SUVITOP 03/12/14

Real bordism

Recall: For each $n \geq 0$ there's an universal rank n v. bundle $\gamma_n \to BU(n)$

$$\begin{tikzcd} \gamma_{n-1} \oplus \mathbb{1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \gamma_n \arrow[d] \\ BU(n-1) \arrow[r] & BU(n) \end{tikzcd}$$

This gives a map on Thom spectra $S^2 \wedge BU(n-1)^{\gamma_{n-1}} \to BU(n)^{\gamma_n}$ forming a spectrum MU.

Each γ_n has a C_2-action given by complex conjugation and we have a map of C_2-equivariant spaces

$$S^{\rho_2} \wedge BU(n-1)^{\xi_{n-1}} \to BU(n)^{\xi_n}$$

regular repn. of C_2

So this patch up to give a C_2-equivariant spectrum $M\mathbb{R}$ (souped up version of MU).

The natural multiplication $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty \Rightarrow$ fgl over $\pi^{C_2}_{*\rho_2} M\mathbb{R}$.

C_2-eq. spaces

$\Rightarrow$ we get induced maps $\pi_{2k} MU \to \pi^{C_2}_{k \cdot \rho_2} M\mathbb{R}$

$\Rightarrow$ nontrivial elements in $\pi^{C_2}_{*} M\mathbb{R}$.

We're going to construct a slice-Whitehead tower of $M\mathbb{R}$.
We're going to understand a nonequiv. version of the story first.

Whitehead tower of MU

Thm (Quillen): $\mathbb{Z}[x_1, x_2, \dots] \xrightarrow{\sim} \pi_* MU \quad |x_i| = 2i$

Algebraic Whitehead tower:

$$\begin{array}{c} \downarrow \\ (x_1^2, x_2, \dots) \\ \downarrow \\ (x_1, x_2, \dots) \\ \downarrow \\ \mathbb{Z}[x_1, \dots] \end{array}$$

ideal of monomials of suff. high degree.

Goal: Use similar ideas to build a topological Whitehead tower.

$x_i : \mathbb{S}^{2i} \to MU$

We can construct a free associative ring spectrum

$$A = \mathbb{S}[x_1, x_2, \dots] = \bigvee_{\substack{m \\ \text{monomial}}} \mathbb{S}[m] \qquad \mathbb{S}[x_1^2 x_2] = \mathbb{S}^2 \wedge \mathbb{S}^2 \wedge \mathbb{S}^4$$

$$\mathbb{S}\left[\prod x_i^{n_i}\right] = \bigwedge_i \mathbb{S}^{n_i \cdot 2i}$$

There are $\mathbb{S}[m_1] \wedge \mathbb{S}[m_2] \longrightarrow \mathbb{S}[m_1 m_2]$

⚠ A is only associative!

There's a ring map $A \longrightarrow MU$ by universal property making MU an A-module spectrum.

Def: $M_{2d} = \bigvee_{\substack{m \text{ monomial} \\ \deg m \geq 2d}} \mathbb{S}[m] \subseteq A$ A-module spectrum

Filtration by A-modules

$$A = M_0 \hookleftarrow M_2 \hookleftarrow M_4 \hookleftarrow \dots$$

and the successive quotients $M_{2d}/M_{2(d+1)} \simeq \bigvee_{\deg m = 2d} \mathbb{S}[m]$ trivial A-module

(Not obvious but true that the colimits are created in spectra)

Def: $K_{2d} = MU \wedge_A M_{2d}$

$\Rightarrow$ We get a tower

$$\begin{tikzcd} MU = K_0 \arrow[d] & K_2 \arrow[l] \arrow[d] & K_4 \arrow[l] \arrow[d] & K_6 \arrow[l] \arrow[d] & \dots \arrow[l] \\ K_0/K_2 & K_2/K_4 & & & \end{tikzcd}$$

Want: K_{2d}/K_{2d+2} = wedge of E-McLane spectra.

$$\frac{MU \wedge_A M_{2d}}{MU \wedge_A M_{2d+2}} \simeq MU \wedge_A \left(M_{2d}/M_{2d+2}\right) \simeq (MU \wedge_A \mathbb{S}) \wedge \left(M_{2d}/M_{2d+2}\right)$$

the action of A on the second factor is trivial

$$\simeq (MU \wedge_A \mathbb{S}) \wedge \bigvee_{\deg m = 2d} \mathbb{S}[m]$$

Lemma (reduction): $MU \wedge_A \mathbb{S} = H\mathbb{Z}$ ⟵ ⚠ Only place in which we have used any property of the x_i.

$$\begin{array}{c} MU \wedge_A \mathbb{S} \\ \wr\wr \\ MU/(x_1, \dots) \end{array}$$

IDEA: Do the same to $M\mathbb{R}$.

