THE TANGLE HYPOTHESIS

Plan:

I. STATEMENT + comparison of cobordism hypothesis.
   equivalent to.

II. APPLICATIONS/IMPLICATIONS.
   e.g. relation to knot invariants.

   specialize: 1-dim.

III. IDEA OF AYALA-FRACLIS PROOF
   method using factorization homology.
   (mostly black box for us.)
(I) STATEMENTS / COMPARISON of COBORDISM.

Refs: Lurie, 4.4; Ayala's MSRI talk.

Cobordism hypothesis: universal property of the bordism category of $k$-manifolds in $\mathbb{R}^\infty$ stabilized.

Q: what properties does the "Bordism category of (framed) manifolds in $\mathbb{R}^n$" have?

\[ \begin{align*}
\text{(so,k)-category} \\
\text{to get the right categorical structure: need to be a bit careful w/ definition,}
\end{align*} \]
DEF. Let $0 \leq k \leq n$. Given a $n$-framed $m$-manifold ($m \leq n$), a $k$-framed submanifold of $M$ consists of the following data:

(i) A submanifold $M_0 \subseteq M$ of codimension $(n-k)$.

(ii) A null homotopy of the Gauss map assoc. to $N_{M_0} \subseteq M$.

Think of this as a $k$-framing of the normal bundle, compatible w/ the $n$-framing on $M$.

$g: M_0 \to G_{n-k, n}$ determined by normal bundle $\subseteq$ stabilized tangent, which has coordinates $z_i \subseteq \IR^n$ canonically.
DEF. Fix $0 \leq k \leq n$ and let $V$ be a framed $(n-k)$-manifold. There is an $(\infty, k)$-category $\mathcal{Tang}_{k,n}$.

(a) objects are compact $k$-framed submanifolds of $V$.

(b) Given objects $M_0, M_1$, a 1-morphism between them is a (compact) $k$-framed submanifold $M \subseteq V \times [0,1]$ with intersection $M_i := M \cap V \times \{i\}$.

(c) More generally, can identify $j$-morphisms ($j \leq k$) with $k$-framed submanifolds $\subseteq V \times [0,1]^j$ satisfying appropriate boundary conditions.
\[ \text{Tang}_{R^n-\ell k} \subseteq \text{Tang}_{k,n} \]

Q: why called Tang? If \( k=1, n=3 \)
get \((0,1)\)-cat of tangles in \( \mathbb{R}^3 \).

related to
Knot theory!
Cobordism hypothesis: $\text{Bord}_k^{tr} \cong \text{free symm. monoidal } (\sigma, \beta)\text{-cat}$ generated by 1 object fully dualizable ob.

$\text{Tang}_{k,n}$: no natural symm mon structure, but there is a natural algebraic structure.

$\text{Tang}_{k,n}$ and the little $(n-k)$-disc operad:

$$D_{n-k} = \mathbb{R}^{n-k}$$

$$\begin{array}{ccc}
\text{D}_{n-k} & \xrightarrow{\text{l-times}} & \text{D}_{n-k} \\
\downarrow & \quad & \quad \downarrow \\
\text{Tang}_{k,n} \times \cdots \times \text{Tang}_{k,n} & \xrightarrow{\text{compatible with differential structure}} & \text{Tang}_{k,n}
\end{array}$$
\[ \text{} \rightarrow \text{Tang}_{\mathfrak{k},n} \text{ is an } \mathbb{E}_{n-k-\text{monoidal}} \text{ (\infty,k)} \text{-category.} \]

[Baez-Dolan Tangle Hypothesis]

\[ \text{Tang}_{\mathfrak{k},n} = \text{free } \mathbb{E}_{n-k-\text{monoidal}} \text{ (\infty,k)} \text{-cat on a fully dualizable object.} \]

More precisely:

Fix \( 0 \leq k \leq n \). Let \( \mathcal{C} \) be an \( \mathbb{E}_{n-k-\text{manoidal}} \text{ (\infty,k)} \text{-category with duals (k<n)} \text{ or adjoints (k=n)}. \) Let \( * = (0 \rightarrow \mathbb{R}^{n-k}) \) in \( \text{Tang}_{\mathfrak{k},n} \). Then evaluation at \( * \) induces an (\infty,k)-equivalence:

\[ \text{Fun}^\otimes (\text{Tang}_{\mathfrak{k},n}, \mathcal{C}) \rightarrow \mathcal{C}^{\sim} \]
**Logical Relationships** (Lurie)

(A) tangle hypothesis $\Rightarrow$ cobordism hypothesis

(B) cobordism hypothesis $\Rightarrow$ tangle hypothesis

\[ \text{w/ singularities} \] \[ \Rightarrow \text{tangle hypothesis} \]

**Idea:**

(A) \[ \mathbb{R}^{n-k} \rightarrow \mathbb{R}^{n-k+1} \]

\[ \exists \]

\[ \text{Tang}_{k,n} \rightarrow \text{Tang}_{k,n+1} \]

in \( (a_k, c_k) \)

\[ \lim_{n \to \infty} \text{Tang}_{k,n} = \text{Bard}_k \]

\[ \text{tangle hypothesis} + (\star) \Rightarrow \text{univ. property of Bard}_k \]
(II) PROOF SKETCH: \( k=1 \) case.
(VIA FACTORIZATION HOMOLOGY)

Ref:
- Ayala-Francis, "The cobordism hypothesis"
- Ayala-Francis-Rozansky, "Factorization homology I: higher categories"
- Ask Misha 😊

You DON'T NEED TO KNOW ABOUT FACTORIZATION HOMOLOGY.

