THE HOMOTOPY GROUPS OF TMF

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1. Introduction

The previous talks of this seminar have built up to the following theorem:

Theorem 1 ("TMF theorem")**.** Let $M_{\overline{ell}}$ be the moduli stack of stable[1] elliptic curves. There is a functor

$$\mathcal{O}^{top} : (\mathrm{Aff}^{et}_{/M_{\overline{ell}}})^{op} \to \mathrm{Alg}_{E_\infty}$$

from the (affine) étale site of $M_{\overline{ell}}$ to E_∞-rings, with the following properties:

- For every affine étale map $\mathrm{Spec}R \to M_{\overline{ell}}$, $\mathcal{O}^{top}(\mathrm{Spec}R)$ is an even-periodic E_∞-algebra with a functorial isomorphism $\pi_0(\mathcal{O}^{top}(\mathrm{Spec}R)) \simeq R$.
- For every map $\mathrm{Spec}R \to M_{\overline{ell}}$ classifying a generalized elliptic curve $C \to \mathrm{Spec}R$, there is a functorial isomorphism

$$\mathrm{Spf}\left(\mathcal{O}^{top}(\mathrm{Spec}R)^0(\mathbb{CP}^\infty)\right) \simeq \hat{C}$$

  between the formal group of $\mathcal{O}^{top}(\mathrm{Spec}R)$ and the formal group $\hat{C}$ of C (that is, the formal completion of C along the zero section with its natural group structure).
- The functor $\mathcal{O}^{top}$ is a sheaf for the étale topology and extends to a sheaf of E_∞-rings on the étale site of $M_{\overline{ell}}$.

Let M_{FG} be the moduli stack of formal groups. In particular, if $M_{\overline{ell}} \to M_{FG}$ is the map which assigns to an elliptic curve its formal group, then we have an isomorphism of sheaves

$$\pi_i \mathcal{O}^{top} = \begin{cases} 0 & \text{if } i \text{ is odd} \\ \omega^{i/2} & \text{if } i \text{ is even} \end{cases},$$

where ω is the *Lie algebra* bundle on $M_{\overline{ell}}$, which is pulled back from M_{FG}. (Strictly speaking, $\pi_i\mathcal{O}^{top}$ is only a sheaf for *affine* étales; in general we should take its sheafification.) We get this because of even periodicity. (A useful reference is [AHS01].)

One defines

$$\mathrm{Tmf} = \Gamma(M_{\overline{ell}}, \mathcal{O}^{top}).$$

In other words, Tmf is defined as the homotopy limit of the various *elliptic spectra* $\mathcal{O}^{top}(R)$ for affine schemes $\mathrm{Spec}R$ étale over $M_{\overline{ell}}$ — thus Tmf maps canonically to any such, and was initially supposed to be the "universal" elliptic cohomology theory. There are other spectra that deserve the name "topological modular forms": for instance, there is the periodic version

$$\mathrm{TMF} = \Gamma(M_{ell}, \mathcal{O}^{top})$$

obtained by taking global sections over the *smooth* locus, and the connective version

$$\mathrm{tmf} = \tau_{\geq 0}(\mathrm{Tmf}).$$

A consequence of this definition is that one has a descent spectral sequence (a special case of the spectral sequence for a cosimplicial spectrum)

$$H^i(M_{\overline{ell}}, \pi_j(\mathcal{O}^{top})) \implies \pi_{j-i}\mathrm{Tmf},$$

which we can also write as

$$H^i(M_{\overline{ell}}, \omega^j) \implies \pi_{2j-i}\mathrm{Tmf}.$$

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[1] In other words, one allows a nodal singularity away from the point at infinity.

The goal of this talk is to explain how to use this spectral sequence to compute $\pi_*\mathrm{Tmf}$, in outline. We'll start by describing the calculation of $H^i(M_{\overline{ell}}, \omega^j)$ (actually, we'll do this for the moduli stack of *smooth* elliptic curves, so really for TMF), using a series of "algebraic Bockstein spectral sequences." Then, we'll explain how (at the prime 3) we can get differentials in the spectral sequence. Mostly, the notes will follow [Bau08] and [Rez07]. Another useful survey is [Hen07].

2. A word from our sponsor

1. **The derived stack $B\mathbb{Z}/2$.** I'd like to start by doing a toy example where there isn't too much actual computation involved, but which highlights some of the global features of what happens for Tmf. Let's consider KO-theory. On the one hand, KO-theory is a cohomology theory going back to the 1950s and 60s with a beautiful geometric description in terms of real vector bundles and Bott periodicity. On the other hand, KO-theory is something that you can describe purely as a homotopy theorist.

Namely, consider K-theory K. This is an even-periodic E_∞-ring associated to the multiplicative formal group $\hat{\mathbb{G}_m}$. It comes with an automorphism

$$\Psi^{-1} : K \to K$$

associated to the automorphism of $\hat{\mathbb{G}_m}$ given by inversion, or by complex conjugation of vector bundles. This gives a $\mathbb{Z}/2$-action on K. One has:

$$KO = (K)^{h\mathbb{Z}/2}.$$

Therefore, there is a homotopy fixed-point spectral sequence that computes π_*KO (in terms of π_*K).

Aside. The $\mathbb{Z}/2$-action on K-theory goes back, in some form, to [Ati66]. The "Real" KR-theory of Atiyah is the equivariant cohomology theory (on $\mathbb{Z}/2$-spaces) represented by K.

The existence of the $\mathbb{Z}/2$-action is a consequence of the Hopkins-Miller theorem, for instance, applied to Morava E-theory E_1 at each prime. Recall that $E_1 = \widehat{K}_p$ is p-completed K-theory and Hopkins-Miller gives E_1 the structure of an E_∞-ring with a $\mathbb{Z}/2$-action coming from the Adams operation Ψ^{-1} (actually, a $\mathbb{Z}_p^\times$-action from all the Adams operations).

We can recover integral K-theory via an arithmetic square

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \prod_p \widehat{K}_p \arrow[d] \\ K_{\mathbb{Q}} \arrow[r] & \left(\prod_p \widehat{K}_p\right)_{\mathbb{Q}} \end{tikzcd} .$$

We have a $\mathbb{Z}/2$-action on $\prod_p \widehat{K}_p$ from Hopkins-Miller, and this gives one on $\left(\prod_p \widehat{K}_p\right)_{\mathbb{Q}}$ by functoriality. Since E_∞-algebras over $\mathbb{Q}$ are described as cdgas and $K_{\mathbb{Q}}$ is the formal cdga $\mathbb{Q}[u^{\pm 1}]$ with $|u| = 2$, we get a $\mathbb{Z}/2$-action on $K_{\mathbb{Q}}$ sending $u \mapsto -u$. The map $K_{\mathbb{Q}} \to \left(\prod_p \widehat{K}_p\right)_{\mathbb{Q}}$ is a map of E_∞-rings with a $\mathbb{Z}/2$-action, by choosing an explicit model over $\mathbb{Q}$. This allows us to build K from the Hopkins-Miller theorem as an E_∞-ring with a $\mathbb{Z}/2$-action, via the above fiber product.

We can also say this in similar terms as for Tmf. Namely, just as we obtain Tmf from the moduli stack of elliptic curves and their formal groups, we can obtain KO from the moduli stack of one-dimensional *tori* (i.e., algebraic groups that become $\mathbb{G}_m$ after sufficient base-change) and their associated formal groups. The moduli stack of tori is precisely $B\mathbb{Z}/2$ since the automorphism group of the torus $\mathbb{G}_m$ is $\mathbb{Z}/2$. In particular, we get a morphism of stacks

$$B\mathbb{Z}/2 \to M_{FG}$$

which, equivalently, comes from the $\mathbb{Z}/2$-equivariant formal group $\hat{\mathbb{G}_m}$.

