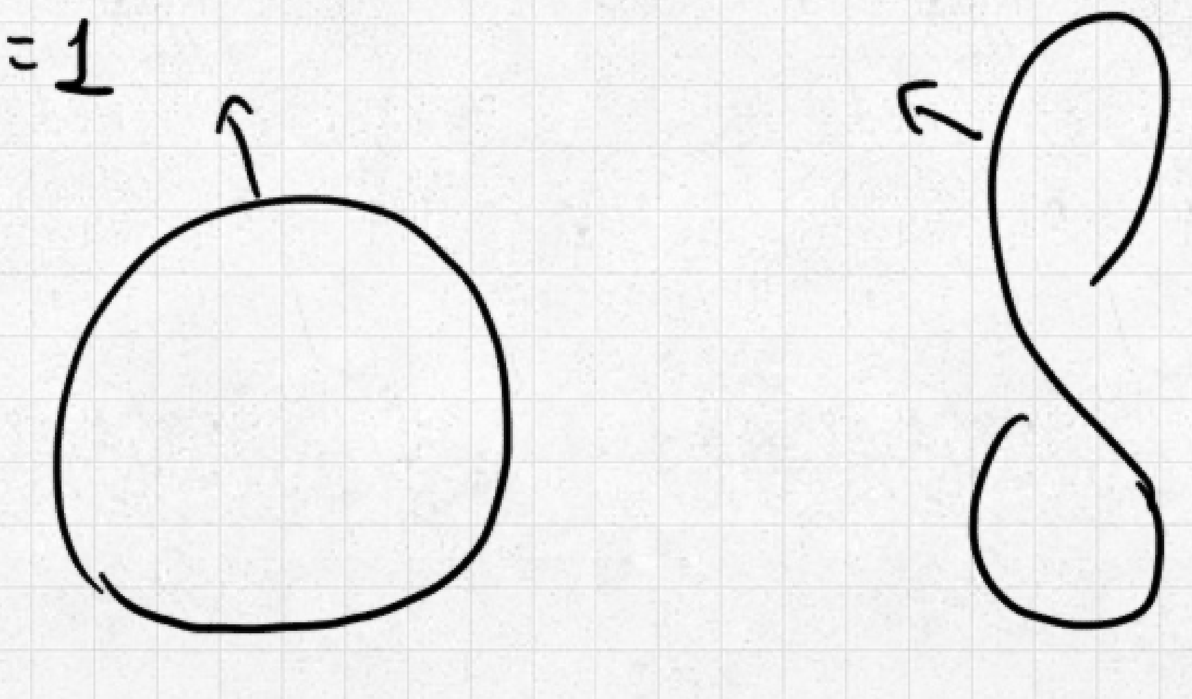
$\boxed{\text{JUVITOP } 24/09/14}$

Pontryagin $\quad \pi_n \mathbb{S} \cong \Omega_n^{fr}$ $\quad$ cobordism group of stably framed manifolds

$f: S^{n+N} \to S^N$ & take the inverse image of a regular manifold

$\Omega_0^{fr} \cong \mathbb{Z}$ $\quad$ (#positive pts − #negative pts)

