CM Abelian Varieties

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Goals

- Define CM abelian varieties and polarizations
- Show how abelian varieties induce K-theory classes
- Show how to construct/parametrize CM abelian varieties
- (Next week) show that CM abelian varieties generate K-theory
Abelian varieties

Definition
An *abelian variety* over $\mathbb{C}$ is a group variety which is a connected complex projective variety.

Proposition
An abelian variety is in fact an abelian group object in the category of varieties.
Example: elliptic curves

- Any smooth projective complex genus 1 curve with a given rational point $O$ is naturally an abelian variety (an *elliptic curve*).
- Elliptic curves over $\mathbb{C}$ are isomorphic as abelian varieties to $\mathbb{C}/L$ for a lattice $L$.
- In fact, any complex abelian variety is of the form $\mathbb{C}^g/L$ (though not all full-rank $L$ work!).
Let $V \cong \mathbb{C}^g$ and $\mathbb{Z}^{2g} \cong L \subset V$ a lattice of rank $2g$.

**Definition**
A skew-symmetric form $E : L \times L \to \mathbb{Z}$ is a Riemann form if the extension $E_{\mathbb{R}} : V \times V \to \mathbb{R}$ satisfies for all $v, w \in V$

(a) $E_{\mathbb{R}}(iv, iw) = E_{\mathbb{R}}(v, w)$, and
(b) If $v \neq 0$, then $E_{\mathbb{R}}(iv, v) > 0$.

When condition (a) holds,

$$H(v, w) = E_{\mathbb{R}}(iv, w) + iE(v, w)$$

defines a Hermitian form on $V$. Then condition (b) says $H$ is positive definite.
Theorem

A complex torus $V/L$ is (the analytification of) an abelian variety over $\mathbb{C}$ iff $L$ admits a Riemann form. Equivalently, there is a positive-definite Hermitian form $H$ on $V$ such that $\Im H(L, L) \subset \mathbb{Q}$.

In this case $H_1(A(\mathbb{C})^{\text{an}}; \mathbb{Z}) \cong L$ has a skew-symmetric form. This is a class in

$$H^2(A(\mathbb{C})^{\text{an}}; \mathbb{Z}) \cong \Lambda^2 L^*.$$ 

When the form is a Riemann form, this class is $c_1(L)$ for an ample line bundle $L$ on $A$. 

Riemann forms
The dual abelian variety

Let \( t_a : A \to A \) denote translation by \( a \in A(\mathbb{C}) \). The **dual abelian variety** \( A^\vee \) of \( A \) is the variety classifying line bundles on \( A \), trivialized at 0, such that \( t_a^* \mathcal{L} \cong \mathcal{L} \). (This is the **Picard scheme** of \( A \).)

By the classifying property, \( A \times A^\vee \) has a universal line bundle \( \mathcal{P} \), the **Poincare bundle**.

Dualization gives an equivalence \( \text{AbVar}^{\text{op}} \to \text{AbVar} \) and satisfies \( A^{\vee\vee} \cong A \).

In the complex case, \( A \cong V/L \). We can take \( A^\vee \) to be \( V^*/L^* \).
Proposition

Let $\mathcal{L}$ be an ample line bundle on an abelian variety $A$. Then

$$\lambda_{\mathcal{L}} : A \rightarrow A^\vee$$

$$x \mapsto t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$$

is an isogeny.

Recall that an isogeny is a surjection of algebraic groups with finite kernel.

An isogeny of the form $\lambda_{\mathcal{L}}$ for some ample $\mathcal{L}$ is called a polarization of $A$.

An isogeny with kernel 0 is an isomorphism, and in this case we call $\lambda_{\mathcal{L}}$ a principle polarization.
There are surjective maps:

\[
\begin{align*}
\text{Line bundles on } A & \quad \downarrow \\
\text{Self-dual morphisms } A \rightarrow A^\vee & \quad \downarrow \\
\text{Skew-symmetric bilinear forms on } H_1(A(\mathbb{C})^{\text{an}}; \mathbb{Z}) & \quad \downarrow
\end{align*}
\]

The composite is the Chern class \( c_1(\mathcal{L}) \in H^2(A(\mathbb{C})^{\text{an}}; \mathbb{Z}) \).
Polarizations

This upgrades to:

\[
\text{Ample line bundles on } A \\
\downarrow \\
\text{Polarizations } A \rightarrow A^\vee \text{ (up to isogeny)} \\
\uparrow \\
\text{Riemann forms on } H_1(A(\mathbb{C})^{an}; \mathbb{Z}) \\
\text{(up to overall rational factor)}
\]

Define a correspondence between abelian varieties \( A \) and \( B \) to be a line bundle on \( A \times B \) with trivializations on \( A \times \{0\} \) and \( \{0\} \times B \) that agree at the origin.