Theorem 2 ("$B\mathbb{Z}/2$ theorem")**.** There is a functor

$$\mathcal{O}^{top} : (\mathrm{Aff}^{et}_{/B\mathbb{Z}/2})^{op} \to \mathrm{Alg}_{E_\infty}$$

with the properties:

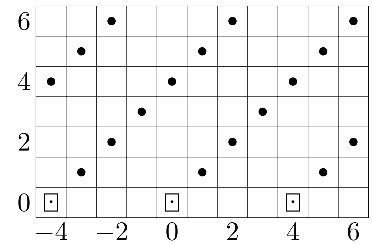

FIGURE 1. The E_2-page for the "descent" spectral sequence for KO-theory

- For every affine étale $\mathrm{Spec}R \to B\mathbb{Z}/2$, $\mathcal{O}^{top}(\mathrm{Spec}R)$ is an even-periodic E_∞-algebra with a functorial isomorphism $\pi_0(\mathcal{O}^{top}(\mathrm{Spec}R)) \simeq R$.
- For every étale map $\mathrm{Spec}R \to B\mathbb{Z}/2$ classifying a one-dimensional torus $G \to \mathrm{Spec}R$, there is a functorial isomorphism

$$\mathrm{Spf}\left(\mathcal{O}^{top}(\mathrm{Spec}R)^0(\mathbb{CP}^\infty)\right) \simeq \hat{G}$$

  between the formal group of $\mathcal{O}^{top}(\mathrm{Spec}R)$ and the formal group $\hat{G}$ of G.
- The functor $\mathcal{O}^{top}$ is a sheaf for the étale topology and extends to a sheaf of E_∞-rings on $B\mathbb{Z}/2$.

One then has:

$$KO = \Gamma(B\mathbb{Z}/2, \mathcal{O}^{top}),$$

and one can study the associated descent spectral sequence — it is exactly the homotopy fixed-point spectral sequence.

Aside. The "$B\mathbb{Z}/2$ theorem" is a consequence of the fact that complex K-theory is an E_∞-ring with a $\mathbb{Z}/2$-action given by complex conjugation, together with the "topological invariance of the étale site" for E_∞-rings. See [LN12].

2. **The descent spectral sequence.** What does this spectral sequence look like? It is

$$H^i(B\mathbb{Z}/2, \omega^j) \implies \pi_{2j-i}KO,$$

where ω is the "Lie algebra" line bundle on $B\mathbb{Z}/2$ (i.e., the sign representation of $\mathbb{Z}/2$). So to get the E_2-page, we need to know the $\mathbb{Z}/2$-cohomology of the trivial representation (over $\mathbb{Z}$), and the $\mathbb{Z}/2$-cohomology of the sign representation. The former is the cohomology of $\mathbb{RP}^\infty$, and the latter is easy to work out, since the cohomology of $\mathbb{Z}/2$ with coefficients in *any* representation is 2-periodic. The result is displayed in Figure 1. The squares denote copies of $\mathbb{Z}$ and the dots denote copies of $\mathbb{Z}/2$.

In other words, the E_2-page can be described algebraically as the bigraded algebra

$$\mathbb{Z}[t^{\pm 1}, \eta]/(2\eta), \quad |\eta| = (1,1), \quad |t| = (4,0).$$

In order to get the differentials, we'll need to compare it with the ANSS for the sphere:

Proposition 3. The above spectral sequence is the K-local Adams-Novikov spectral sequence for KO. In particular, it receives a map from the classical ANSS for the sphere S^0, and the representative Hopf map $\eta \in \pi_1(S^0)$ goes to the class called η.

This isn't obvious, and seems to require some non-formal input. In order to check that something is a valid resolution for the K-local Adams spectral sequence, we need to know, for example, that

$$K_*KO \to K_*K$$

is an injection. It isn't clear what K_*KO is or how we might compute it. In fact, it turns out that one can run the homotopy fixed-point spectral sequence as $K \otimes KO = (K \otimes K)^{h\mathbb{Z}/2}$ (where the $\mathbb{Z}/2$-action is on the second factor) — a priori this would only be obvious if K were a finite spectrum. The calculus of stacks lets you organize this nicely.

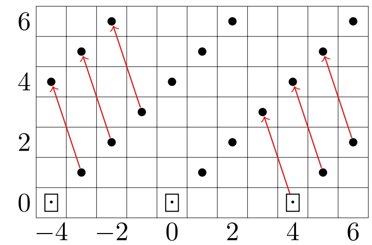

FIGURE 2. The differentials in the "descent" spectral sequence for KO-theory

However, one can produce a *map* from the ANSS of the sphere to the descent spectral sequence without it, which is actually what we need right now. Namely, the homotopy fixed point spectral sequence comes from the cosimplicial object

$$K \rightrightarrows \prod_{\mathbb{Z}/2} K \substack{\to\\\to\\\to} \prod_{\mathbb{Z}/2\times\mathbb{Z}/2} K \dots$$

whose totalization is KO. It suffices to work with the associated *semicosimplicial* object by a cofinality argument: the semisimplicial category is cofinal in the simplex category (see §6.5 of [Lur09]).

The ANSS comes from the cosimplicial, or even semicosimplicial, object

$$MU \rightrightarrows MU^{\otimes 2} \substack{\to\\\to\\\to}$$

and we can produce a map of semicosimplicial objects, which corresponds on homotopy groups to the map of Hopf algebroids

$$(MU_*, MU_*MU) \to (K_*, \prod_{\mathbb{Z}/2} K_*)$$

realizing the map of stacks

$$B\mathbb{Z}/2 \to M_{FG}.$$

(In fact, in the homotopy category, we can promote everything to a map of cosimplicial things quite directly.)

This provides the desired map of spectral sequences. An explicit algebraic calculation in the cobar construction can be used to verify that η comes from the Hopf map as desired.

Aside. It is probably possible to produce the map directly as a map of cosimplicial spectra, but I am not quite sure how to do this, unless we know that $KO \to K$ is a Galois extension (see below).

Consequently, since $\eta^4 = 0$ in the stable stems, $\eta^4 = 0 \in \pi_*(KO)$ and a differential must kill η^4. This forces:

Proposition 4. The first differential is d_3, and $d_3(t) = \eta^3$.

In figure 2, we display the differentials in the descent spectral sequence. For dimensional reasons, the spectral sequence collapses at E_4 and we get the homotopy groups familiar from Bott periodicity:

$$\pi_* KO = \mathbb{Z}[\eta, \alpha, \beta^{\pm 1}]/(2\eta, \eta^3, \alpha^2 = 4\beta),$$

where α is represented by $[2t]$ and β by $[t^2]$ where t is as in the E_2 page.

3. **General features of the ss.** Let's note some of the general features of this spectral sequence, which will carry over to the one for tmf:

- The E_2 term (which was the cohomology of some stack, and here of the group $\mathbb{Z}/2$) had infinite towers and non-nilpotent elements. This isn't too surprising: the cohomology of a finite group is always infinite.
- Although the (Bott) periodicity element β in degree 2 does not even make it to the E_2-page of the spectral sequence, β^2 is in E_2, and β^4 lives all the way till E_∞. In particular, a sufficiently high power of the even periodicity element β survives, and KO exhibits periodicity

(of a longer nature). This holds for any homotopy fixed-point spectrum of a Landweber-exact, even-periodic ring spectrum under a finite group action. For instance, TMF is 576-fold periodic.
- Most importantly, the spectral sequence *degenerates* at a finite stage with a *flat* vanishing line. One can see this as follows: there is a *pro-spectrum* coming from the cosimplicial tower that computes $K^{h\mathbb{Z}/2}$; however, this pro-spectrum turns out to be a constant pro-spectrum. This implies the desired claim.
- In particular, for *any* (possibly infinite) spectrum X, one has:
$$(K \otimes X)^{h\mathbb{Z}/2} \simeq KO \otimes X.$$
Therefore, for *any* spectrum X, one has a descent spectral sequence that computes KO_*X.

Let us sketch proofs of some of these assertions.

Proposition 5. The pro-spectrum associated to the cosimplicial object
$$\mathrm{Fix}^\bullet(\mathbb{Z}/2, K) : K \rightrightarrows \prod_{\mathbb{Z}/2} K \substack{\to\\\to\\\to} \prod_{\mathbb{Z}/2\times\mathbb{Z}/2} K \dots$$
(which computes homotopy fixed points) is constant. In particular, for any spectrum X, the natural map $KO \otimes X \to (K \otimes X)^{h\mathbb{Z}/2}$ is an equivalence.

The two statements are equivalent: the pro-spectrum associated to the cosimplicial object $\mathrm{Fix}^\bullet(\mathbb{Z}/2, K)$ can be *identified* with the functor $X \mapsto (K \otimes X)^{h\mathbb{Z}/2}$ on spectra.

Proof. One can deduce this as a general consequence of the nilpotence theorem and its associated consequences (in particular, the Hopkins-Ravenel smashing theorem [Rav92]), and then prove an analog for TMF. Something like this is apparently "well-known" to the experts but I don't know of a source—ask me after the talk if you're curious.