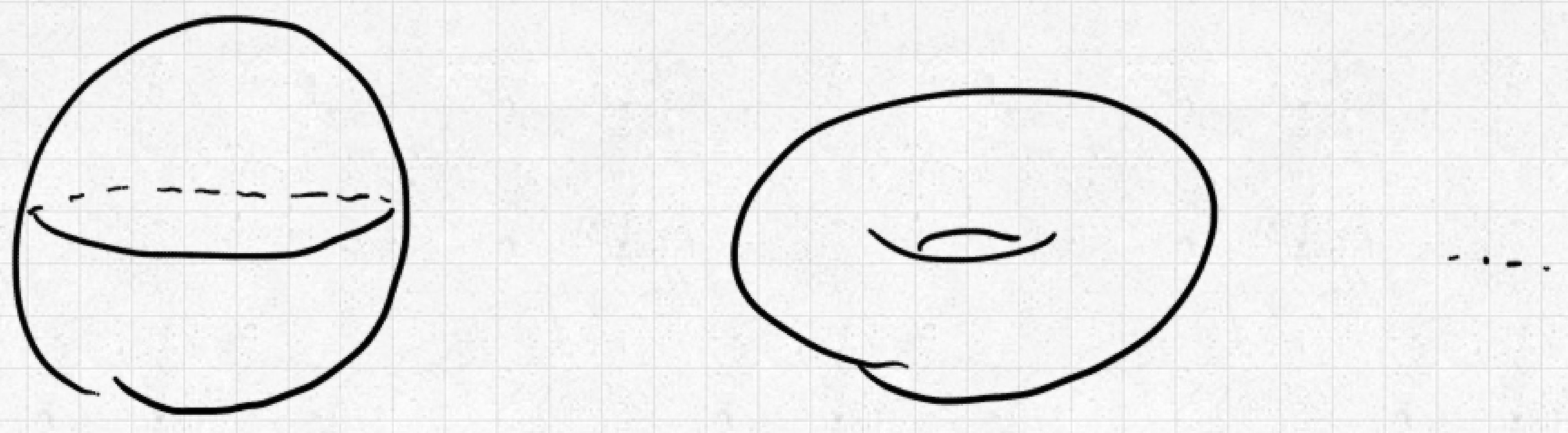
$n=1$



$\pi_1 \mathbb{S} \simeq \mathbb{Z}/2$

$n=2$

Σ :



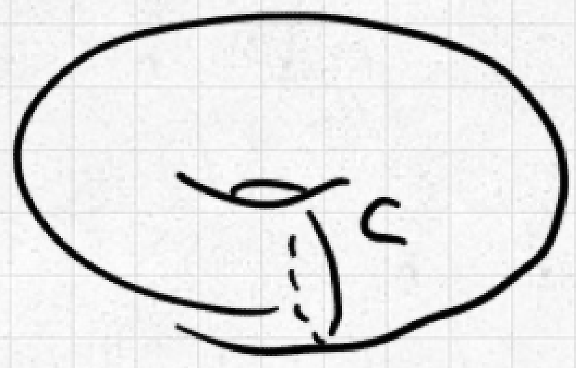
$g=0$ $\qquad$ $g=1$

Let's first suppose $g=0$. $\exists$ a framing of Σ for which $|\Sigma| = 0$ in Ω_2^{fr}

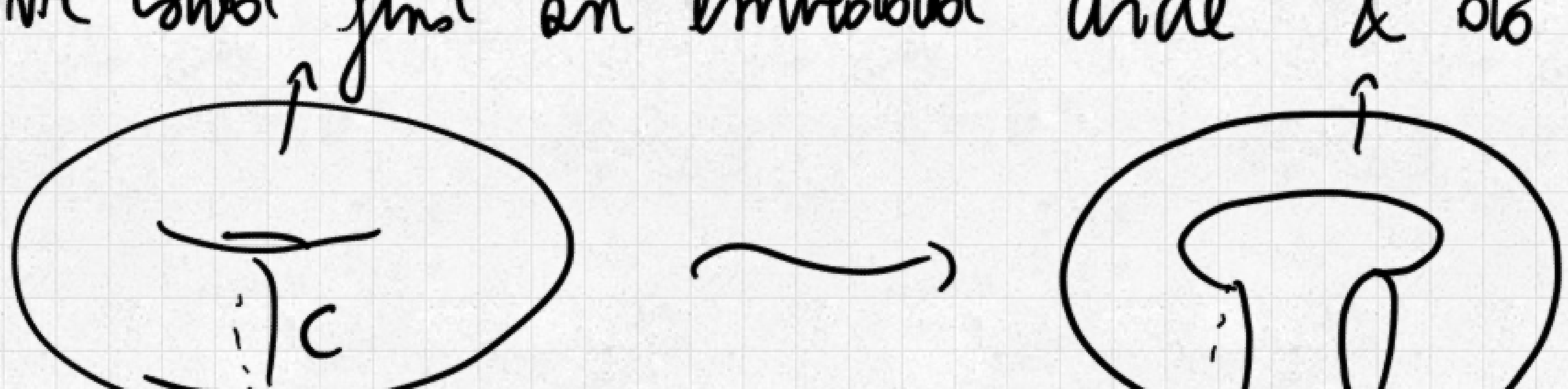
Two different framings differ by $\Sigma \to GL_N\mathbb{R}$. Since $\pi_2(\text{Lie grp}) = 0$