Then polarizations are furthermore in bijection with symmetric correspondences on \( A \times A \) which pull back on the diagonal to an ample bundle on \( A \).

This is a “symmetric positive-definite bilinear form” on \( A \).
The moduli stack of principally polarized abelian varieties $\mathcal{A}_g$ classifies complex abelian varieties $A$ of dimension $g$, equipped with a principle polarization $\lambda : A \rightarrow A^\vee$. 
The moduli stack of abelian varieties

A complex PPAV is specified by a full-rank lattice \( L \subset \mathbb{C}^g \) with an identification \( \lambda : \mathbb{C}^g / L \sim (\mathbb{C}^g)^* / L^* \). By principleness this gives \( L \sim L^* \). Choosing \( L \) to contain the real basis \( e_1, \ldots, e_g \) of \( \mathbb{C}^g \), then \( L \) and \( \lambda \) are specified by the lift of \( \lambda \),

\[
\tau : \mathbb{C}^g \to (\mathbb{C}^g)^*
\]

which, by the properties of the Riemann form, satisfies

\[
\tau^T = \tau \quad \text{and} \quad \Re \tau \text{ is positive definite}.
\]

Let \( \mathbb{H}_g \) denote the space of such \( \tau \) (the Siegel upper half-space). Change of basis preserving the Riemann form preserves the underlying variety and polarization, so

\[
\mathcal{A}_g(\mathbb{C}) \cong \mathbb{H}_g / \text{Sp}_{2g}(\mathbb{Z})
\]
The map to $K$-theory

Let $L_\lambda$ be the pullback of the Poincare bundle under the graph morphism $A \to A \times A^\vee$. The map

$$
\mathcal{A}_g(\mathbb{C}) \longrightarrow \mathcal{SP}(\mathbb{Z})
$$

$$(A, \lambda) \longmapsto (H_1(A(\mathbb{C})^{\text{an}}; \mathbb{Z}), c_1(L_\lambda))$$

extends to a map of groupoids.

So we get

$$
|\mathcal{A}_g(\mathbb{C})| \to |\mathcal{SP}(\mathbb{Z})| \to \Omega^\infty K\text{Sp}(\mathbb{Z})
$$

and adjointly

$$
\Sigma^\infty_+ |\mathcal{A}_g(\mathbb{C})| \to K\text{Sp}(\mathbb{Z}).
$$

We wish to show this map is surjective on (mod $q$) homotopy. In fact, there is a nice subgroupoid of $\mathcal{A}_g(\mathbb{C})$ for which this is true.
We wish to construct maps $\Sigma_+^\infty(B(\mathbb{Z}/q)) \to \text{KSp}(\mathbb{Z})$ (to get “CM classes” in $\text{KSp}_*(\mathbb{Z}; \mathbb{Z}/q)$ as images of the Bott element).

We get these from maps $B(\mathbb{Z}/q) \to \mathcal{A}_g(\mathbb{C})$, equiv. $\mathbb{Z}/q$ actions on PPAVs.

Such an action can be realized in PPAVs admitting an action of $\mathcal{O}_q \doteq \mathbb{Z}[\zeta_q]$. These are principally polarized abelian varieties with complex multiplication.
Complex multiplication

Definition
Let $A$ be an abelian variety of dimension $g$. If $\text{End}(A) \otimes \mathbb{Q}$ has a commutative $\mathbb{Q}$-subalgebra of dimension $2g$, then $A$ is said to have complex multiplication.

What do these look like?
Definition

A CM field $E$ is a number field which is a totally imaginary extension of a totally real field $E_+$.  
($E = E_+(\sqrt{d})$, where every algebra map $E_+ \to \mathbb{C}$ has real image and takes $d$ to a negative number.)