Here is a simple direct argument: consider the collection of *finite* spectra X such that $\mathrm{Fix}^\bullet(\mathbb{Z}/2, K) \otimes X$ is constant as a pro-object (i.e., equivalent to the constant pro-object given by its homotopy inverse limit). This is a *thick* subcategory: that is, it is closed under finite limits and colimits and retracts; we want to show that it contains S^0. The "theorem of Reg Wood" $KO \otimes \Sigma^{-2}\mathbb{CP}^2 = K$ can be proved by observing that we have an equivalence of spectra acted upon by $\mathbb{Z}/2$:
$$K \otimes \Sigma^{-2}\mathbb{CP}^2 \simeq K \oplus K$$
where the $\mathbb{Z}/2$-action on the latter is by flipping the factors. To check this, compute $K_*(\mathbb{CP}^2)$ with its $\mathbb{Z}/2$-action.

The associated homotopy fixed point cosimplicial object for $K \otimes \Sigma^{-2}\mathbb{CP}^2$ is therefore constant: it is a split cosimplicial object. So our thick subcategory contains $\Sigma^{-2}\mathbb{CP}^2$. Now a thick subcategory of finite spectra containing $\Sigma^{-2}\mathbb{CP}^2$ contains the sphere. The thick subcategory theorem ([HS98]) tells us that it contains S^0.[2] □

Corollary 1. The map $KO \to K$ is a faithful Galois extension (in the sense of [Rog08]).

Sketch of the proof of Proposition 3. The homotopy fixed-point (or descent) spectral sequence for KO-theory comes from the following cosimplicial resolution of KO-theory:
$$\mathrm{Fix}^\bullet(\mathbb{Z}/2, K) : K \rightrightarrows \prod_{\mathbb{Z}/2} K \substack{\to\\\to\\\to} \prod_{\mathbb{Z}/2\times\mathbb{Z}/2} K \dots.$$

The canonical Adams resolution for KO is obtained from the "Amitsur complex" which runs
$$\mathrm{Am}(K)^\bullet : K \rightrightarrows K^{\otimes 2} \substack{\to\\\to\\\to} \dots$$
by tensoring with KO. We don't have to use this one, though. The defining property of the resolution for the K-ASS is that the associated cosimplicial object obtained by applying K_*-homology is split (which is true) and resolves $K_*(KO)$ (which can be checked from the homotopy fixed point spectral sequence for $K_*(KO)$).

[2] It's fun to show that a thick subcategory containing $\Sigma^{-2}\mathbb{CP}^2$ contains the sphere directly from $\eta^4 = 0$; this illustrates what nilpotence has to do with thick subcategories (and is a prototype for the general Hopkins-Smith result).

□

There are analogs of all this for TMF. Localized at a prime p, the moduli stack of smooth elliptic curves admits finite étale Galois covers by affine schemes (given by the moduli of elliptic curves with a level N structure, for $p \nmid N$) and N not too small. The associated elliptic spectrum may be written $\mathrm{TMF}(N)$ for "topological modular forms of level N," and we have

$$\mathrm{TMF} = (\mathrm{TMF}(N))^{h\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})}.$$

These too are Galois covers.

But maybe it's time to start actually talking about TMF.

3. Cubic curves and the Weierstrass Hopf algebroid

Let's now discuss the spectral sequence

$$H^i(M_{ell}, \omega^{\otimes j}) \implies \pi_{2j-i}\mathrm{TMF}$$

for computing TMF. The input data is the moduli stack M_{ell} and its cohomology, so we need a way of computing that — this will be a lot more difficult than for KO. It's easier to start with the moduli stack of *cubic* curves. The basic and handy reference for this, and for integral modular forms, is [Del75].

Definition 1. A **cubic curve** over a scheme S is a morphism $p : X \to S$ with a section $e : S \to X$ such that Zariski locally on S, X is given by an equation in $\mathbb{P}^2_S$

$$y^2 + a_1xy = x^3 + a_2x^2 + a_4x + a_6$$

with $e : S \to X$ the line at ∞.

(Equivalently, p is a proper, flat morphism with a section contained in the smooth locus, whose fibers are geometrically integral curves of arithmetic genus one.)

There is a natural moduli stack M_{cub} of cubic curves. The moduli stack of *smooth* elliptic curves M_{ell} is cut out by the nonvanishing of the discriminant $\Delta \in H^0(M_{cub}, \omega^{12})$. In particular, to compute $H^*(M_{ell}, \omega^{\otimes})$, we can compute $H^0(M_{cub}, \omega^{\otimes})$ and invert Δ. So we'll start with this.

Aside. In fact, there is a morphism $M_{cub} \to M_{FG}$, and a spectral sequence

$$H^i(M_{cub}, \omega^j) \to \pi_{2j-i}\mathrm{tmf};$$

this is the Adams-Novikov spectral sequence for tmf. I haven't been able to find in the literature a good explanation of this that doesn't already rely on computations in Tmf. In [Bau08], this is used to work out $\pi_*\mathrm{tmf}$ directly.

In order to compute the cohomology of M_{cub}, we can use a *presentation* of M_{cub} by a Hopf algebroid. Strictly speaking, we note that M_{cub} has a $\mathbb{G}_m$-torsor M°_{cub} over it, given by the moduli stack of cubic curves together with a trivialization of the Lie algebra. (This $\mathbb{G}_m$-torsor is equivalent to a line bundle, namely ω.) The cohomology of the tensor powers of ω on M_{cub} is equivalent to the cohomology of the structure sheaf on M°_{cub}.

Aside. In general, a $\mathbb{G}_m$-action on something serves to record a grading. It's therefore safe to pass to a $\mathbb{G}_m$-torsor over a stack (or scheme) to compute cohomology.

In order to compute the cohomology of M°_{cub}, consider the map

$$\mathrm{Spec}\mathbb{Z}[a_1, a_2, a_3, a_4, a_6] \to M^\circ_{cub}$$

classifying the Weierstrass elliptic curve

$$y^2 + a_1xy = x^3 + a_2x^2 + a_4x + a_6$$

over $T = \mathrm{Spec}\mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$, together with the trivialization $-x/y$ of the tangent space. This turns out to be a faithfully flat cover, and it is $\mathbb{G}_m$-equivariant if we grade things so that

$$|x| = 4, \quad |y| = 6, \quad |a_i| = 2i.$$

In fact, we can recover the moduli stack M°_{cub} from the simplicial scheme

$$\cdots \substack{\to \\ \to \\ \to} T \times_{M^{\circ}_{cub}} T \rightrightarrows T.$$

Since the diagonal of M_{cub} is affine, this is a simplicial object in affine schemes. In fact, it is a groupoid object: that is, it is a *Hopf algebroid.* Let's describe it.

The scheme $T \times_{M^{\circ}_{cub}} T$ classifies the universal isomorphism between Weierstrass curves (respecting the trivialization of the tangent space). The universal such isomorphism is given by

$$x \mapsto x + r \tag{1}$$

$$y \mapsto y + sx + t, \tag{2}$$

where

$$|r| = 4, \quad |s| = 2, \quad |t| = 6.$$

It follows that

$$T \times_{M^{\circ}_{cub}} T = \mathrm{Spec}\mathbb{Z}[a_1, a_2, a_3, a_4, a_6][r, s, t],$$

and the moduli stack M°_{cub} can be presented by the following Hopf algebroid.

Definition 2. The **Weierstrass Hopf algebroid** (A, Γ) keeps track of Weierstrass equations and isomorphisms between them. Here

$$A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6], \quad \Gamma = A[r, s, t];$$

the right unit is given by

$$\eta_R(a_1) = a_1 + 2s \tag{3}$$

$$\eta_R(a_2) = a_2 - sa_1 + 3r - s^2 \tag{4}$$

$$\eta_R(a_3) = a_3 + ra_1 + 2t \tag{5}$$

$$\eta_R(a_4) = a_4 - sa_3 + 2a_2r - (t + rs)a_1 + 3r^2 - 2st \tag{6}$$

$$\eta_R(a_6) = a_6 + ra_4 + r^2a_2 + r^3 - ta_3 - t^2 - rta_1 \tag{7}$$

The coproduct is given by the composition law for isomorphisms; thus

$$\Delta(r) = r \otimes 1 + 1 \otimes r \tag{8}$$

$$\Delta(s) = s \otimes 1 + 1 \otimes r \tag{9}$$

$$\Delta(t) = t \otimes 1 + t \otimes 1 + s \otimes r. \tag{10}$$

The grading has already been specified above.