$\Rightarrow [\Sigma, GL_N\mathbb{R}] \simeq 0$

So if $g=0$ $\quad |\Sigma| = 0$. What about $g>0$?



Suppose we could find an embedded circle & do surgery



This lowers the genus but doesn't change the cobordism class.

There's an obstruction to extend the framing given by the class on the S^1 of surgery data. $|C| \in \Omega_1^{fr}$

<u>The surgery obstruction</u>: $\varphi : H^1(\Sigma; \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$

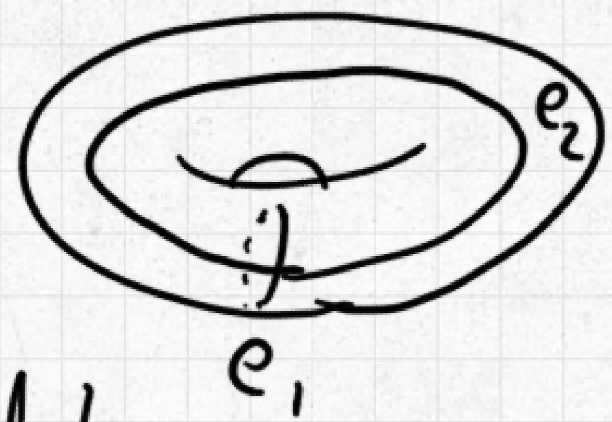
Pontryagin argued: there's always sth in $\ker \varphi \Rightarrow$ we can do always surgery

$\Rightarrow \Omega^2_{fr} = 0$ $\boxed{\text{FALSE}}$

What's the mistake? φ is not linear: it is quadratic. In fact it is a quadratic refinement of the $\cap$ pair

$$\varphi(x+y) - \varphi(x) - \varphi(y) = \langle x \cup y, [\Sigma] \rangle .$$

Let's take $g = 1$



$H_1(\Sigma) = \mathbb{Z}/2 \cdot e_1 \oplus \mathbb{Z}/2\, e_2$

There are only four possibilities

0	0	0	0
e_1	1	0	1
e_2	0	1	1
e_1+e_2	0	0	1

we don't like it! Obstruction!

A quadratic function is determined by the Arf invariant

$$e^{\pi i \,\mathrm{Arf}(\varphi)} = \frac{1}{\sqrt{\# H^1(\Sigma)}} \sum_{x \in H^1(\Sigma)} e^{\pi i \varphi(x)} \qquad (\underline{\text{democratic invariant}})$$

So for any (framed) manifold $2n$ M we get $\varphi : H^n(M; \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$

and $\mathrm{Arf}(\varphi)$ is the $\underline{\text{Kervaire invariant of } \varphi}$.

