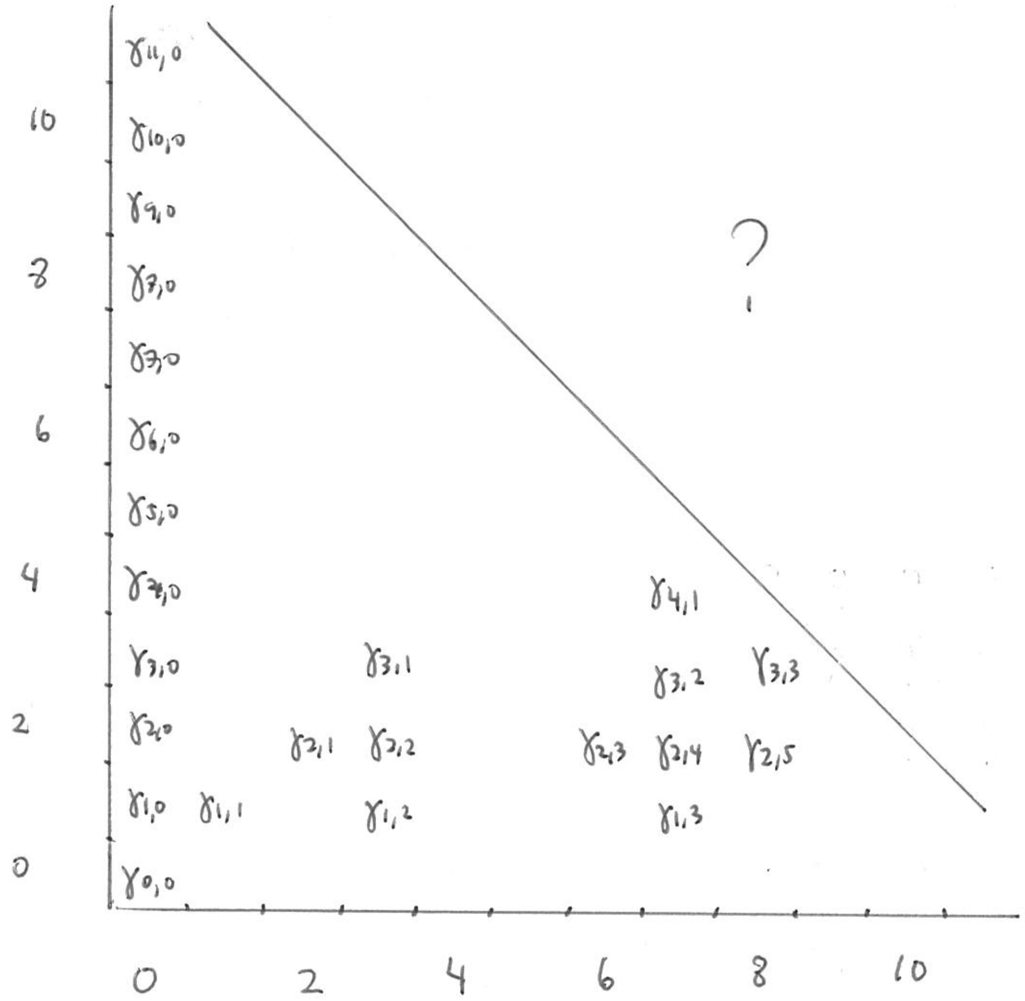
Arrows go $(1,-1)$ because internal degree (t) is preserved.

Patterns

- 0^{th} stem has a generator for each $s \geq 0$
- Filtration degree $s=1$ has generators in each 2^i-1 stem, $i \geq 1$.
  (Proof: recall that $\mathcal{A}$ is generated by indecomposables $\{Sq^{2^i}\}$ as an algebra.)

The dual of $g_{1,i}$ is called h_i.

E_2 page for $t \leq 11$



Multiplicative structure?

<u>Lem.</u> The product $h_i \cdot \gamma_{s,n}$ contains the summand $\gamma_{s+1,m}$ iff $\partial_{s+1}(g_{s+1,m}) = \sum_j a_j g_{s,j}$ contains the summand $Sq^{2^i} g_{s,n}$.

Examples

(a) $h_0 \cdot \gamma_{s,0} = \gamma_{s+1,0}$?

$\partial_{s+1}(g_{s+1,0}) = \underline{Sq^1 g_{s,0}}$

So by induction $\gamma_{s,0} = h_0^{s+1}$.

(b) $h_1^2 = \gamma_{2,1}$, i.e. $h_1 \cdot \gamma_{1,1} = \gamma_{2,1}$?

$\partial_2(g_{2,1}) = Sq^3 g_{1,0} + \underline{Sq^2 g_{1,1}}$

$\Rightarrow h_1^2 = \gamma_{2,1}$

(c) $h_1^3 = \gamma_{3,1}$, i.e. $h_1 \cdot \gamma_{2,1} = \gamma_{3,1}$?