Given a Hopf algebroid, there is a *cobar complex* which one can use to compute its cohomology. Unfortunately, the Weierstrass Hopf algebroid is rather unwieldy, with a whole bunch of free variables. It's more efficient to localize at a prime and then use simpler presentations of the stack. (Analogy: to compute with the ANSS, it's easier to use BP than MU.)

Example 1. When we invert 6, the moduli stack of cubics becomes very simple. By completing the square and the cube, we can put any cubic in the form

$$y^2 = x^3 + Ax + B.$$

The only isomorphisms between cubics in this form over a $\mathbb{Z}[1/6]$-algebra come from the $\mathbb{G}_m$-action: that is, $y \mapsto u^3y$ and $x \mapsto u^2x$. It follows that

$$M^{\circ}_{cub}[1/6] = \mathrm{Spec}\mathbb{Z}[1/6][A, B],$$

so there is no higher cohomology when 6 is inverted (a fact which would not be obvious from the Weierstrass Hopf algebroid).

In particular, the descent spectral sequence degenerates with the terms concentrated on the bottom line; the homotopy groups of TMF when 6 is inverted are given simply by $\mathbb{Z}[1/6][A, B, \Delta^{-1}]$ where Δ is the discriminant.

4. The moduli stack M_{cub} at $p = 3$

The Weierstrass Hopf algebroid described in the previous section is one way of presenting M_{cub}. It is the groupoid object one gets from the faithfully flat cover $\mathrm{Spec}\mathbb{Z}[a_1, a_2, a_3, a_4, a_6] \to M_{cub}^\circ$. The key insight is that the specific Hopf algebroid is not as important as the stack, and the stack can be presented at a specific prime (the interesting ones are $p = 2, p = 3$) in simpler ways. Henceforth, we will focus on the prime 3, and by M_{cub} or M_{cub}° we will tacitly mean the localization at 3.

For example, consider the Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

When we have inverted 2, we can "complete the square" to eliminate a_1 and a_3. After a faithfully flat base change, we can perform a translation in x to eliminate a_6. This motivates, though does not quite prove:

Proposition 6. The map $\mathrm{Spec}\mathbb{Z}_{(3)}[a_2, a_4] \to M_{cub}^\circ$ classifying the cubic curve

$$y^2 = x^3 + a_2x^2 + a_4x$$

is a flat cover of M_{cub}°.

What we *have* shown is that the map is a cover: any map to M_{cub}° lifts, fppf locally, to $\mathrm{Spec}\mathbb{Z}_{(3)}[a_2, a_4]$. In particular, the map of stacks is surjective. To check flatness, we consider a map $\mathrm{Spec}R \to M_{cub}^\circ$ classifying a cubic curve, and consider the fiber product

$$\mathrm{Spec}\mathbb{Z}_{(3)}[a_2, a_4] \times_{M_{cub}^\circ} \mathrm{Spec}R.$$

If $\mathrm{Spec}R \to M_{cub}^\circ$ classifies a Weierstrass equation, the above fiber product is the universal change of coordinates that makes its a_1, a_3, a_6 vanish. One can directly compute this to be a finite, flat R-algebra.

Example 2. For example, if $R = \mathbb{Z}_{(3)}[a_2, a_4]$ itself, then we find

$$\mathrm{Spec}\mathbb{Z}_{(3)}[a_2, a_4] \times_{M_{cub}^\circ} \mathrm{Spec}\mathbb{Z}_{(3)}[a_2, a_4] = \mathbb{Z}_{(3)}[a_2, a_4, r]/(r^3 + a_2r^2 + a_4r).$$

This states that the isomorphisms of Weierstrass equations with $a_1 = a_3 = a_6 = 0$ are precisely given by x-translations by elements r satisfying the above cubic: this is easy to check by playing with the equations themselves.

This calculation tells us something important:

Proposition 7. $\mathrm{Spec}\mathbb{Z}_{(3)}[a_2, a_4] \to M_{cub}^\circ$ is a finite flat cover of rank three.

We thus obtain a more economical Hopf algebroid for computing $H^*(M_{cub}^\circ, \mathcal{O})$. The associated unnormalized cobar complex begins

$$\mathbb{Z}_{(3)}[a_2, a_4] \overset{\eta_L - \eta_R}{\to} \mathbb{Z}_{(3)}[a_2, a_4][r]/(r^3 + a_2r^2 + a_rr) \to \dots.$$

This is much more sensible. (Actually, can get something even smaller if we used the normalized cobar complex.)

Aside. There are a host of covers of the moduli stack of elliptic curves which come from "moduli of elliptic curves with some level structure." The particular cover here comes from the moduli stack of elliptic curves together with a nonzero point of order 2 (and a trivialization of the Lie algebra). It isn't obvious, though, that what we get actually extends to a three-fold cover of the moduli stack of *cubic* curves.

Let's just set down the formulas for this Hopf algebroid.

$$\eta_R(a_2) = a_2 + 3r \tag{11}$$

$$\eta_R(a_4) = a_4 + 2a_2r + 3r^2 \tag{12}$$

$$\Delta(r) = r \otimes 1 + 1 \otimes r. \tag{13}$$

5. The Bockstein spectral sequence

1. **The general setup.** At the prime 3, we now have a sensible Hopf algebroid (A, Γ) for calculating the cohomology of M°_{cub}. Still, it will be convenient to employ a further collection of spectral sequences to compute $H^*(M^\circ_{cub}, \mathcal{O})$. These spectral sequences are "algebraic Bockstein spectral sequences" and were apparently first developed by Miller and Novikov for computing the E_2-page of the Adams-Novikov spectral sequence.

To set this up, let (A, Γ) be a graded, connected Hopf algebroid, presenting a stack $\mathfrak{X}$. Let $x \in A_n$ be an invariant, homogeneous element: $\eta_R(x) = x$. Then the ideal (x) is invariant, and cuts out a closed substack of the stack associated to (A, Γ). The Bockstein spectral sequence uses the (x)-adic filtration to give rise to a spectral sequence starting from the cohomology of the stack cut out by (x) to the cohomology of the whole stack.

Theorem 8 (Miller-Novikov)**.** There is a multiplicative spectral sequence

$$H^*_{A/(x),\Gamma/(x)}(A/(x), \Gamma/(x))[t] \implies H^*(A, \Gamma),$$

where we use $H^*(A, \Gamma)$ to refer to $H^*(\mathfrak{X}, \omega^{\otimes *})$ for $\mathfrak{X}$ the stack presented by (A, Γ).

For example, if $x = p$ is a prime number, then this is the classical Bockstein spectral sequence from mod p cohomology to $\mathbb{Z}_p$-cohomology. To obtain the spectral sequence, consider the cosimplicial ring

$$A \rightrightarrows \Gamma \overset{\to}{\underset{\to}{\to}} \Gamma \otimes_A \Gamma \dots$$

whose cohomology computes $H^*(A, \Gamma)$ (via the cobar complex). Give it the x-adic filtration to get a filtered cosimplicial ring. The associated graded is the cobar complex for the Hopf algebroid $(A/(x), \Gamma/(x))$, and the spectral sequence we get is the spectral sequence for a filtered chain complex. In particular, we can work out the differentials by looking at the boundary maps in the cobar complex (they are Bocksteins and higher Bocksteins).

In many cases, when there is a grading and when we are dealing with *connected* objects, the convergence is very good. If x has positive degree, for example, the filtration we get is a finite filtration in each degree, and we have strong convergence.

2. **Filtering M_{cub}.** In the Hopf algebroid $(A, \Gamma) = (\mathbb{Z}_{(3)}[a_2, a_4], \mathbb{Z}_{(3)}[a_2, a_4, r]/(r^3 + a_2 r^2 + a_4 r))$, we note the sequence of invariant ideals

$$I_1 = (3), \quad I_2 = (3, a_2), \quad I_3 = (3, a_2, a_4).$$

These correspond to the closed substacks cut out by 3, the Hasse invariant v_1, and then v_2. (All that is left at the end is the "cuspidal" cubic $y^2 = x^3$, whose formal group is the additive one.) The strategy is to compute the cohomology of these closed substacks, starting from the smallest and working one's way up, using the Bockstein spectral sequence. The differentials in the Bockstein spectral sequence are Bocksteins and come from a known filtration of the cobar complex, so they can be calculated explicitly.