<u>Ed Brown</u>: $2n$-dim. framed manifold $\alpha \in H^n(M; \mathbb{Z}/2)$

$$\alpha : M \to K(\mathbb{Z}/2, n) \quad \in \Omega^{2n}_{fr}(K(\mathbb{Z}/2, n)) = \pi^s_{2n}(K(\mathbb{Z}/2, n))$$

And it can be computed:

$$\Sigma^\infty K(\mathbb{Z}/2, n) \longrightarrow \Sigma^n H\mathbb{Z}/2 \xrightarrow{Sq^{n+1}} \Sigma^{2n+1} H\mathbb{Z}/2 \quad \text{is a fiber sequence}$$

$\Rightarrow \pi^s_{2n}(K(\mathbb{Z}/2, n)) = \mathbb{Z}/2$. So we have

$$\varphi : H^n(M; \mathbb{Z}/2) \longrightarrow \pi^s_{2n}(K(\mathbb{Z}/2, n)) \cong \mathbb{Z}/2$$

Kervaire invariant problem: For which dimensions $\exists$ a framed manifold M w/ Kervaire invariant 1.

This question corresponds to a question in htpy thy.

Browder: A manifold M of Kervaire invariant 1 exists if $n = 2^{j+1}-2$ and in that case iff the element h_j^2 survives the ASS
$(\exists \theta_j \in \pi_{2^{j+1}-2}(\mathbb{S}))$

The classical ASS $\quad \mathrm{Ext}^{s,t}_{\mathcal{A}^*}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s}(\mathbb{S})^{\wedge}_2$

$\mathrm{Ext}^1 = \mathrm{Hom}(I/I^2, \mathbb{F}_2)$ $\qquad I/I^2$ has basis $Sq^1, Sq^2, Sq^4, \dots$

$\mathrm{Ind}(\mathcal{A}^*)$ $\qquad h_j$ sends Sq^{2^i} to 0 if $i \neq j$

h_j persists iff $\exists$ an element of Hopf invariant 1 in $\pi_{2^j-1}(\mathbb{S})$
(Adams: only for $j = 0, 1, 2, 3$)

exist

$\theta_0, \ \theta_1, \ \theta_2, \ \theta_3 \qquad \theta_4 \ \ \theta_5 \qquad \theta_6 \, ?$

$h_0^2 \quad h_1^2 \quad h_2^2 \quad h_3^2 \qquad h_4^2 \ \ h_5^2$

$S^0 \times S^0 \quad S^1 \times S^1 \quad S^2 \times S^2 \quad S^3 \times S^3$

$\boxed{\theta_i \text{ does not exists for } i \geq 7}$ (Hopkins, Hill, Ravenel)

The proof

Classical ASS $\qquad\qquad$ ANSS

$$\begin{tikzcd}
\mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2) \arrow[d] & \mathrm{Ext}_{MU_*MU}(MU_*, MU_*) \arrow[l] \arrow[dl, dotted] \\
\mathbb{F}_2 &
\end{tikzcd}$$

In $\mathrm{Ext}_{MU_*MU}(MU_*, MU_*)$ there's a lot of elements that live above h_j^2
This group was computed by Miller-Ravenel-Wilson
The important part is that there are elements of the form x, $v_1 x$, $v_2 x$
on the 2-line (no higher chromatic elems)

<u>odd primes</u>: Suppose we have a FGL over R and a finite group of symmetries

$G \to \mathrm{Aut}(R) \Rightarrow \mathrm{Ext}^{s,t}(MU_*, MU_*) \to H^*(G, R)$

Suppose $R = \pi_* E$ $\quad E \circlearrowleft G$

$$\begin{tikzcd} \mathrm{Ext}^{s,t}(MU_*, MU_*) \arrow[r] \arrow[d, Rightarrow] & H^*(G,R) \arrow[d, Rightarrow] \\ \pi_* \mathbb{S} \arrow[r] & \pi_* E^{hG} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Ext}_{MU_*MU}(MU_*, MU_*) \arrow[r] \arrow[d] & H^*(G;R) \arrow[d, "\chi \text{ extending Kervaire}"] \\ \mathrm{Ext}_{A_*}(\mathbb{F}_2, \mathbb{F}_2) \arrow[r] & \mathbb{Z}/2 \hookrightarrow \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

