$\boxed{\text{SUVITOP } 1/10/14}$

Framed cobordism & htpy spheres

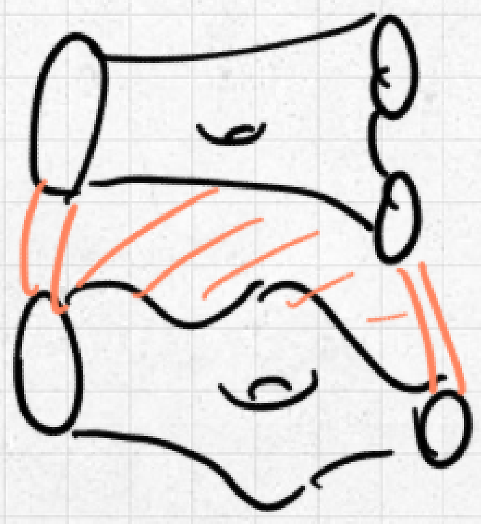
All manifolds by default will be closed & oriented

<u>Def</u>: M_1, M_2 are <u>h-cobordant</u> ($M_1 \sim_h M_2$) if $\exists\, W$ $(n+1)$-manifold w/ boundary s.t. $\partial W \cong M_1 \sqcup -M_2$ & $M_i \hookrightarrow W$ are htpy equivalence

$\Theta_n = \{\text{htpy } n\text{-spheres}\}/\text{h-cobordism}$

<u>Thm (Smale)</u>: If M_1, M_2 are simply connected n-manifolds, $n \geq 5$
$\Rightarrow$ every h-cobordism between M_1 & M_2 is a cylinder.

h-cobordism plays well w/ connected sum ($\#$)



Moreover we have inverses:

$M \# (-M) \sim_h S^n$

$\Rightarrow$ Θ_n is a grp under $\#$.

We can construct $\Theta_n \to \operatorname{coker} J_n$. Let M be a stably framed manif. Embed $M \hookrightarrow S^N$. By Pontryagin-Thom construction we get

$S^N \longrightarrow S^{N-\dim M}$

⚠ This map depends on the framing

<u>Def</u>: M is s-parallelizable if $TM \oplus \varepsilon^n$ is trivial

Homotopy spheres are s-parallelizable ($K^n(S^n) = \mathbb{Z}$)

So we have a map $(M, \varphi) \xrightarrow{PT} \pi_n \mathbb{S}$ stably framed manifold

It can be shown that framed cobordant manifold have the same image

$$PT(M \# N) = PT(M) + PT(N)$$

The image by two different framings differ by an element in $\mathrm{Im}\, J_n$ so we have a map $\qquad J: SO \to \Omega^\infty \Sigma^\infty S^0$

$$\Theta_n \longrightarrow \pi_n \mathbb{S} / \mathrm{Im}\, J_n$$

Let bP_{n+1} be the kernel of this map, that is the group of htpy spheres bounding stably framed manifold.

$\Rightarrow bP_{n+1}$ finite $\Leftrightarrow$ Θ_n finite.

Surgery theory

M manifold $\quad S^p \times D^{q+1} \hookrightarrow M \quad p+q=n \quad$ we can do surgery and replace that by $D^{p+1} \times S^q$

Doing surgery we can kill htpy groups of M up to dimension k if $n = 2k+1$.

We can also do <u>framed surgery</u>. This does not change the framed cobordism class.

(That is we use surgery data that respect the natural framing of $S^p \times D^{q+1}$)

Again we can kill all htpy grps up to k doing framed surgery.

We want to study bP_{2k}. The Kervaire invariant is an obstruction to extend the killing of htpy grps to dimension k.

For k even we get a surjective map $\mathbb{Z} \to bP_{2k}$ which we can study using Adam's work on $\mathrm{Im}\, J$.

$$\mathbb{Z} \twoheadrightarrow bP_{2k}$$
$$t \longmapsto [\partial W] \quad \text{sign } H^k(W) = 8t$$

For k odd we have $\mathbb{Z}/2 \twoheadrightarrow bP_{2k}$

$t \longmapsto [\partial W]$ if $c(W)=t$ Kervaire invariant of W

So we have a map $c_k : \pi_{2k}\mathbb{S} \to \mathbb{Z}/2$

Fix W framed $2k$-manifold w/ boundary which is $(k-1)$-connected.

The intersection pairing $\langle , \rangle$ on $H_k(W;\mathbb{Z}/2)$ is skew symm & unimodular

$\Rightarrow \exists \{\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r\}$ symplectic basis $\left(\langle , \rangle = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \right)$

$\forall \alpha \in H_k W \simeq \pi_k W \Rightarrow \alpha : S^k \to W$

$$\alpha^* TW \simeq TS^k \oplus \nu(\alpha)$$

and a stable framing on W induces a stable framing of $\nu(\alpha)$.

$$\varepsilon^N \simeq \alpha^*(\varepsilon^m \oplus TW) \simeq \underbrace{\varepsilon^m \oplus TS^k}_{\simeq \varepsilon^{m+k}} \oplus \nu(\alpha)$$

and this induces a class in $\pi_k(V_{2k-N,\, k-N}) \cong \mathbb{Z}/2$ depending only on α.

So we have a map $\phi : H_k W \longrightarrow \mathbb{Z}/2$

<u>Thm</u>: For $k=3,7$ $\phi(\alpha)=0 \iff \nu(\alpha)$ is trivial

For $k=3,7$, $\nu(\alpha)$ is trivial $\phi(\alpha)=0 \iff \alpha : S^k \times D^{n-k} \hookrightarrow M$ can be framed

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Arf invariant

Let (V, ψ) be a v. space $/\mathbb{F}_2$ w/ a quadratic form ψ

(Recall $\psi : V \to \mathbb{F}_2$ is quadratic iff $\psi(x+y) - \psi(x) - \psi(y)$ is bilinear)

Suppose $\langle -, - \rangle = \psi(- + -) - \psi(-) - \psi(-)$ is nondegenerate $\Rightarrow$ we can fix a symplectic basis $\{x_i, y_i\}$

$$\mathrm{Arf}(V, \psi) = \sum \psi(x_i)\psi(y_i) \qquad \text{(independent of the basis)}$$

Thm: $\phi : H_k(W; \mathbb{Z}/2) \to \mathbb{Z}/2$ is a quadratic form refining the intersection pairing.

Def: The Kervaire invariant of W is $\mathrm{Arf}(H_k(W; \mathbb{F}_2), \phi)$.

Thm: W is framed cobordant to a contractible manifold iff $c(W) = 0$.

Kervaire invariant 1 problem: Is $bP_{2k} = 0$?

$c : \pi_{2k} \mathbb{S} \to \mathbb{Z}/2$ Is c 0?