SOME EXERCISES ON ÉTALE COHOMOLOGY

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(1) Let k be a field, and let n be an integer. Suppose that n is invertible in k, and that k contains a primitive n-th root of unity. Prove that the set of $\mathbb{Z}/n\mathbb{Z}$-extensions of k is naturally in bijection with $k^\times/k^{\times,n}$.
(Hint: use the Kummer sequence, along with classical Hilbert 90.)

(2) Let X be a scheme.
   (a) Write $\mathbb{G}_a$ for the abelian presheaf on $X_{\text{ét}}$ given by $Y \mapsto \mathcal{O}_Y(Y)$. Show that $\mathbb{G}_a$ is a sheaf.
   (b) Let X be a scheme over $\mathbb{F}_q$. Prove that the Artin–Schreier sequence
$$0 \to \underline{\mathbb{F}_q} \to \mathbb{G}_a \to \mathbb{G}_a \to 0$$
   is an exact sequence of abelian sheaves on $X_{\text{ét}}$, where the morphism $\mathbb{G}_a \to \mathbb{G}_a$ sends $x \mapsto x^q - x$.
   (c) Let k be a field. Show that $H^1_{\text{ét}}(\operatorname{Spec} k; \mathbb{G}_a) = 0$.
   (Hint: use the normal basis theorem and Shapiro's lemma.)
   (d) Let k be a field of characteristic $p > 0$. Deduce a description of the set of $\mathbb{Z}/p\mathbb{Z}$-extensions of k similar to that in (1).

(3) Let X be a 1-dimensional regular integral scheme. Prove that the divisor sequence is an exact sequence of abelian sheaves on $X_{\text{ét}}$.

(4) Let X be a connected normal noetherian scheme, and let A be a torsion-free abelian group. Show that $H^1_{\text{ét}}(X; \underline{A}) = 0$. Infer that we shouldn't directly take étale cohomology with $\underline{A}$-coefficients for A in $\{\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_\ell, \mathbb{Q}_\ell\}$.

(5) Let X be a connected smooth proper curve over an algebraically closed field k, and let q be an odd prime power. Suppose that q is invertible in k. Compute $K_i^{(\beta)}(X; \mathbb{Z}/q\mathbb{Z})$.
(Hint: use the Thomason spectral sequence.)

(6) (This may require knowledge of class field theory.)
Let K be a number field, let X be an open subscheme of $\operatorname{Spec} \mathcal{O}_K$, and write S for the set of places of K not in X. (So S is finite and contains all archimedean places.) Show that $H^0_{\text{ét}}(X; \mathbb{G}_m) = \mathcal{O}_U(X)^\times$, $H^1_{\text{ét}}(X; \mathbb{G}_m) = \operatorname{Pic} X$, and that we have an exact sequence
$$0 \to H^2_{\text{ét}}(X; \mathbb{G}_m) \to \bigoplus_{v \in S} H^2_{\text{ét}}(\operatorname{Spec} K_v; \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z} \to H^3_{\text{ét}}(X; \mathbb{G}_m) \to 0.$$
(Hint: use the divisor sequence, argue again that $Rj_*\mathbb{G}_m = j_*\mathbb{G}_m$, and use the computation of $H^2_{\text{ét}}(\operatorname{Spec} F; \mathbb{G}_m)$ for local and global fields F. See (II 2.1) in Milne's Arithmetic Duality Theorems.)

(7) Read (II 2.9) in Milne's Arithmetic Duality Theorems. Use this knowledge, along with the Thomason spectral sequence, to compute $K_i^{(\beta)}(X; \mathbb{Z}/q\mathbb{Z})$ in terms of Galois cohomology for $X = \operatorname{Spec} \mathbb{Z}[\frac{1}{q}]$ or $X = \operatorname{Spec} \mathbb{Z}[\zeta_q][\frac{1}{q}]$ (where q is an odd prime).

Bott element addendum by Nat (follow ups to rmk 2.6 in the paper)

(1) By saying what they do on modules, define K-theory maps $K(\mathbb{Z}; \mathbb{Z}/q) \to K(\mathbb{Z}[\mu_q]; \mathbb{Z}/q) \to K(\mathbb{Z}; \mathbb{Z}/q)$ exhibiting a split of $K(\mathbb{Z}[\mu_q]; \mathbb{Z}/q)$ with $K(\mathbb{Z}; \mathbb{Z}/q)$ as a direct summand and, using the $(\mathbb{Z}/q)^\times$

action on $K(\mathbb{Z}[\mu_q];\mathbb{Z}/q)$ show that a Bott element $\beta \in K_2(Z[\mu_q];\mathbb{Z}/q)$ has $q-1$st power landing in $K(\mathbb{Z};\mathbb{Z}/q)$. q is a prime.

(2) Use the Bockstein spectral sequence to show that given β^{p-1} in mod p homotopy, $\beta^{p^k(p-1)}$ exists in mod p^k homotopy