Γ Lubin-Tate group of height n on $\overline{\mathbb{F}}_2 \Rightarrow \mathrm{Aut}(\Gamma)$ has a largest finite subgroup. In fact we want cyclic subgp. If the height is 1 there's only $\mathbb{Z}/2$. For $n=2$ there's a $\mathbb{Z}/4$. For $n=3$ we get $\mathbb{Z}/2$ and for $n=4$ we get $\boxed{\mathbb{Z}/8}$. Since you don't want v_1, v_2 to be invertible (χ kills $v_1 x$, $v_2 x$ but leaves x). We can hope for $n=4$

<u>Purely algebraic question</u>

A: $\mathbb{Z}_2[\zeta_8]$ ring of integers $\qquad f \mapsto f'(0)$
Lubin-Tate: constructed a fgl F over A s.t. $\mathrm{End}(F) \cong A$
So $\mu_8 \subseteq A^\times = \mathrm{Aut}(F)$.

Detection theorem: $\exists\, \chi : H^*(\mathbb{Z}/8; A_*) \longrightarrow \mathbb{Q}/\mathbb{Z}$ making the diagram commute

Pf: Easy but long (need to do computations for seeing the commutativity) $\square$

Let $E = E_4$ w/ the std action of $\mathbb{Z}/8$ given by its action on A

$$\begin{tikzcd}
\mathrm{Ext}_{MU_*MU}(MU_*, MU_*) \arrow[r] \arrow[d, Rightarrow] & H^*(\mathbb{Z}/8; \pi_* E) \arrow[d, Rightarrow] \\
\pi_* \mathbb{S} \arrow[r] & \pi_* E^{h\mathbb{Z}/8}
\end{tikzcd}$$

① $\pi_* E^{h\mathbb{Z}/8}$ is periodic w/ period 256

② $\pi_{-2} E^{h\mathbb{Z}/2} = 0$

② is the "innovative" one, let's look at it in more detail

$$\pi_* E_4 = W\mathbb{F}_2[[u_1, \dots, u_3]][u^{\pm 1}] \qquad |u_i| = 0 \quad |u| = -2$$

How does $\mathbb{Z}/8$ acts on $\pi_* E_4$

If G is finite w/ cyclic p-Sylow subgroup, you can find $u_1, \dots, u_{n-1}, u$ s.t. G preserves their v. space span and the action is the one on the Dieudonné module

Case $\mathbb{Z}/8$ on $\pi_* E_4$ is the induced sign representation $\mathrm{Ind}_{\mathbb{Z}/2}^{\mathbb{Z}/8}(\mathrm{sign})$

$4 + 8\mathbb{Z}$ acts by $(-1)^m$ on $\pi_{2m} E_4$

The next idea is to take apart E_4 in $\mathbb{Z}/8$-equivariant htpy theory

We can find algebra generators $v_1, \dots, v_4$ s.t. $v_i \mapsto v_{i+1}$, $v_4 \mapsto -v_1$

We can find $\mathbb{Z}/2$-equivariant maps detecting v_1

$$S^{\mathrm{sign}} \longrightarrow E_4$$

Well, let's induce up $\mathbb{Z}/4 \times_{\mathbb{Z}/2} S^{\mathrm{sign}} \longrightarrow E_4$ picking up $v_1, \dots, v_4$

Using the norm map we have these $\hat{S} \longrightarrow E_4$

(slice tower)

and we want a filtration of E_4 w/ associated graded $H\underline{\mathbb{Z}} \wedge \hat{S}$

Easy check: For all the $\hat{S}$ that come up

$$\pi_{-2}(H\underline{\mathbb{Z}} \wedge \hat{S}) = 0$$

$$\Rightarrow \pi_{-2}\left(E_4^{h\mathbb{Z}/8}\right) = 0$$

In the paper there's no E_4, because we need to make a modification of MU to set things up and that modified MU is enough for the Kervaire inv't 1.