First, though, let's note some general features of what we should get about the cohomology:

- The stack M°_{cub} admits a three-fold flat cover by an affine scheme. Consequently, the only torsion in the cohomology can be of order three.
- The global sections of $\mathcal{O}_{M^\circ_{cub}}$ are called *integral modular forms* (localized at 3).[3] The ring of integral modular forms can be calculated ([Del75]) to be isomorphic to

  $$\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 = 1728\Delta),$$

  where Δ is the discriminant and c_4, c_6 are certain modular forms. One can explicitly write (see [Del75]) formulas for these.

[3]Why "modular forms"? A *modular form* of weight $2k$ in the classical sense of the word can be described as a function from the space of lattices in $\mathbb{C}$ to $\mathbb{C}$ satisfying certain holomorphicity and homogeneity conditions. But an elliptic curve over $\mathbb{C}$ with a trivialization of its tangent space is the same thing as a lattice. So a modular form in the classical sense can be identified as a function from elliptic curves equipped with a trivialization of the Lie algebra to $\mathbb{C}$ satisfying certain holomorphicity and homogeneity conditions. These conditions are encoded algebraically here.

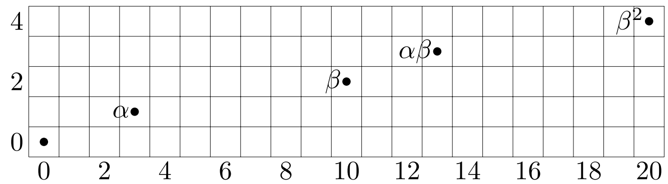

FIGURE 3. The cohomology of $B\alpha_3$, with Adams indexing

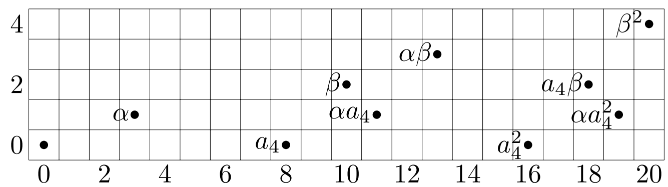

FIGURE 4. The BSS for the cohomology of $(A/I_2, \Gamma/I_2)$

- There has to be nontrivial cohomology of M°_{cub} (or tmf would be rather uninteresting). One can see this by noting the existence of a "mod 3" modular form (the Hasse invariant) which is not liftable to characteristic zero. (By contrast, for primes $p > 3$, the Hasse invariant is liftable to characteristic zero, via the Eisenstein series E_{p-1}.)

3. Let's start by computing the cohomology of the Hopf algebroid $(A/I_3, \Gamma/I_3)$. This is the moduli stack of cubic curves with additive reduction — i.e., whose formal group is locally isomorphic to the additive one. As a Hopf algebroid, this is

$$(\mathbb{Z}/3, \mathbb{Z}/3[r]/(r^3)),$$

where r is primitive. In other words, it is the classifying stack of the group scheme α_3.

Using a minimal resolution, one computes the cohomology of this stack:

Proposition 9. The cohomology ring of $B\alpha_3$ (as presented by the above Hopf algebroid) is $\mathbb{Z}/3[\alpha, \beta]/(\alpha^2)$ where:

- $\alpha \in H^{1,4}$ is represented by $[r]$ in the cobar complex.
- $\beta \in H^{2,12}$ is represented by $[r^2|r] - [r|r^2]$ in the cobar complex, and $\beta = \langle \alpha, \alpha, \alpha \rangle$.

The result is displayed in Figure 3. In all diagrams, we use *Adams indexing*: the spot (s, t) refers to $H^{s,s+t}$.

Note that α and β are defined in terms of r and rely only on the fact that r is primitive; thus they lift to the cohomology of M°_{cub} (which means that the Bockstein differentials exiting them are all trivial in future BSSes).

4. Let's now work our way up to the cohomology of $(A/I_2, \Gamma/I_2)$. To get here, we use the Bockstein spectral sequence. To get the Bockstein spectral sequence, we just add a free variable a_4 to the picture we got in the previous section. This is displayed in Figure 4.

The Bockstein spectral sequence starts with this picture. All the differentials go up by one and to the left by one, which sounds at first strange except that the BSS actually has a *tri*grading that I am ignoring. In any event, the fact that α, β lift as cohomology classes (they lift all the way to M°_{cub}) means that the BSS degenerates. We find:

Proposition 10. The cohomology ring of $(A/I_2, \Gamma/I_2)$ (as presented by the above Hopf algebroid) is $\mathbb{Z}/3[\alpha, \beta, a_4]/(\alpha^2)$ where:

- $\alpha \in H^{1,4}$ is represented by $[r]$ in the cobar complex.
- $\beta \in H^{2,12}$ is represented by $[r^2|r] - [r|r^2]$ in the cobar complex, and $\beta = \langle \alpha, \alpha, \alpha \rangle$.
- $a_4 \in H^{0,8}$ is represented by a_4 in the cobar complex (i.e., it is an invariant element of the ground ring).

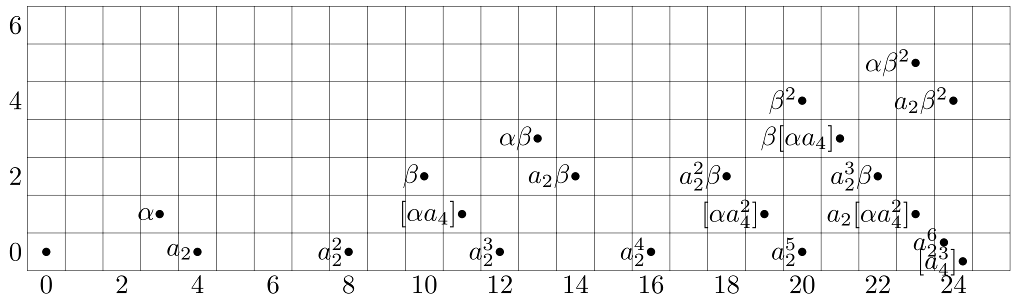

FIGURE 5. The BSS for the cohomology of $(A/I_1, \Gamma/I_1)$ after the first round of differentials

And so the picture of the BSS is the picture of the cohomology.

5. Let's now continue our way up to the cohomology of $(A/I_1, \Gamma/I_1)$. The passage to here is the hard part.

We start with the diagram of the cohomology of $(A/I_2, \Gamma/I_2)$, and adjoin a free variable a_2 in position $(4, 0)$. This time, there are differentials:

- As before, α, β are permanent cycles that survive the BSS.
- However, now a_4 is not a permanent cycle. Looking at the boundary of a_4 in the cobar complex shows that the Bockstein differential has

$$a_4 \mapsto 2a_2\alpha,$$

since α is represented by $[r]$ in the cobar complex. The spectral sequence is multiplicative, so that determines the first page of differentials in the BSS.

In particular, we can enumerate the cycles and boundaries:

- The cycles are the subring generated by $a_2, \alpha, \beta, a_4\alpha, a_4^2\alpha, a_4^3$.
- The boundaries are the ideal (in the ring of cycles!) generated by $a_2\alpha$ and $a_4a_2\alpha = d(a_4^2)$.

While a_4 does not survive, αa_4 does, and represents a class in the next page denoted $[\alpha a_4]$. Also, a_4^3 does, thanks to the power of the Frobenius.

We display in Figure 5 what happens *after* the first round of BSS differentials. We observe that $a_2[\alpha a_4] = 0$ but $[\alpha a_4^2]$ generates a free $\mathbb{Z}/3[a_2]$-module summand. The whole thing is periodic with $[a_4^3]$.

Now let's try to consider the *second* round of differentials. The question is where $[a_4^3]$ and $[\alpha a_4^n]$ go.

- $[\alpha a_4]$ is represented in the cobar complex by $a_4[r]$. The cobar boundary of this is

$$2a_2[r|r].$$

Therefore, if we consider $a_4[r] - a_2[r^2]$, then this is equal to $a_4[r]$ modulo terms of positive a_2-filtration, and is an honest cycle mod 3. It follows that $[\alpha a_4]$ is a cycle.
- By contrast, $[\alpha a_4^2]$ is no longer a cycle. This is represented in the cobar complex by $a_4^2[r]$, and the cobar boundary of this is $(\eta_L(a_4^2) - \eta_R(a_4^2))[r]$ or

$$\left((a_4^2 - (a_4 + 2a_2r)^2\right)[r] = -a_2a_4[r|r] - a_2^2[r^2|r].$$

Let's try to modify this by another cocycle of positive (a_2)-filtration to get something whose boundary has even smaller (a_2)-filtration, and hope that this calculation doesn't have too many sign errors. Namely, observe that

$$d(a_2a_4[r^2]) = 2a_2^2[r|r^2] + 2a_2a_4[r|r] = -(a_2^2[r|r^2] + a_2a_4[r|r]),$$

and therefore

$$d(a_4^2[r] - a_2a_4[r^2]) = a_2^2\left([r|r^2] - [r^2|r]\right).$$

Unwinding the definition of the spectral sequence, this means that the *higher* Bockstein on $[\alpha a_4^2]$ is hitting precisely $a_2^2\beta$.

