Factorization algebra as an extended TFT

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This are notes for a talk given during the Juvitop seminar in Fall 2020. The main references are

- Lurie’s paper “On the classification of Topological Field Theories”
- Scheimbauer’s Ph.D. thesis “Factorization Homology as a Fully Extended Topological Field Theory”

Throughout, we let $(\mathcal{C}, \otimes)$ be a symmetric monoidal $\infty$-cat; we assume that $\mathcal{C}$ admits sifted colimits and that $\cdot \otimes \cdot$ preserves them in each component. We fix a natural number $n \in \mathbb{N}$.

1 $E_n$-algebras

We start by recalling the definition of an $E_n$-algebra. Let $\text{Man}_{fr}^n$ the symmetric monoidal $\infty$-category obtained as the coherent nerve of the topological category with

- objects: framed $n$-dimensional smooth manifolds;
- morphism spaces $\text{Emb}_{fr}^n(M,N)$ of framed embeddings $M \hookrightarrow N$ between manifolds.
- The monoidal product is given by disjoint union.

We denote by $\text{Disk}_{fr}^n \subset \text{Man}_{fr}^n$ the full $\infty$-subcategory spanned by those framed $n$-manifolds which are isomorphic to a finite disjoint union $\bigsqcup_{\text{finite}} \mathbb{R}^n$ (with the standard framing).

**Definition 1.** An $E_n$-algebra in $\mathcal{C}$ is a symmetric monoidal functor from $\text{Disk}_{fr}^n$ to $\mathcal{C}$.

If $A: \text{Disk}_{fr}^n \to \mathcal{C}$ is an $E_n$-algebra, we say that $A(\mathbb{R}^n) \in \mathcal{C}$ is the underlying object of $A$; by abuse of notation, we also denote it by $A$. Unraveling the definition, we see that $A$ is equipped with multiplication maps

\[ A^\otimes k = A \otimes \cdots \otimes A \simeq A(\mathbb{R}^n \sqcup \cdots \sqcup \mathbb{R}^n) \longrightarrow A(\mathbb{R}^n) = A \]  \hspace{1cm} (1)
parameterized by the space of framed embeddings $\text{Emb}^{fr}(\mathbb{R}^n \sqcup \cdots \sqcup \mathbb{R}^n, \mathbb{R}^n)$ of $k$ disks into a bigger disk.

For example, for $n = 1$, the space $\text{Emb}^{fr}(\coprod^k \mathbb{R}^1, \mathbb{R}^1)$ is homotopy equivalent to the discrete set of permutations of $\{1, \ldots, k\}$; hence the multiplication maps are parameterized by choosing a linear order on the input copies of $A$. Unraveling the coherence conditions, one sees that an $E_1$-algebra in $C$ is precisely an associative (and unital) algebra object.

The goal of this talk is to explain a construction which takes as input an $E_n$-algebra $A$ and produces an $n$-dimensional topological field theory $\text{Bord}^n_0 \to \text{Alg}_n$ with values in an $(\infty, n)$-category $\text{Alg}_n$ which we can heuristically describe as follows:

- The objects of $\text{Alg}_n$ are $E_n$-algebras;
- a 1-morphism $A \to A'$ between $E_n$-algebras is an $A$-$A'$-bimodule, i.e., an $E_{(n-1)}$-algebra $B$ on which $A$ and $A'$ act compatibly from the left and from the right, respectively;
- a 2-morphism

$$
\begin{array}{c}
B \\
A \quad C \\
A'
\end{array}
\xrightarrow{\subseteq} 
\begin{array}{c}
C \\
B'
\end{array}
$$

between two bimodules $B, B'$: $A \to A'$ is a $B$-$B'$-bimodule $C$, by which we mean an $E_{(n-2)}$-algebra $C$ on which $A, A', B, B'$ act compatibly from the top, the bottom, the left and the right, respectively;
- 3-morphisms are bimodules of bimodules of $E_n$-algebras;
- $\ldots$
- $n$-morphisms are (bimodules of)$^n$ of $E_n$-algebras.

Composition in $\text{Alg}_n$ is defined as a suitable tensor product which, for instance, sends an $A$-$A'$-bimodule $B$ and an $A'$-$A''$-bimodule $B'$ to the $A$-$A''$-bimodule $B \otimes_{A'} B'$.

The topological field theory associated to the $E_n$-algebra $A$ is supposed to send the point $\mathbb{R}^0 \in \text{Bord}^n_0$ to the object $A \in \text{Alg}_n$. According to the cobordism hypothesis, this property uniquely characterizes this TFT. We will in fact give an explicit formula to compute this TFT on an arbitrary $k$-morphism $M$ in $\text{Bord}^n_0$ ($0 \leq k \leq n$) as the factorization homology $\int_M \times \mathbb{R}^{n-k} A$; heuristically, we take the local data encoded by the $E_n$-algebra $A$ and “integrate” it over the $n$-manifold $M \times \mathbb{R}^{n-k}$.
2 Factorization Algebras

The first challenge is to give a rigorous definition of the $(\infty, n)$-category $\text{Alg}_n$.

