Nilpotence detecting Lubin-Tate theories at heights 0, 1

We will prove the following result, which implies Nullstellensatz at height 1.

**Theorem.** Let $R \in \text{CA}_g(S_{T_0})$. \exists a nilpotence detecting map $R \rightarrow E(K)$ for $K$ a reduced Krull dimension 0 ring.

First we'll prove the analogous result at height 0.

**Theorem.** Let $R \in \text{CA}_g(S_{T_{10}})$. \exists nilp def map $R \rightarrow K[[t^\pm]]$ for some $K$ reduced Krull dimension 0.

**Idea.** Keep modifying $R$ to make it closer to $K[[t^\pm]]$ while at every step making sure the new object detects nilpotence.

Keep doing until new ring looks like what you want.

Nilpotence detecting maps are closed under

- (transfinite) compositions
- colakechongt

so a small object argument shows that for any collection of nilpotence detecting maps $\alpha_i : A_i \rightarrow B_i$,

$\exists$ $R \rightarrow S$ detecting nilpotence s.t.

$A_i \rightarrow S$ \text{ is } $\alpha_i \downarrow S$.

$B_i$ \text{ "right lifting property"}
We will choose 3 maps
\[
\alpha_1 = \mathbb{Q} \to \mathbb{Q}[x^\pm]
\]
\[
\alpha_2 = \mathbb{Q}[x] \to \mathbb{Q}
\]
\[
\alpha_3 = \mathbb{Q}[t] \to \mathbb{Q} \times \mathbb{Q}[t^\pm]
\]

\(\alpha_1\) detects nilpotence...

\(\alpha_2\) does too; it is
\[
\text{Q modules w/ locally nilpotent end}
\]
\[
\text{forget}
\]

\(\alpha_3\) does too: if \(f: X \to Y\) has \(f \circ m_k = 0\)
\[
\Rightarrow f \circ m_{-k} = 0
\]
\[
\Rightarrow k > 0
\]
\[
\Rightarrow f \circ \alpha = 0
\]
\[
\alpha > 0.
\]

\(S \perp \alpha_1 \Rightarrow S\) is 2-periodic.

\(\alpha_2 \Rightarrow S\) is even.

\(\alpha_3 \Rightarrow \pi_0 S\) reduced Krull dimension.

So we have \(R \to S\) detecting nilpotence w/S satisfying these.

\(S \to \prod S_x\) detects nilpotence. Suffices to show \(r_0 S_x \cong \prod_0 S_x[t^\pm]\) so that
\[
\prod S_x \cong (\prod_0 S_x)[t^\pm] \text{ and we're done.}
\]
The sheaf $\Pi_0 S_x = \Pi_0 S_x [t^{-1}]$ which is formally smooth, by $S_x$ is a field, so there is no obstruction to building a section $\varphi \in \Gamma_0 S_x$ which composing with map to $S_x$ and tensoring with $Q[t^{-1}] \to S_x$ gives an equivalence.

\[
E(U_0^p) = KU_p
\]

$E(U_0^p) \to E(U_0^p)/p$ detects nilpotence by nilpotence thm.

$\Rightarrow$ we can assume WLOG $E \in \text{CAlg}(\text{Mod}(E)_{K(U)})$.

\[\text{Thm} \exists \text{ a fully faithful functor } \text{CAlg} \rightarrow \text{CAlg}_E\]

$E(-) : \text{Perf}_{U_0^p} \to \text{CAlg}_E$

with right adjoint given by

\[
R \to (\Pi_0 R/p)_p = R^b = \lim_{\rightarrow} \Pi_0 R/p \to \Pi_0 R/p \to \Pi_0 R/p
\]

$\text{PS } \text{Perf}_{U_0^p}$ has cotangent complex vanishing

so giving a map amounts to something on $\Pi_0/p$.

The counit is an isomorphism

$E(R^b) \to R$. $R$ is $E(A) \otimes -$ this is an isomorphism.

We'll run a small abstruse argument & use the following to see the result is what we want.