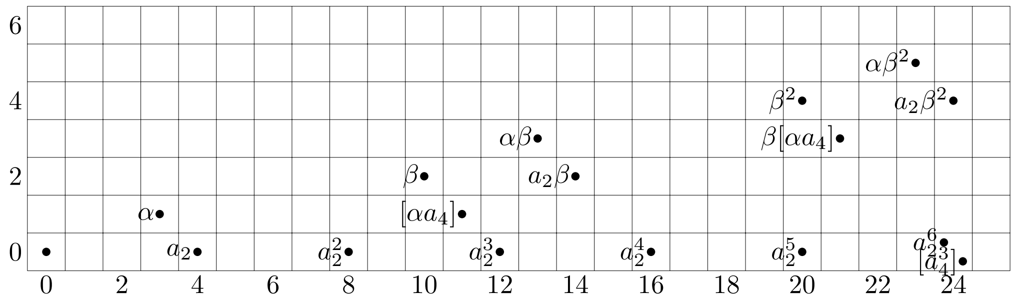

FIGURE 6. The BSS for the cohomology of $(A/I_1, \Gamma/I_1)$ after the second round of differentials (and the final thing)

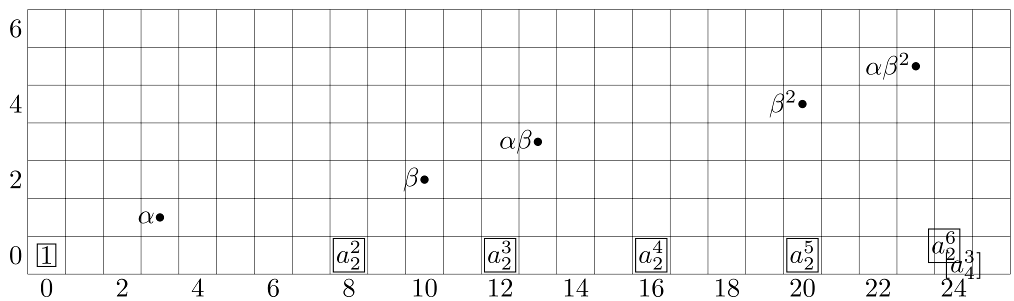

FIGURE 7. The cohomology of M_{cub} at 3

- It is easy to see that a_4^3 can go nowhere, since there is nothing for it to kill. In fact, $a_2[\alpha a_4^2]$ is the only option, and we have just seen that it is not a cycle.

In other words, (since $[\alpha_4^3]$ is a permanent cycle in the BSS) we find that $[a_4^n\alpha]$ is a permanent cycle for $n \equiv 0, 1 \mod 3$, while at $n = 2$ is kills $a_2^2\beta$.

This means in the spectral sequence, the ideal $a_2^2\beta$ is now killed, and it is displayed in Figure 6.

There are no more differentials. Observe that there is a periodicity with period $[a_4^3]$.

6. In the previous section, we computed the cohomology of $(A/(3), \Gamma/(3))$, and found that it was generated by classes $a_2, \alpha, \beta, [\alpha a_4], [a_4^3]$. Here $[\alpha a_4], \alpha$ had square zero, and $\beta a_2^2 = 0$.

Now we have to run a final Bockstein spectral sequence to get the cohomology of (A, Γ). The Bockstein spectral sequence is really trigraded, with multiplication by 3 jutting out of the page and giving an infinite cube spectral sequence. We won't draw that. Suffice it to say:

- α, β are known to be permanent cycles.
- $d(a_2) = 3\alpha$.
- $d([\alpha a_4]) = \pm 3\beta$ or otherwise β would have order more than three (which is impossible, as we saw); of course, this can be checked in the cobar complex too.
- $[a_4^3]$ is, for dimensional reasons, a permanent cycle.

Observe that a_2^2 survives, as does a_2^3.

The final output of the spectral sequence is displayed above in Figure 7. Here, modulo signs, a_2^2 is represented by c_4, a_2^3 by c_6 (mod something divisible by 3), and a_3^4 is represented by Δ.

Theorem 11. The cohomology of M_{cub} is given by

$$\mathbb{Z}_{(3)}[\alpha, \beta, c_4, c_6, \Delta]/I,$$

where I is the ideal generated by the relations

$$3\alpha = 3\beta = 0 \tag{14}$$

$$\alpha^2 = 0 \tag{15}$$

$$c_4\alpha = c_4\beta = c_6\alpha = c_6\beta = 0 \tag{16}$$

$$c_4^3 - c_6^2 = 1728\Delta, \tag{17}$$

The last statement is a known result about integral modular forms. We note in particular that the cohomology exhibits a Δ-periodicity (by contrast, if we invert c_4, we get nothing in higher cohomology).

6. The homotopy groups of TMF

In the previous section, we computed the cohomology groups of M_{cub} at 3. We find as a result that the E_2-page of the descent spectral sequence for TMF is given by

$$\mathbb{Z}_{(3)}[\alpha, \beta, c_4, c_6, \Delta^{\pm 1}]/I,$$

where I is the ideal generated by the same relations. In this section, we will determine the differentials in the spectral sequence (called the *elliptic spectral sequence* by [Bau08]) and thus describe $\pi_*\mathrm{TMF}$. It will turn out that the entire spectral sequence is a consequence of a single differential, which will come out of the homotopy groups of spheres.

Proposition 12. c_4, c_6 are permanent cycles in the elliptic spectral sequence at $p = 3$ (that is, they survive to elements in $\pi_*\mathrm{TMF}$). Ditto for α, β.

Proof. By comparison with the ANSS at $p = 3$, α is in the image of the Hurewicz homomorphism (under the class in the homotopy groups of spheres usually called α). We get $\beta = \langle \alpha, \alpha, \alpha \rangle$ since the same relation is true in $\pi_*(S^0)$.

To see that c_4, c_6 are permanent cycles, we first show that they are in the image of the *transfer* from the three-fold cover of M_{ell} given by $\left(\mathrm{Spec}\mathbb{Z}_{(3)}[a_2, a_4][\Delta^{-1}]\right)/\mathbb{G}_m$. To see this, we may pass to the 6-fold Galois cover of M_{ell} by imposing a full level 2 structure (i.e., a choice of two distinct nonzero points of order two on an elliptic curve). This gives a Galois cover

$$M_{ell}[2] \to M_{ell}$$

which is Galois with group S_3. The claim is that c_4, c_6 are in the image of the transfer (trace) from this 6-fold cover; this is equivalent to the statement about the 3-fold cover as we are localized at 3. This follows from the next lemma since c_4, c_6 kill the torsion in the cohomology of the moduli stack.

Once one sets up an appropriate *topological* theory of the transfer (trace maps) for E_∞-rings, this proposition should be a corollary of the algebraic observation just made, because c_4, c_6 arise in the image of the transfer from "topological modular forms of level 2."

□

Lemma 13. Let M be a $\mathbb{Z}_{(3)}[S_3]$-module. Let $x \in M$; then $x \in M^{S_3}$ is a norm if and only if $x\beta = 0$ where $\beta \in H^2(S_3; \mathbb{Z}_{(3)})$ is a certain 3-torsion element.

Proof. We can replace S_3 by the cyclic group C_3. In this case, given $x \in H^0(C_3, M)$, to say that x is a norm is to say that x maps to zero in the *Tate cohomology* $\hat{H}^0(C_3, M)$. The Tate cohomology of C_3 with any coefficients is periodic with periodicity operator given by $\beta \in H^2(C_3, \mathbb{Z}) = H^2(S_3; \mathbb{Z}_{(3)})$. □

In Figure 8, we display the elliptic spectral sequence. The ideal in the zero-line generated by c_4, c_6 is not shown, because these are permanent cycles by the above proposition and they annihilate the torsion of the spectral sequence. Instead of writing α, we use a line of slope 1/3 to indicate that two classes are connected by a multiple of α.