$\partial_3(g_{3,1}) = Sq^4 g_{2,0} + \underline{Sq^2 g_{2,1}} + Sq^1 g_{2,2}$

$\Rightarrow h_1^3 = \gamma_{3,1}$.

(d) $h_0 h_2 = \gamma_{2,2}$, i.e. $h_0 \cdot \gamma_{1,2} = \gamma_{2,2}$?

$\partial_2(g_{2,2}) = Sq^4 g_{1,0} + Sq^2 Sq^1 g_{1,1} + \underline{Sq^1 g_{1,2}}$

$\Rightarrow h_0 h_2 = \gamma_{2,2}$

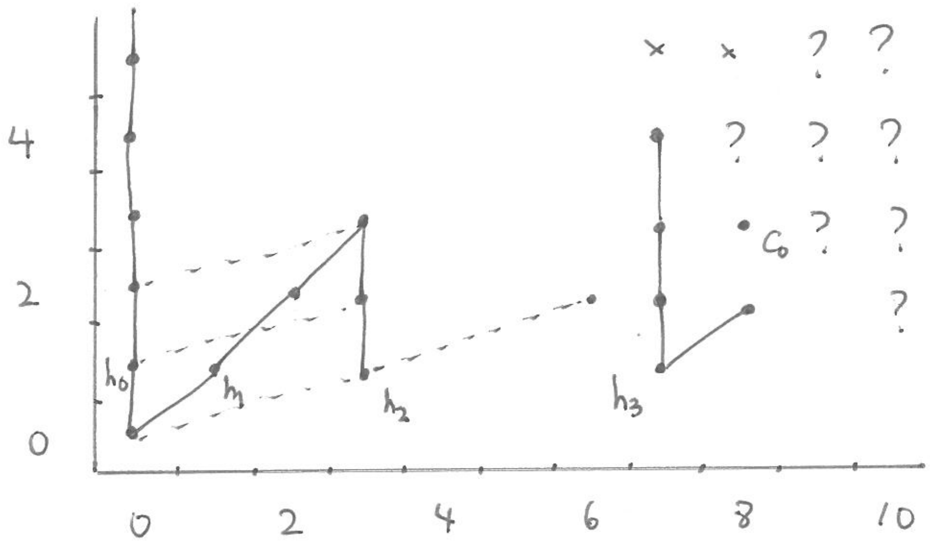
(e) $h_0^2 h_2 = \gamma_{3,1}$, i.e. $h_0 \cdot \gamma_{2,2} = \gamma_{3,1}$?

$\partial_3(g_{3,1}) = Sq^4 g_{2,0} + Sq^2 g_{2,1} + \underline{Sq^1 g_{2,2}}$

$\Rightarrow h_0^2 h_2 = \gamma_{3,1}$

(c) + (e) $\Rightarrow$ $h_0^2 h_2 = h_1^3$.

Get:



To compute E_∞ page, nice to know $E_2^{s,t} = 0$ for large s (for fixed $t-s$).

Thm (Adams vanishing '66)

$\mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2) = 0$ for $0 < t-s < 2s - \varepsilon$, where

$$\varepsilon = \begin{cases} 1 & s \equiv 0,1 \mod 4 \\ 2 & 2 \mod 4 \\ 3 & 3 \mod 4 \end{cases}$$

Lem. $E_2^{s,t} = E_\infty^{s,t}$ for $t \leq 11$

Pf. h_i, $0 \leq i \leq 3$ represent essential homotopy classes (they're the Hopf maps η, ν, σ); they are infinite cycles. So $d_r(h_i) = 0$ $\forall r \geq 2$. Also, $d_r(c_0)$ lies in Adams' vanishing range. □.

Thm

$$\pi_i(S^0)^\wedge_2 = \begin{cases} \mathbb{Z}/2 & i = 1,2,6 \\ \mathbb{Z}/8 & i = 3 \\ 0 & i = 4,5 \\ \mathbb{Z}/16 & i = 7 \end{cases}$$

Pf. Multiplication by h_0 corresponds to multiplication by 2. To see this, reduce to the case $\pi_0(S^0) = \mathbb{Z}$ by the naturality of the smash product pairing. □.

Rmk Can also deduce multiplicative structure of $\pi_*(S^0)$ from the multiplicative structure of Ext_A.

Rmk (Facts about the Hopf-Steenrod invariants h_i)

- Saw that h_0, h_1, h_2, h_3 are permanent cycles, detecting Hopf maps
- (Adams '60) h_j, $j > 3$ support nontrivial differentials.
- (Browder) h_j^2 related to Kervaire invariants.

(b) $\pi_*(MU)$

MU – spectrum for complex cobordism.

Thm (Milnor) $\pi_*(MU) = \mathbb{Z}[x_1, x_2, \dots]$ with $x_i \in \pi_{2i}(MU)$.

