Cobordism Hypothesis Talk:

We've reduced CH to Bord^+ \to \to Bord^+, now reduce this.

Let's start w/ length 2: B_1 \to B_2 s.m. from I-cat to a 2-cat.

Need to describe Map_{B_2}(F(x), F(y)).

a priori this is functorial in B_2^{op} \times B_2, but if we assume B_2 has duals then Map_{B_2}(F(x), F(y)) = Map_{B_2}(1, F(x', y)) - functorial in B_1!

So let M(x) = Map_{B_2}(1, F(x)), then x \mapsto M(x)

is a functor B_1 \to Cat(\omega_1).

Detour: We can "unstraighten" the above functor.

Grotendieck Construction:

Let B, M be as above, then

Groth(B, M) has objects (x, y), x \in B, y \in M(x)

mapping space (x, y) to (x', y') & classifying space for data (F, \alpha), F \in Map_B(x, x'), \alpha \in Map_M(x)(F, 1, y')

Have a functor Groth(B, M) \to B, \ (x, y) \mapsto X s.t.

X \in B, Groth(B, M)_X = M(X), & these fibers vary functorially in B,

in the sense that for all F: X \to Y in B, &

F \in Groth(B, M)_X, \exists ! \Phi \in Groth(B, M)_Y & \exists F: X \to Y

\Phi \mapsto ess. uniquely determined by X & F

Prop 3.3.24: M \mapsto Groth(B, M) determines an equivalence between

a) Functors B \to Cat(\omega_0)^1 (Lax s.m.)

b) oCat^+ Fibs \tau : Z \to B. (s.m.)

Corollary (Prop 3.3.28): B_1 \to B_2 as above is equivalent to

a s.m. coCat Fib \tau : Z \to B_1.
Now for higher $n$.

\[ B_1 \to B_2 \to \ldots \to B_n \]  

ess surj. S.m. Factors, $B_i$ has duals

\[ F_i : B_i \to B_i \]

Let \( M_i : B_i \to \text{Cat}(0, i-1) \), \( M_i(x) = \text{Map}_{B_i}(1, F_i(x)) \)

More general version of the GC gives \( \Pi : \text{Cat} \to \text{F}_0 \)

w/ \( i \) on \( (\infty, i-1) \)-cat

& have ess surj's S.m. Factors \( \xi_2 \to \xi_3 \to \ldots \to \xi_n \)

\[ \downarrow \]

\[ \text{B}_i \]

\text{IF } \xi_2 \text{ has duals then we've basically just reduced } n \text{ by 1.}

**Prop. 3.3 29:** \( B_i \to B_{i-1} \) as before, \( \text{TRA} \):

1. All maps \( 1 \to X \) in \( B_{i-1} \) have left adjoints.
2. The cat \( \text{Cat} \) has duals \( \text{Grd}(B_{i-1}, 1) \)

Now let \( \text{G}_i : \xi_{i-1} \to \xi_i \), & \( \text{N}_i : \xi_i \to \text{Cat}(\omega, i-2) \)

\( N_i(x) = \text{Map}_{\xi_i}(1, \text{G}_i(x)) \)

Notice these maps are all relative to \( B_i \) so they are determined by

maps on the fibers \( \text{Fib}(\xi \to B_i) = M_i(1) = \text{Map}_{B_i}(1, F_i(1)) = \text{Map}_{\xi_i}(1, 1) \)

\( =: S_{2i} \)

so new seq \( S_{2B_1} \to S_{2B_2} \to \ldots \to S_{2B_n} \)

Goal: keep going until \( n = 2 \), then use previous result.

**Def.** A skeletal seq of length \( n \) is a diagram

\[ B_1 \xrightarrow{F_1} B_2 \xrightarrow{F_2} \ldots \xrightarrow{F_n} B_n \]

such that

1. \( B_n \) a s.m. \( X \)-cat
2. \( F_i \) is \((k-1)\)-connective
3. \( B_i \) has duals
4. For \( 2 \leq k \leq n \) every map \( q : 1 \to X \) in \( S^{k-2}B_k \) has a LA
We've seen how to reduce.

Prop 3.3.1: Fix a SS $F: B_1 \rightarrow B_2$ of len 2. Run the above discussion gives an equivalence.

1) $SS$ of len 1 $B_1 \rightarrow B_2 \rightarrow ... \rightarrow B_n$, leg $w/f F$

2) $SS$ of len $1$ $q:B_2 \rightarrow C_2 \rightarrow ... \rightarrow C_n$

So we can reduce $SS$ to 1-cat data, but how do we organize all that data?

Def: A categorical chain complex of length $n$ consists of:

a) A seq of s.m. cats $F_i: \mathbb{Z} \rightarrow \mathbb{Z}$ is a seq, between 1-cats $\mathbb{E}$ $\mathbb{E}$ and each $F_i$ has duals.

b) $z_k = \mathbb{E}_k \times \mathbb{E}_{k-1}$

Construction: Given a SS $B_1 \rightarrow B_2 \rightarrow ... \rightarrow B_n$, associate a C.C.C.

$z_k = \mathbb{E}_k \rightarrow \mathbb{E}_{k-1}$
$z_1 = \mathbb{E}_1$, $1 \leq k \leq n$
$p_n = \mathbb{C} \mathbb{S}^{2k-2} F_{k-1}$

ex:

\{ $B_1 \rightarrow \mathbb{E}$ \}
\{ $\mathbb{E} \rightarrow B_2$ \}
\{ $\mathbb{E} \mathbb{S} \rightarrow 2B_2$ \}

Finally: Applying all of this to bord $\rightarrow$ bord $\rightarrow ... \rightarrow$ bord

get C.C.C. \{ $\text{Cob}_0^\infty (A) \rightarrow \text{Cob}_1^\infty (A-1)$ \}

\text{Cob}_0^\infty (A) \text{ the } k-1 \text{ manifolds with boundary, } \text{ mapping spaces close to classifying spaces for bordisms of bordisms in } \text{Cob}_2 (A-1) \text{ are } \text{closed } (k-2) \text{- manifolds.}