Proposition 14. The first differential in the spectral sequence is a d_5, and

$$d_5(\Delta) = \pm\beta^2\alpha.$$

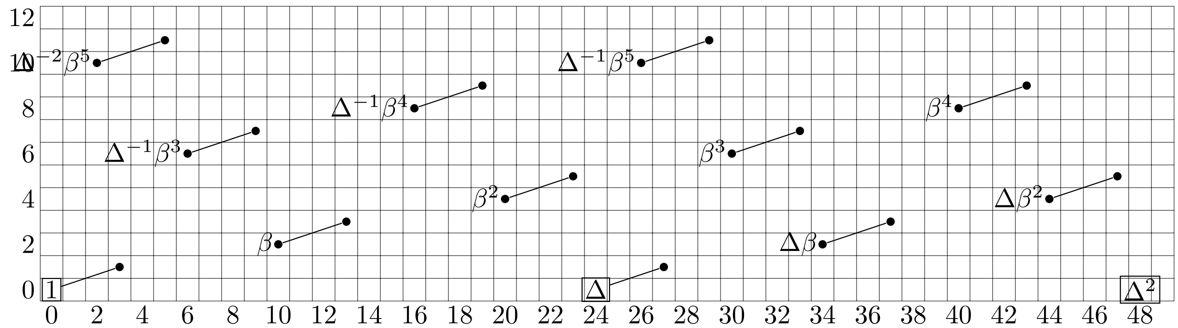

FIGURE 8. The E_2-page of the elliptic spectral sequence at $p = 3$

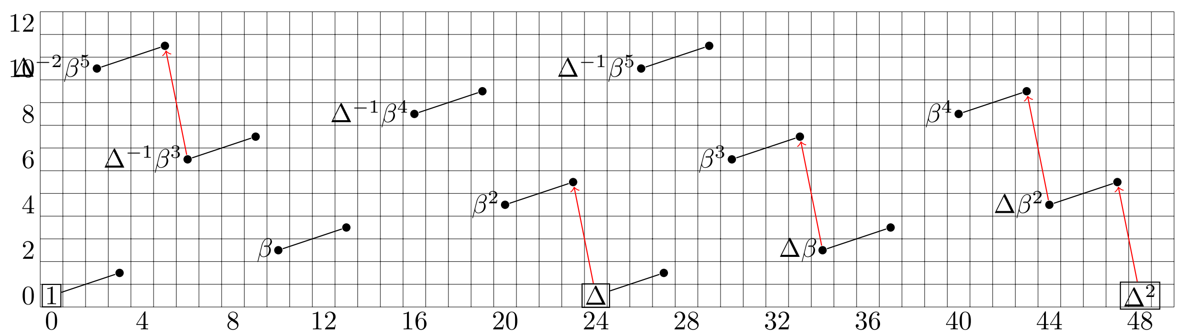

FIGURE 9. d_5 in the elliptic spectral sequence at $p = 3$

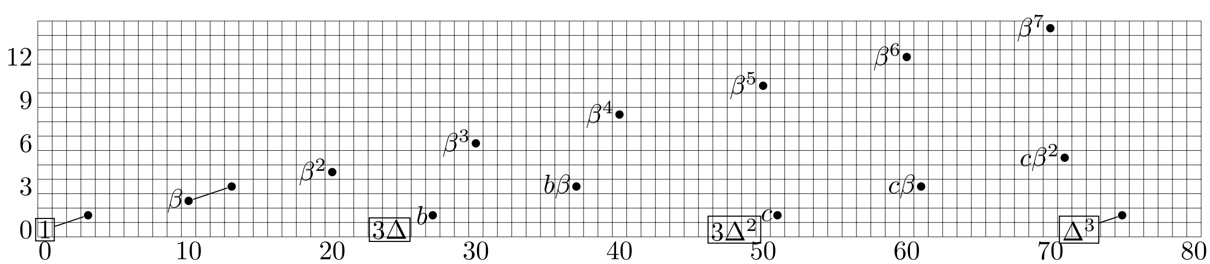

FIGURE 10. The elliptic spectral sequence at $p = 3$ after d_5

Proof. There are no differentials before d_5 for dimensional reasons. One knows that $\beta^3\alpha = 0$ in the homotopy groups of spheres; this is the "Toda relation" ([Tod68]) and must therefore hold in TMF as well. In particular, a differential must kill $\beta^3\alpha$ in the elliptic spectral sequence. The only possibilities for elements which can kill $\beta^3\alpha$ are already visible in the spectral sequence, and we get

$$d_5(\Delta\beta) = \pm\beta^3\alpha.$$

This forces the desired differential. □

To recap:

- The survivors of this spectral sequence include $\alpha, \beta, [3\Delta], b = [\alpha\Delta], c = [\alpha\Delta^2], \Delta^3$, and in fact are the ring generated by these.
- We have the relations $\alpha\beta^2 = 0$ and $\Delta\beta^2\alpha = 0$. $b\beta^2 = 0$ and $[3\Delta]$ kills all the torsion.

Figure 10 displays the elliptic spectral sequence after the d_5 differential; we observe that this forces $\alpha\beta^2 = 0 \in \pi_*(\mathrm{TMF})$ rather than simply up to higher filtration (in fact, there is nothing in the column that used to be $\beta^2\alpha$, even from towers starting at other powers of Δ). Observe that there is now a

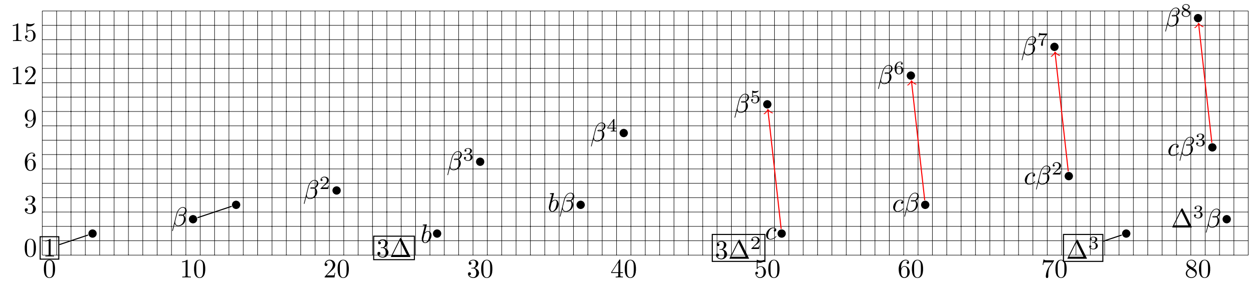

FIGURE 11. The d_9 in the elliptic spectral sequence at $p = 3$

periodicity generated by Δ^3 rather than Δ. We are not done, though. The powers of β continue, while we know that β is nilpotent in $\pi_*(\mathrm{TMF})$: in fact, the spectral sequence should degenerate at a finite stage with a *horizontal* vanishing line, which we certainly do not have right now.

However, if we play with Massey products, we can say a few things. First, let's recall the equation

$$\beta = \langle \alpha, \alpha, \alpha \rangle$$

which is already valid in the homotopy groups of spheres, and which we saw in the cohomology of the moduli stack of elliptic curves. Consider that

$$\beta^3 = \beta^2 \langle \alpha, \alpha, \alpha \rangle \subset \left\langle \beta^2\alpha, \alpha, \alpha \right\rangle .$$

Since $\alpha^2\beta$ is nullhomotopic in $\pi_*\mathrm{TMF}$, we find that (in $\pi_*\mathrm{TMF}$) that

$$\beta^3 \in \langle 0, \alpha, \alpha \rangle = \alpha\pi_{27}\mathrm{TMF}.$$

In particular, since $\beta^3 \in \pi_*\mathrm{TMF}$ is nonzero (clearly it is a permanent cycle and nothing can kill it in the spectral sequence), we find that there is a nonzero class in $\pi_{27}\mathrm{TMF}$ such that when multiplied by α, it gives β^3. This class must have filtration lower than β^3 and a look at the spectral sequence now shows:

Proposition 15. The class b is a permanent cycle and converges to an element of $\pi_{27}TMF$ with $b\alpha = \beta^3$ (at least modulo terms of higher filtration).

In fact, we will see that there are no terms of higher filtration. For now, we just need to know that b survives.

The transfer argument showed that 3Δ and $3\Delta^2$ survive in the spectral sequence. If Δ^3 supports a differential, we know that at least it is a very long differential (certainly not until d_{10}).