One approach makes use of factorization algebras, which we introduce now.

Let $X$ be a topological space. Denote by $\mathcal{U}_X$ the colored operad\(^1\) with

- colors/objects are the open subsets of $X$;
- there is a unique operation/morphism $U_1, \ldots, U_k \to U$, whenever $U_1, \ldots, U_k \subseteq \cup U \in \mathcal{U}_X$ are pairwise disjoint subsets of $U \in \mathcal{U}_X$.

**Definition 2.** A prefactorization algebra $F$ on $X$ with values in $\mathcal{C}$ is an $\mathcal{U}_X$-algebra in $\mathcal{C}$, i.e. a map of $\infty$-operads $\mathcal{U}_X \to \mathcal{C}$.

Unraveling the definition, $F$ assigns an object $F(U) \in \mathcal{C}$ to each open subset $U \in \cup U_X$, and a morphism $F(U_1) \otimes \cdots \otimes F(U_k) \to F(U)$, whenever $U_1, \ldots, U_k$ are pairwise disjoint open subsets of $U \in \mathcal{U}_X$; it needs to be functorial in the obvious sense.

A factorization algebra is a prefactorization algebra $F$ which additionally satisfies

1. If $U_1, \ldots, U_k \in \mathcal{U}_X$ are pairwise disjoint open subsets of $X$, the induced map $F(U_1) \otimes \cdots \otimes F(U_k) \simeq F(U_1 \sqcup \cdots \sqcup U_k)$ is an equivalence (in particular, for $k = 0$, the object $F(\emptyset)$ is identified with the monoidal unit of $\mathcal{C}$).

2. A suitable descent condition, which allows the value $F(U)$ to be computed as a colimit of values on sufficiently well behaved open covers. We shall not spell it out here.

A factorization algebra $F$ on $X$ is called **locally constant**, if the inclusion $F(D) \to F(D')$ is an equivalence, whenever $D \subseteq D'$ are both (homeomorphic to) $\mathbb{R}^n$.

3 Factorization homology

The following construction shows how an $E_n$-algebra gives rise to a factorization algebra on each framed $n$-manifold $M$.

**Construction 1.** Let $A : \text{Disk}^n \to \mathcal{C}$ be an $E_n$-algebra. We denote by

\[
\left( \int A \right) : \text{Man}_n^F \to \mathcal{C}
\]  

\(^1\)One way to think of a colored operad is a “multi-category” which has objects (usually called colors) and between them not just 1-to-1-morphisms $x \to y$, but also many-to-1-morphisms $(x_1, \ldots, x_k) \to y$. It needs to satisfy the suitable analogs of associativity and unitality.

\(^2\)Each symmetric monoidal (\infty-)category is canonically an (\infty-)operad, by declaring a multi-morphism $x_1, \ldots, x_k \to y$ to simply be a morphism $x_1 \otimes \cdots \otimes x_k \to y$ in $\mathcal{C}$.
the left Kan extension of $A$ and call it **factorization homology with coefficients in $A$**. For any framed $n$-manifold $M$, it is computed explicitly by the pointwise formula:

$$\int_M A := \operatorname{colim} \left( \text{Disk}^\text{fr}_n/M \to \text{Disk}^\text{fr}_n A \to C \right)$$

(4)

where $\text{Disk}^\text{fr}_n/M$ denotes the overcategory of all possible embeddings of disjoint disks into $M$. By construction, $\int_M A$ is functorial along embeddings of manifolds; hence in particular along inclusions of open subsets. Moreover one can check that the $\infty$-category $\text{Disk}^\text{fr}_n/M$ is sifted, hence the monoidal product in $C$ commutes with the limit in (4); a direct calculation produces a canonical identification

$$\left( \int_{U_1} A \right) \otimes \cdots \otimes \left( \int_{U_k} A \right) \cong \left( \int_U A \right)$$

(5)

whenever $U = U_1 \sqcup \cdots \sqcup U_k$ arises as a pairwise disjoint union of open subsets $U_1, \ldots, U_k$ of $M$. This exhibits $\int_{\subseteq M} A$ as a factorization algebra on $M$. It is locally constant because the inclusion $D \subseteq D'$ of two disks is an equivalence in the $\infty$-category $\text{Man}^\text{fr}_n$. 

An important special case arises when we consider $M = \mathbb{R}^n$. In this case, we have $\int_{\subseteq}^n A = A$ and in fact the factorization algebra $\int_{\subseteq \subset \mathbb{R}^n} A$ on $\mathbb{R}^n$ encodes the same data as the $E_n$-algebra $A$. More precisely we have the following theorem.

**Theorem 1** (Lurie). The assignment $A \mapsto \int_{\subseteq} A$ assembles to an equivalence of $\infty$-categories between $E_n$-algebras in $C$ and locally constant factorization algebras on $\mathbb{R}^n$ with values in $C$.