A CM algebra is a product of CM fields.  
A CM order is an order in a CM algebra which is stable under complex conjugation.

An order in $E$ is a full rank integral sublattice; a free $\mathbb{Z}$-subalgebra $\mathcal{O}$ such that $\mathcal{O} \otimes \mathbb{Q} \cong E$.

If $A$ has complex multiplication, then $\text{End}(A)$ is a CM order.
A PPAV with complex multiplication is specified by the following data:

1. $\mathcal{O}$ a CM order (with CM algebra $E$)
2. $\alpha$ an $\mathcal{O}$-submodule of $E$ such that $\alpha \otimes \mathbb{Q} \cong E$
3. $\Phi : \mathcal{O} \otimes \mathbb{R} \xrightarrow{\sim} \mathbb{C}^g$ an algebra isomorphism
4. A purely imaginary element $u$ in $E$

(1-3) give $\mathcal{O} \otimes \mathbb{R}/\alpha$ the structure of a complex torus. Then (4) induces a Riemann form on $\alpha$ by

$$\alpha \times \alpha \rightarrow \mathbb{Q}$$

$$(x, y) \mapsto \text{Tr}_{E|\mathbb{Q}}(\overline{x}uy).$$
Realizing the CM classes

If we fix a CM order \( \mathcal{O} \) in a CM field \( E \) and an isomorphism \( \Phi : \mathcal{O} \otimes \mathbb{R} \xrightarrow{\sim} \mathbb{C}^g \), then the remaining data can be assembled into a groupoid \( \mathcal{P}_E^- \) with objects

\[ \mathcal{O} \text{-lattices in } E \text{ with a Riemann form.} \]

This is equivalent to the groupoid of rank 1 projective \( \mathcal{O} \)-modules with skew-Hermitian form valued in \( \text{Hom}_\mathbb{Z}(\mathcal{O}, \mathbb{Z}) \). On a lattice which is an ideal of \( \mathcal{O} \), the skew-Hermitian form is given by

\[ (x, y) \mapsto [z \mapsto \text{Tr}_{E|\mathbb{Q}}(\overline{x}uyz)]. \]

Using the skew-Hermitian notation from previous talks, we have

\[ \mathcal{P}_E^- \cong \mathcal{P}(\mathcal{O}, \text{Hom}_\mathbb{Z}(\mathcal{O}, \mathbb{Z}), f \mapsto (x \mapsto -f(\overline{x}))). \]
Realizing the CM classes

The CM theory then gives a functor

\[ \text{ST} : \mathcal{P}_E^- \rightarrow \mathcal{A}_g(\mathbb{C}). \]

The composite map with \( \mathcal{A}_g(\mathbb{C}) \rightarrow \mathcal{SP}(\mathbb{Z}) \):

\[ \mathcal{P}_E^- \rightarrow \mathcal{SP}(\mathbb{Z}) \]

is of particular interest. On the full subgroupoid of \( \mathcal{O} \)-lattices with integral-valued Riemann form, this simply realizes the lattice as a \( \mathbb{Z} \)-module and the Riemann form as a skew-symmetric form.
Specialize to $\mathcal{O} = \mathbb{Z}[\zeta_q]$, the ring of integers of $K_q = \mathbb{Q}(\zeta_q)$. Fix an isomorphism $\Phi: \mathcal{O} \otimes \mathbb{R} \sim \mathbb{C}^g$.

**Lemma**

*Let $\mathfrak{b}$ be a fractional ideal of $\mathcal{O}$. Then there exist lattices $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathcal{P}^-_{K_q}$ such that $[\mathfrak{a}_1 \mathfrak{a}_2] = [\mathfrak{b} \overline{\mathfrak{b}}^{-1}]$ in the ideal class group of $\mathcal{O}$. This is used for...*
Conclusion

Proposition (Next week)

Take $E = K_q$. The map

$$P_E^- \to SP(\mathbb{Z})$$

induces maps

$$\pi_{4k-2}^s(|P_E^-|; \mathbb{Z}/q) \to KSp_{4k-2}(\mathbb{Z}; \mathbb{Z}/q)$$

which are surjective for all $k \geq 1$. 