Slice-Whitehead tower for $MU^{((G))}$

Let $G = C_{2^n}$. There are adjoint functors

$$N_{C_2}^G : (C_2\text{-equivariant ring}) \rightleftarrows (C_{2^n}\text{-equivariant ring}) : i^*_{C_2}$$

Def: $MU^{((G))} = N_{C_2}^G M\mathbb{R}$

Remember: We have maps $\bar{r}_k : S^{k\rho_2} \to M\mathbb{R} \to i^*_{C_2} MU^{((G))}$

We can form $\mathbb{S}[\bar{r}_k]$ = free associative ring on $\bar{r}_k \to i^*_{C_2} MU^{((G))}$

So we get a map $N_{C_2}^G(\mathbb{S}[\bar{r}_k]) \to MU^{((G))}$

$$\begin{array}{c} N_{C_2}^G(\mathbb{S}[\bar{r}_k]) \\ \| \\ \mathbb{S}[G\cdot\bar{r}_k] \end{array}$$

If I have a collection $\{\bar{r}_i : \mathbb{S}^{i\rho_2} \to M\mathbb{R}\}$

$$A := \bigwedge_{i\geq 1} \mathbb{S}[G\cdot\bar{r}_i] = \operatorname{colim}_N \bigwedge_{i=1}^{N} \mathbb{S}[G\cdot\bar{r}_i] \longrightarrow MU^{((G))}$$

Concretely

$$A = \bigvee_{f: J \to \mathbb{N}_0} \mathbb{S}^{|f|\rho_{\mathrm{stab}(f)}}$$

where $J = (G/C_2) \amalg (G/C_2) \amalg \cdots$

G acts on $\mathrm{Map}(J, \mathbb{N}_0)$ by conjugation

So A is a wedge of rep. spheres.

Def: $M_{2d} = \bigvee_{\substack{f: J \to \mathbb{N}_0 \\ |f| \dim \rho_f \geq 2d}} \mathbb{S}^{|f|\rho_{\mathrm{stab}(f)}}$ A-module

$$K_{2d} = M_{2d} \wedge_A MU^{((G))}$$

So that's our proposed Whitehead tower.

$$MU^{((G))} = K_0 \leftarrow K_2 \leftarrow K_4 \leftarrow \cdots$$

$$K_{2d}/K_{2d+2} \simeq \left(MU^{((G))} \wedge_A \mathbb{S}\right) \wedge \Big(\underbrace{\bigvee_{|g| \dim St(g) = 2d} S^{|g| \rho_{St\, g}}}_{\widehat{W}_{2d}} \Big)$$

$\widehat{W}_{2d}$ = wedge of repr. spheres

<u>Thm (Reduction)</u>: For appropriately chosen $\overline{r}_i$:

$$MU^{((G))} \wedge_A \mathbb{S} = H\underline{\mathbb{Z}}$$

<u>Recall</u>: <u>Slice cells</u>: $G_+ \wedge_H \mathbb{S}^{m\rho_H}$

We say it's • free if $H = e$
• isotropic otherwise

$\widehat{W}_{2d}$ is a wedge of isotropic even-dimensional slice cells

<u>Corollary</u>: $MU^{((G))}$ has a Whitehead tower w/ quotients of the form
$H\underline{\mathbb{Z}} \wedge$ (wedge of even-dimensional isotropic slice cells)

Last time we proved:

<u>Lemma</u>: For each $\hat{\mathbb{S}}$ regular isotropic slice cell

$$\pi_k^G(H\underline{\mathbb{Z}} \wedge \hat{\mathbb{S}}) = 0 \text{ for } -4 < k < 0$$

<u>Corollary</u>: $\pi_i^G(MU^{((G))}) = 0$ for $-4 < i < 0$ (Slice spectral sequence)

⚠ Whitehead tower $\neq$ slice tower

<u>From Whitehead to slice</u>

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
K_2/K_4 \arrow[r] & MU^{((G))}/K_4 \arrow[d] \\
K_0/K_2 \arrow[r] & MU^{((G))}/K_2 \arrow[d] \\
 & *
\end{tikzcd}$$

(Conjectural slice tower)

Why is this the slice tower?

Prop: $MU^{((G))}/K_{2d+2} \cong P^{2d} MU^{((G))}$

Proof:

$$\begin{tikzcd} K_{2d+2} \arrow[r] & MU^{((G))} \arrow[r] \arrow[d] & P^{2d} MU^{((G))} \\ & MU^{((G))}/K_{2d+2} & \end{tikzcd}$$

FACT 1: $K_{2d+2} \geq 2d$

colimit of slices of dim $> 2d$

FACT 2: $MU^{((G))}/K_{2d+2} < 2d+1$

Given the two facts the prop follows.

Pf 1: Let $\mathcal{B} = \{ B \mid B \text{ is } A\text{-module},\ B \wedge_A M_{2d} > 2d \}$

We want to show $MU^{((G))} \in \mathcal{B}$.

In fact $\mathcal{B}$ contains $G/H_+ \wedge A$ & closed under colims.

Pf 2: Induction using

$$H\underline{\mathbb{Z}} \wedge \hat{W}_{2d} \longrightarrow MU/K_{2d+2} \longrightarrow MU/K_{2d} \quad (\exists \text{ by red. thm.})$$

$MU^{((G))}/K_2 \simeq H\underline{\mathbb{Z}}$ (red. thm) and we're done.