Here’s the deal with the Bott element. First, remember that we have this spectrum ku, connective complex $K$-theory, whose zeroth space is $BU \times \mathbb{Z}$. So we have, for a space $X$, that

$$\pi_0 \text{Map}(\Sigma^\infty_+ X, \text{ku}) = \pi_0 \text{Map}(X, BU \times \mathbb{Z})$$

is the group of virtual vector bundles on $X$. In particular, $\pi_2 \text{ku}$ is the group of virtual vector bundles on $S^2 = \mathbb{C}P^1$, and we have a favorite one, $\mathcal{O} (1)$, coming from the canonical line bundle. We usually shift it to have rank 0 and use $\beta := [\mathcal{O}(1)] - 1$. So we’ve got this diagram

$$\mathbb{C}P^1 \to \mathbb{C}P^\infty = BS^1 = BU(1) \to BU,$$

defining the Bott element.

On the other hand, we can include a cyclic group $C_q$ into $S^1$ as the $q$th roots of unity, i.e. the kernel of the map $S^1 \to S^1$ given by $z \to z^q$. So we have a map

$$BC_q \to BS^1$$

We also have a little cofiber sequence

$$S^1 \to S^{1}/q \to S^{1}/q \to S^2.$$ 

Consider the diagram:

$$\begin{array}{ccc}
S^{1}/q & \to & BC_q \\
\downarrow & & \downarrow \\
S^2 & \to & BS^1 \\
\downarrow & & \downarrow \\
& & BS^1 \\
\end{array}$$

The long composite $S^1/q \to BS^1$ is null, $BS^1$ represents $H^2(-, \mathbb{Z})$ and $H^2(S^1/q) = \mathbb{Z}/q$, so can fill in the dotted map. (Alternatively, you could use the fact that $\pi_1 BS^1 = 0$ and use the cofiber sequence).

Taking suspension spectra, we then get the following diagram:

$$\begin{array}{ccc}
\Sigma S^1/q & \to & \Sigma^\infty_+ BC_q \\
\downarrow & & \downarrow \\
\Sigma^2 S & \to & \Sigma^\infty_+ BS^1 \to \text{ku} \\
\end{array}$$

Now, it turns out that $S/q$ is dualizable, so that

$$\text{Map}(S/q, -) \simeq D(S/q) \otimes (-),$$
for some spectrum $D(S/q)$. Which spectrum? Well, we can work it out by dualizing the defining cofiber sequence:

$$\Sigma^{-1}S/q \to S \xrightarrow{\cdot q} S \to S/q$$

becomes

$$\Sigma D(S/q) \leftarrow S \leftarrow S \leftarrow D(S/q)$$

so that

$$D(S/q) = \Sigma^{-1}S/q.$$ 

In particular:

$$\pi_n(X/q) := \pi_n(X \otimes S/q) = \pi_0\text{Map}(\Sigma^nS, X \otimes S/q) = \pi_0\text{Map}(\Sigma^{n-1}S/q, X).$$

Revisiting our diagram, we have produced an element $\beta \in \pi_2((\Sigma^n BC_q)/q)$ which lifts the mod $q$ reduction of $\beta$ in $\pi_2 ku$.

If we knew that the mod $q$ homotopy groups had a ring structure, we would then get a split injection

$$\mathbb{Z}/q[\beta] \to \pi_*((\Sigma^n BC_q)/q).$$

Now, $\Sigma^n BC_q$ is a ring (it’s like the ‘group ring’ of the space $BC_q$, which is a topological abelian group), but $S/q$ actually can’t be made into a highly structured ring. However, when $q \geq 3$, we can define a pairing

$$S/q \otimes S/q \to S/q$$

which is unital; and that’s enough to get a ring structure on the homotopy groups, which is enough to define all those powers $\beta^n$ of the Bott element.