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Suppose \( A \in \text{CH}(\mathbb{F}) \) is

1. even
2. \( E(A^6) \rightarrow R \) is surjective
3. \( R^6 \) is Krull dim 0 reduced.

\[ \Rightarrow R \cong E(R^6) \]

**Proof:**

3. \( E(R^6) \rightarrow R \) is inj onto be \( R^6 \rightarrow R^6 \) and cannot have a kernel or else an idempotent would split off. Thus \( E(R^6) \cong R \).

So we use 3 maps

1. \( E[x_i^2] \rightarrow E \Rightarrow 1 \) if \( \bot \)
2. \( E[t^{1 \over 2}, \alpha] \rightarrow E[t^{1 \over 2}, \alpha] \times E \Rightarrow 3 \) if \( \bot \)

Need to check 1) detects nilpotence.

\[ E[x_i] \rightarrow E \]

\[ \downarrow x^{-1} \quad \downarrow \]

\[ R \rightarrow R[x_0] \]

This detects nilpotence.

\[ \text{Fib} (R \rightarrow R[x_0]) \] is a free \( R \)-module

in odd degrees be free \( K(1) \)-local \( E \)-ring is even.

More generally if \( x_{-1} \) isn't sent to zero (eg id map)

\[ \text{fib} (R \rightarrow \text{pushout}) \] has a filtration w/ associated graded free \( R \)-module in odd degrees.

In this situation, you can slow
Finally need to construct a map forcing 2),
\[ A^h = \colim A \xrightarrow{F_r} A \xrightarrow{F_{rd}} A \xrightarrow{F_{rd}} \]
\[ \mathbb{F}_p \{ \mathbb{F}_p \}^3 = \text{free } \mathcal{S}\text{-ring on } x/y_p. \]
To force 2) we produce a map \( g \)
\[ g : E \{ x/y_p \}^3 \rightarrow E(\mathbb{F}_p \{ x/y_p \}^3) \text{ such that on } \]
\[ \Pi_0 / \pi_p \text{ it is } \]
\[ \mathbb{F}_p \{ x/y_p \}^3 \rightarrow \mathbb{F}_p \{ x/y_p \}^3. \text{ If } g \perp S, \]
\[ \Pi_0 S / \pi_p \text{ has surj Frobenius } \Rightarrow E(R^b) \rightarrow \]
\[ \text{surj on } \Pi \tilde{e}. \]
Such a map is easily seen to detect nilpotence;
\[ \mod p, \mathbb{F}_p \{ x/y_p \}^3 \text{ is a free module over } \mathbb{F}_p \{ x/y_p \}^3 \]
(it is \( p \)-completely \( \mathcal{S} \)-faithfully flat)
\( \Pi_0 \) is naturally a \( \mathcal{S} \)-ring, and \( \Pi_0(E(A)) \) is the
cofree \( \mathcal{S} \)-ring on \( A \).
\[
\begin{align*}
\left[ E \{x \bar{x} \}, \mathcal{E}(\mathbb{F}_p \{x^2\}) \right]_{\text{Gal}_E} \\
\Pi_0 \mathcal{E}(\mathbb{F}_p \{x^{1/2}\}) = \mathcal{W}(\mathbb{F}_p \{x^{1/2}\}) \\
\text{Hom}_{\text{ring}}(\mathbb{Z}_p \{x^2\}, \mathcal{W}(\mathbb{F}_p \{x^{1/2}\})) \\
\text{Hom}_{\text{ring}}(\mathbb{Z}_p \{x^2\}, \mathbb{F}_p \{x^{1/2}\}) \\
\text{fl}_{\text{perf}_{\text{FP}}} (\mathbb{F}_p \{x^{1/2}\}, \mathbb{F}_p \{x^{1/2}\}) \ \text{id}
\end{align*}
\]

So we're done.

We need this cofreeness result at higher heights, which is the main technical ingredient in the proof.

Proving the cofreeness result will involve understanding operations on K(n)-local En-algebras well.

- Eunice's talk: $\Pi$-algebras generalize $\mathcal{S}$-rings to higher heights.
- Notolie's talk: Nilpotence detection
- Aron's talk: explaining proof at higher heights
- David/Not/Adela's talks build up to proving cofreeness.