SUVITOP 10/12/14

The reduction theorem & the gap theorem

Thm (Reduction thm): There is a w.e. between $(G = C_{2^n})$

$$h : MU^{((G))} \wedge_A \mathbb{S} \xrightarrow{\sim} H\underline{\mathbb{Z}}_{(2)}$$

$$A = \mathbb{S}[\bar{r}_1, \bar{r}_2, \dots] \to MU^{((G))}$$

Proof: By induction on $\# G = g$. When $g = 2$, this is Quillen's thm on FGL.

In the induction step we can mash w/ the isotropy separation sequence

$$\begin{array}{ccccc} EP_+ \wedge (MU^{((G))} \wedge_A \mathbb{S}) & \longrightarrow & MU^{((G))} \wedge_A \mathbb{S} & \longrightarrow & \widetilde{EP} \wedge (MU^{((G))} \wedge_A \mathbb{S}) \\ \downarrow & & \downarrow & & \downarrow \\ EP_+ \wedge H\underline{\mathbb{Z}}_{(2)} & \longrightarrow & H\underline{\mathbb{Z}}_{(2)} & \longrightarrow & \widetilde{EP} \wedge H\underline{\mathbb{Z}}_{(2)} \end{array}$$

The map on the left is an equivalence by the induction hypothesis (check on fixed pts for proper subgroups. So our goal is to show that the map on the right is an equivalence. So we just need to check if it is a w.e. on the G-fixed pts.

$$\pi_*^G h : \pi_* \Phi^G(MU^{((G))} \wedge_A \mathbb{S}) \longrightarrow \pi_* \Phi^G(H\underline{\mathbb{Z}}_{(2)})$$

STEP 1: These are isomorphic as abstract group.

$$\pi_* \Phi^G H\underline{\mathbb{Z}}_{(2)} = \mathbb{Z}/2[b] \quad |b| = 2$$

$$\pi_* \Phi^G (MU^{((G))} \wedge_A \mathbb{S}) = \begin{cases} \mathbb{Z}/2 & * \text{ even} \\ 0 & * \text{ odd} \end{cases}$$

$$\pi_* \Phi^G (MU^{((G))} \wedge_A \mathbb{S}) = \pi_* (\underbrace{\Phi^G MU^{((G))}}_{= MO} \wedge_{\Phi^G A} \underbrace{\Phi^G \mathbb{S}}_{= \mathbb{S}})$$

since Φ^G is symm. monoidal

So we need to figure out what $\Phi^G A$ is.

Claim: $\Phi^G A = \mathbb{S}[\Phi^G(N_{C_2}^G \bar{r}_1), \dots] = \mathbb{S}[\Phi^{C_2} \bar{r}_1, \dots]$

<u>Claim</u>: We only need to show that $\pi_*^G h$ is surjective when $* = 2^k$

$$\pi_{2^k} h : \pi_{2^k}\left(MU^{((4))} \big/ (G \cdot \bar{r}_{2^k-1}) \right) \longrightarrow \pi_{2^k} \Phi^G H\underline{\mathbb{Z}}_{(2)}$$

<u>Proof</u>: Binary

Let $c_k = 2^k - 1$, $M_k = MU^{((4))} \big/ (G \cdot \bar{r}_{c_k})$

We have a restriction isomorphism

$$\pi^G_{c_k \rho_G + 1}\left(\widetilde{EP} \wedge M_k\right) \xrightarrow{\sim} \pi^G_{c_k+1}\left(\widetilde{EP} \wedge M_k\right)$$

induced by the inclusion of representations $S^{c_k+1} \hookrightarrow S^{c_k \rho_G + 1}$

So we want to show $N\bar{r}_{c_k} \in \pi^G_{c_k \rho_G} MU^{((4))}$ lifts $b^{2^{k-1}}$

$$\begin{tikzcd}
EP_+ \wedge MU^{((4))} \arrow[r] \arrow[d] & MU^{((4))} \arrow[r] \arrow[d] & \widetilde{EP} \wedge MU^{((4))} \arrow[d] \\
EP_+ \wedge M_k \arrow[r] \arrow[d] & M_k \arrow[r] \arrow[d] & \widetilde{EP} \wedge M_k \arrow[d] \\
EP_+ \wedge H\underline{\mathbb{Z}}_{(2)} \arrow[r] & H\underline{\mathbb{Z}}_{(2)} \arrow[r] & \widetilde{EP} \wedge H\underline{\mathbb{Z}}_{(2)}
\end{tikzcd}$$

$N\bar{r}_{c_k}$ vanishes in here

$\Longrightarrow$

$\Rightarrow$ it comes from some element in $\pi^G_{c_k \rho_k}\left(EP_+ \wedge MU^{((4))}\right)$.

This goes to 0 in $\pi^G_{c_k \rho_k}(M_k)$ $\Rightarrow$ it comes from somewhere in $\pi^G_{c_k \rho_G + 1}\left(\widetilde{EP} \wedge M_k\right)$ that gets mapped to some element in $\pi^G_{c \rho_G + 1}\left(EP_+ \wedge H\underline{\mathbb{Z}}_{(2)}\right)$

$$\begin{tikzcd}
\pi^G_{c_k \rho_G + 1}\left(\widetilde{EP} \wedge M_k\right) \arrow[r] \arrow[d] & \pi^G_{c_k \rho_G}\left(EP_+ \wedge M_k\right) \arrow[d] \\
\pi^G_{c_k \rho_G + 1}\left(\widetilde{EP} \wedge H\underline{\mathbb{Z}}_{(2)}\right) \arrow[r] & \pi^G_{c_k \rho_G}\left(EP_+ \wedge H\underline{\mathbb{Z}}_{(2)}\right)
\end{tikzcd}$$

isom when rest. to the inclusion

We need to show also that the image is nonzero, but we do not have time.

Gap theorem: $\Omega^{-i}(pt) = 0$ for $-4 < i < 0$

Periodicity says $\Omega^{-i}(pt) \simeq \Omega^{-i-2^{j+1}}(pt)$ for $j > 6$

Detection says that every Kervaire invariant one element should have nonzero image in $\Omega^{-i-2^{j+1}}(pt)$.

$$\Omega^{-i}(pt) = \pi_i \Omega = \pi_i \Omega_0^{hC_8} = \pi_i \left((D^{-1} MU^{((C_8))})^{hC_8} \right)$$

$$= \pi_i^{C_8} \left(D^{-1} MU^{((C_8))} \right) = \pi_i^{C_8} \left(\operatorname{hocolim} \Sigma^{-jl\rho_{C_8}} MU^{((C_8))} \right)$$

$$= \operatorname{colim} \pi_i^{C_8} \Sigma^{-jl\rho_{C_8}} MU^{((C_8))}.$$

Slice spectral sequence: $\pi_i P_n^n \Sigma^{-jl\rho_{C_8}} MU^{((C_8))} \Rightarrow \pi_{i-n}^{C_8} \Sigma^{-jl\rho_{C_8}} MU^{((C_8))}$

$$\pi_i \Sigma^{-jl\rho_{C_8}} P^{n+8jl}_{n+njl} MU^{((C_8))}$$

slice thm

$$\pi_i \Sigma^{-jlC_8} \begin{cases} 0 & n \text{ odd} \\ H\underline{\mathbb{Z}} \wedge W & n \text{ even} \end{cases}$$

isotropic

$$= \pi_i \begin{cases} 0 \\ H\underline{z} \wedge \omega \end{cases} = 0 \quad \text{if } -4 < i < 0 \quad \text{(cell lemma)}$$

isotropic