INPUT: "almost a theorem"

Conjecture (Ayala-Francis).

There is a fully faithful functor

\[ \int : \text{Alg}_{k-1} (\text{Cat}_{\infty,1}) \overset{\text{duals}}{\longrightarrow} \text{coPS} (\text{Mfd}_{\infty,1}^{\text{spr}}) \]

(defined by taking factorization homology).
NOTATION

- $\mathcal{K}$ an $(\infty,1)$ $E_n$ algebra map
- $\int_{\mathcal{K}: \text{Mfd}_{n-1,n}^{\text{stf}}} (-) : \text{Spaces}$

- for $Z$ a framed $n$-mfd or $n-1$-mfd.
  (more generally: object of $\text{Mfd}_{n-1,n}^{\text{stf}}$)
- $\int_Z \mathcal{K} = \text{factorization homology}$ of $Z$ w/ coeffs in $\mathcal{K}$

There's an actual definition of this...

The category $\text{Mfd}_{n-1,n}^{\text{stf}}$ heuristically

$\text{Obs} = \text{framed } n\text{-mfds w/ codim: 1 defects}$

1-mors = Zig-zags of certain kinds.
Key Properties.

\( + \) \[ \int_{\mathbb{R}^{n-1}} \mathcal{X} = \mathcal{X}^\sim \]

\[ \int_{\mathbb{R}^n} \mathcal{X} \cong \text{End}_{\mathcal{X}}(\mathcal{I}) \]

\( + \) (Lemma.) The co-presheaf

\[ \int_{(-)} \text{Tang}_{\ast, n} \text{ is representable by } \mathbb{R}^{n-1}. \]

\[ \hom_{\text{Mfd}_{n-1, n}}(\mathbb{R}^{n-1}, -) \]

Yoneda embedding of \( \mathbb{R}^{n-1} \) into co-presheaves.
DE DUCE T A N G L E

\[ \text{hom} (\Theta \circ \text{Tang}_{d,n}, \mathcal{X}) \overset{\cong}{\Rightarrow} \text{hom} \left( \int \text{Tang}_{d,n}, \int \mathcal{X} \right) \]

(\mathbb{R}^{n-1}, \int \mathcal{X})

\[ \text{SII} \text{ REPRESENTABLE (H)} \]

\[ \text{SII YONEDA LEMMA} \]

\[ \int \mathcal{X} \]

\[ \mathcal{X} \]

KEY TAKEAWAYS:

- Factorization homology leads you to the “right” cat to study Tang_{d,n}

- Pretty easy / formal - gives significant machinery

- Deduce (\omega,1) cobordism.
Applications.

Refs: Ayala's MSRI talk. (see juwiptop website for link!)

Actually doesn’t require tangle hypothesis—just correct formulation of the key conj. on factorization homology.

A compact, framed tangle $L \subset \mathbb{R}^n$ determines a morphism $\mathbb{Z}^*(L)$ in $\mathcal{M}_{\text{str}}^{n-n}$ from $\mathbb{R}^{n-1} \to \mathbb{R}^n$.

\[
\begin{array}{ccc}
\mathbb{R}^{n-1} & \leftarrow & L \times \mathbb{R}^{n-1} \\
& \longrightarrow & \mathbb{R}^n
\end{array}
\]

$\Rightarrow$ map of spaces $\mathcal{X} \xrightarrow{\int \mathcal{X}(L)} \text{End}_{\mathcal{X}}(\mathcal{Y})$.

Fix dualizable object $y$ in $\mathcal{X}$:

$\Rightarrow L \mapsto \int_{\mathcal{X}(L)} \mathcal{X}(y) \in \text{End}_{\mathcal{X}}(\mathcal{Y})$

$\text{link invariant}$
• Varying \( X \), dualizable objects lots of invariants.
• Computed geometrically by cutting up links.

**Ex:** \( k=1, \ n=3 \).
SPECIFICALLY. (Still $k=1$, $n=3$, tangles in $\mathbb{R}^3$.)

Any quantum group $U_q G$ has a cat. of f.d. reps w/ a natural braided monoidal structure.

\[ \text{End}_{\text{Rep} (U_q G)} (1) \cong \mathbb{C}(q) \]

$\text{Rep} 1 \rightarrow$ concrete knot invariant w/ basic constituents algebraically defined.

$\text{SL}_2$: Jones polynomial.

Note: can do explicitly from data of Ribbon cat, but this generalizes/gives conceptual framework.
Q: how reasonable is the Ayala-Francis conj?

Q: Is the category $\text{Mfd}_{sfr}^{n-1,n}$ somehow related to something Lurie does to deduce the tangle hypothesis from singularities cobordism hypothesis?

Q: how easy is the $k>1$ generalization?

Q: how computable are these knot invariants, really?