The next order of business is c. We have

$$\beta^5 = \beta^2\beta^2 \langle \alpha, \alpha, \alpha \rangle \subset \left\langle \alpha\beta^2, \alpha, \alpha\beta^2 \right\rangle = \langle 0, \alpha, 0 \rangle = 0.$$

Therefore, something must kill β^5. The only possibility is that

$$d_9(c) = \pm\beta^5.$$

We draw the ninth differential next:

After d_9, there is room for no more differentials. We conclude with the desired computation:

Theorem 16. We have

$$\pi_*\mathrm{TMF}_{(3)} = \mathbb{Z}_{(3)}[c_4, c_6, [3\Delta], [3\Delta^2], \Delta^{\pm 3}, \alpha, \beta, b]/J$$

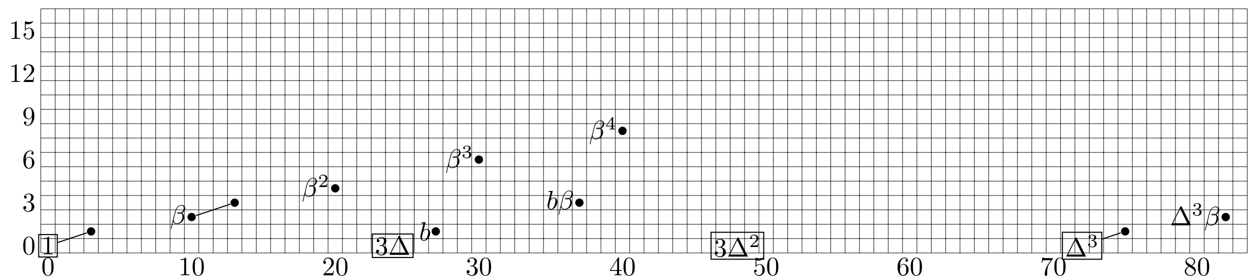

FIGURE 12. The homotopy groups of TMF (at $p = 3$)

where J is the ideal generated by the relations

$$c_4^3 - c_6^2 = 576[3\Delta] \tag{18}$$

$$[3\Delta]^2 = 3[3\Delta^2] \tag{19}$$

$$3\alpha = 3\beta = 3b = 0 \tag{20}$$

$$\alpha[3\Delta] = \alpha[3\Delta^2] = \beta[3\Delta] = \beta[3\Delta^2] = 0 \tag{21}$$

$$\alpha\beta^2 = \beta^5 = 0 \tag{22}$$

$$c_4\alpha = c_4\beta = c_4 b = c_6\alpha = c_6\beta = c_6 b = 0 \tag{23}$$

(24)

Once again, we find that the spectral sequence degenerates at a finite stage (at E_{10}) with a horizontal vanishing line. Let $\mathrm{Spec} R \to M_{ell}$ be an étale cover (localized at 3, as always). Then we can form a simplicial diagram

$$\dots \substack{\to\\ \to\\ \to} \mathrm{Spec} R \times_{M_{ell}} \mathrm{Spec} R \rightrightarrows \mathrm{Spec} R$$

and consequently a cosimplicial diagram in E_∞-rings

$$\mathcal{O}^{top}(\mathrm{Spec} R) \rightrightarrows \mathcal{O}^{top}(\mathrm{Spec} R \times_{M_{ell}} \mathrm{Spec} R) \substack{\to\\ \to\\ \to} \dots$$

whose homotopy limit is, by definition, TMF. The descent spectral sequence is the homotopy spectral sequence for this cosimplicial E_∞-rings. However, the degeneration of the descent ss at a finite stage with a horizontal vanishing line reflects the fact that the seemingly infinite homotopy limit here is really a finite one. The pro-object associated to the Tot tower is a constant pro-object. Moreover, TMF exhibits its own periodicity, here of order 72 (when localized at 3).

We remark that there are maps

$$\pi_* S^0 \to \pi_* \mathrm{TMF} \to MF_*[\Delta^{-1}]$$

where $MF_* = H^*(M_{cub}, \omega^{\otimes *})$ is the ring of integral modular forms. The first map is the Hurewicz homomorphism and the second map is the edge homomorphism in the descent spectral sequence. Surprisingly, most of the torsion in $\pi_*\mathrm{TMF}$ (at 3, at least) comes from $\pi_* S^0$, and the torsion-free part is still pretty close to MF_*. This perhaps justifies the name "topological modular forms."

7. FURTHER REMARKS

1. $p = 2$**.** At the prime 2, the calculation is handled somewhat similarly, at least in outline. However, the details are much more complicated. Rather than a two-fold cover of the moduli stack of cubics, one has to use an *eight-fold* cover (given by a elliptic curves with a $\Gamma_1(3)$ structure) to compute the Hopf algebroid cohomology. Then, when computing $\pi_*\mathrm{TMF}$, the modular forms c_4, c_6 support quite a few differentials, too—not just Δ. For inspiration, see [Hop08].

In short, both the computation of the cohomology of the stack and the differentials is far more complicated, but it is worked out in [Bau08].

2. **The classical Adams spectral sequence.** It is also possible to compute $\pi_*\mathrm{tmf}$ using the classical Adams spectral sequence. Namely, it is known that

$$H^*(\mathrm{tmf};\mathbb{Z}/2) \simeq \mathcal{A}//\mathcal{A}_2,$$

where $\mathcal{A}$ is the (mod 2) Steenrod algebra and $\mathcal{A}_2 \subset \mathcal{A}$ is the 64-dimensional subalgebra generated by $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4$. The "change-of-rings isomorphism" enables one to calculate the E_2-page of the classical ASS for $\mathrm{tmf}_{(2)}$ via

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_2}(\mathbb{Z}/2,\mathbb{Z}/2) \implies \pi_{t-s}\mathrm{tmf}_{(2)}.$$

This is a more complicated analog of the fact that

$$H^*(bo;\mathbb{Z}/2) \simeq \mathcal{A}//\mathcal{A}_1,$$

from which one can compute $\pi_* bo$ as in chapter 3 of [Rav86]. Computing Ext over $\mathcal{A}_2$ is *much* more complicated than making the computations for bo, and the ASS for tmf is a mess.

Aside. It is not possible to have spectra whose cohomology is $\mathcal{A}//\mathcal{A}_n$ for $n > 2$. This is a consequence of the solution to the Hopf invariant one problem.

Aside. It was initially (incorrectly) believed that there was no spectrum with the cohomology $\mathcal{A}//\mathcal{A}_2$.

At the prime 3, the cohomology of tmf is not cyclic over the Steenrod algebra, but one can still run the ASS. See [Beh].

As far as I know, to prove such things (or in general anything about tmf) requires some prior knowledge about the homotopy groups of Tmf—that is, I don't think you can get away from the Adams-Novikov (or descent) spectral sequence for Tmf.

3. **The ANSS for** tmf**.** The ANSS for tmf is well-understood. Recall that the ANSS runs

$$\mathrm{Ext}^{s,t}_{MU_*MU}(MU_*, MU_*\mathrm{tmf}) \implies \pi_{t-s}\mathrm{tmf}.$$

It is known that the E_2-page of this spectral sequence is isomorphic to $H^*(M_{cub}, \omega^{\otimes j})$. More is true: the Hopf algebroid

$$(MU_*\mathrm{tmf}, MU_*(MU \otimes \mathrm{tmf}))$$

presents a stack equivalent to the moduli stack M_{cub} of pointed cubic curves. This result is used in [Bau08].

There is an analog of this result, too, for KO-theory. The ANSS for bo can be described using the moduli stack of "quadratic equations and translations."

I know a rather convoluted argument for this result, but it too relies on computations in $\pi_*\mathrm{Tmf}$ (specifically, the existence of a "gap theorem") in the homotopy groups. Does anyone know how to prove this directly? Please enlighten me!

4. **Duality.** Given the homotopy groups of tmf (computed using the ASS or ANSS above), one can work out the homotopy groups of TMF via $\mathrm{TMF} = \mathrm{tmf}[\Delta^{-1}]$. Obtaining the homotopy groups of Tmf is more subtle. The following result enables us to do so:

Theorem 17 (Stojanoska [Sto12])**.** The Anderson dual of Tmf is $\Sigma^{21}\mathrm{Tmf}$.

The result does not immediately tell us how to compute, for instance, $\pi_{-1}\mathrm{Tmf}$. However, we have:

Theorem 18 (Gap theorem)**.** $\pi_i\mathrm{Tmf} = 0$ for $-21 < i < 0$.

It would be interesting if there were a way to prove this result directly from duality. The computations done above are sufficient to prove the gap theorem at $p = 3$, though.

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