Pf. We have an ASS in homology:

$$E_2^{s,t} = \underset{\substack{\uparrow \\ \text{dual Steenrod algebra.}}}{\mathrm{Ext}^{s,t}_{\mathcal{A}_p^\vee}}(\mathbb{Z}/p, H_*(MU, \mathbb{Z}/p)) \Rightarrow \pi_{t-s}(MU)^\wedge_p$$

Prop^n $H_*(MU; \mathbb{Z}/p) \cong P_* \otimes \mathbb{Z}/p[\{u_i\}_{i \neq p^k-1}]$ $\quad (|u_i| = 2i)$

where $P_* \subset \mathcal{A}_p^\vee$ is $P_* = \begin{cases} \mathbb{Z}/2[\xi_1^2, \xi_2^2, \dots] & p > 2 \\ \mathbb{Z}/p[\xi_1, \xi_2, \dots] & p > 2. \end{cases}$

So $E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p^\vee}(\mathbb{Z}/p, P_* \otimes \mathbb{Z}/p[\{u_i\}_{i \neq p^k-1}])$

$\cong \mathrm{Ext}^{s,t}_{\mathcal{A}_p^\vee}(\mathbb{Z}/p, P_*) \otimes \mathbb{Z}/p[\{u_i\}_{i \neq p^k-1}]$

change of rings iso $\longrightarrow$ $\cong \mathrm{Ext}^{s,t}_{E}(\mathbb{Z}/p, \mathbb{Z}/p) \otimes \mathbb{Z}/p[\{u_i\}_{i \neq p^k-1}]$

where $\underset{\substack{\uparrow \\ \text{exterior algebra.}}}{E} = \mathcal{A}_p^\vee // P_* := \mathcal{A}_p^\vee \otimes_{P_*} \mathbb{Z}/p$

$\Rightarrow E_2^{s,t} \cong \mathbb{Z}/p[a_0, a_1, \dots] \otimes \mathbb{Z}/p[\{u_i\}_{i \neq p^k-1}]$ $\quad |u_i| = (0, 2i)$

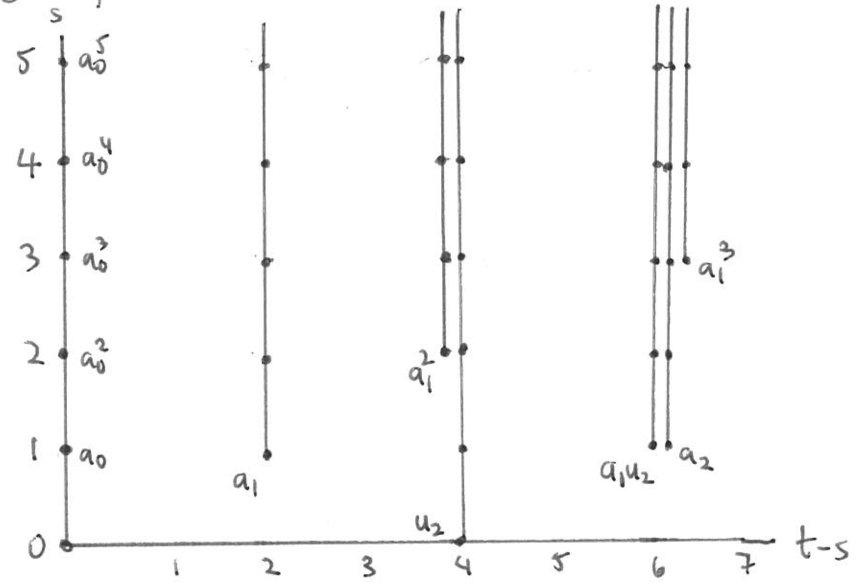
$|a_i| = (1, 2p^i - 1)$

(as an algebra)

Observe: E_2 page generated solely by even-dim'l classes.

So $E_2 = E_\infty$.

e.g. $p=2$



$\Rightarrow$ suspect

$$\pi_*(MU) = \mathbb{Z}[x_1, x_2, \dots]$$

$$|x_i| = 2i$$

(again multiplication by a_0 corresponds to multiplication by p)

Still need to piece together global structure from $\pi_*(MU)^\wedge_p$, but this is just algebra. □.

3] The Adams-Novikov Spectral Sequence.

<u>Idea</u>: There is nothing really special about E-M spectra H. Can rederive SS using some other E_∞ ring spectrum E, with E_*E taking the role of $\mathcal{A}^\vee$.

1. But to take derived functors Ext, need the category of (graded, left) E_*E-comodules to be abelian.
   $\rightsquigarrow$ require E to be <u>flat</u>.
2. Also, for convergence, need analogue of p-completion. i.e. given a spectrum X, need another spectrum $\hat{X}$, with a map $X \to \hat{X}$ inducing isos on E_*-homology and s.t. $\hat{X}$ has an Adams tower with $\underleftarrow{\mathrm{holim}}\, \hat{X}^s \simeq *$.

This is called the <u>E-completion</u> of X.

Thm (Adams-Novikov SS)

If E is flat and X has a E-completion, there is a SS

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{E_*E}(E_*(Y), E_*(X)) \Rightarrow \pi^{Y}_{t-s}(\hat{X}).$$

Example. $E = MU$ works.

This was used in HHR's proof of the detection theorem in their solution of the Kervaire invariant problem.