### 4 Stratified factorization algebras

To systematically encode the (bimodules of ...) which make up the $(\infty, n)$-category $\text{Alg}_{\infty,n}$, it is convenient to study a stratified variant of factorization algebras.

Let $X$ be a topological space. A stratification of $X$ consists of an ascending chain $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_l = X$ of closed subspaces. The **index** of an open subset $U \subset X$ is the smallest $i$ such that $U \cap X_i \neq \emptyset$.

**Definition 3.** Let $X$ be a topological space with stratification $X_*$. A factorization algebra $F$ on $X$ is called **locally constant with respect to the stratification**, if the inclusion $D \subseteq D'$ induces an equivalence $F(D) \cong F(D')$ whenever $D$ and $D'$ are disks of the same index $i$ which both remain connected when intersected with $X_i$.

Note that we get the previous notion of locally constancy with respect to the trivial stratification $\emptyset \subset X$.

Finally, let us remark that factorization algebras which are locally constant with respect to stratifications can be pushed forward along suitable maps
f : X → Y of stratified spaces by declaring f_*F(U) = F(f^{-1}(U)) for each open subset U ⊂ Y.

5 The Morita category

We can now finally say, at the very least, what the morphisms are in the (∞, n)-category Alg_n.

For each k, a k-morphism in Alg_n is a factorization algebra on R^n which is locally constant with respect to the stratification

S^k : ∅ ⊂ ⋯ ⊂ ∅ ⊂ R^n_{-k} × \{0\}^k ⊂ ⋯ ⊂ R^{n-1} × \{0\} ⊂ R^n. \tag{6}

Inside the stratified space (6) we find the two subspaces

R^{n-k} × (-∞, 0) × R^{k-1} ⊂ R^n \quad \text{and} \quad R^{n-k} × (0, +∞) × R^{k-1} ⊂ R^n \tag{7}

which are both isomorphic as stratified spaces to (R^n, S^{k-1}). Thus we can restrict each factorization algebra F on (R^n, S^k) to two factorization algebras on (R^n, S^{k-1}) which we declare to be the source and target (k - 1)-morphism of F, respectively.

The composition in Alg_n can be roughly described as follows: Given two composable k-morphisms E \xrightarrow{F} E' \xrightarrow{F'} E'', we can reparameterize them and glue them to a factorization algebra on the stratified space

∅ ⊂ ⋯ ⊂ ∅ ⊂ R^{n-k} × \{-1, 1\} × \{0\}^k ⊂ ⋯ ⊂ R^{n-k+1} × \{0\}^k ⊂ ⋯ ⊂ R^{n-1} × \{0\} ⊂ R^n, \tag{8}

where E, E', E'' are identified with the restriction to

R^{n-k} × (-∞, -1) × R^{k-1}, \tag{9}

R^{n-k} × (-1, 1) × R^{k-1}, \tag{10}

R^{n-k} × (+1, +∞) × R^{k-1}, \tag{11}

respectively. This factorization algebra can then be pushed forward along the map R^n → R^n which in the (n - k + 1)-th coordinate sends [-1, 1] to 0 and identifies

+1 : (-∞, -1] \xrightarrow{=} (-∞, 0] \quad \text{and} \quad -1 : [+1, +∞) \xrightarrow{=} [0, +∞). \tag{12}

6 Factorization homology as a TFT

Finally we sketch how to make \int_A into a functor of (∞, n)-categories

\left( \int_A \right) : \text{Bord}_n^{fr} \to \text{Alg}_n. \tag{13}
If we are given a $k$-morphism $N$ in $\text{Bord}^n_{fr}$, we can consider the factorization algebra $\int_{-\subseteq M} A$ on $M := N \times \mathbb{R}^{n-k}$. For $k = 0$, i.e., $N = \mathbb{R}^0$ gives rise to the factorization algebra $\int_{-\subseteq \mathbb{R}^n} A$ which is exactly the object corresponding to $A$ in $\text{Alg}_n$.

For $k \neq 0$, we have to push forward along a suitable map to a stratified space by choosing appropriate collars. For example, if we are given a 1-morphism, i.e. a cobordism $N$ between $N_0$ and $N_1$, we can choose collars

\[ N_0 \times (-\infty,0] \hookrightarrow N \hookleftarrow N_1 \times [0,\infty) \quad (14) \]

and define a map $f : N \to \mathbb{R}$ as follows:

- on the collars it is given by projecting onto $(-\infty,0]$ or $[0,\infty)$, respectively;
- all other points go to 0.

Finally, we can define the value of our TFT on $N$ to be the factorization algebra obtained by pushing $\int_{-\subseteq N \times \mathbb{R}^{n-1}}$ forward along $f \times \text{id} : N \times BR^{n-1} \to \mathbb{R}^n$.

The construction for higher $k$ is similar by repeatedly choosing collars in $M := N \times \mathbb{R}^{n-k}$ and then pushing forward along an analogous collapse map $M \to \mathbb{R}^n$, where the right side is stratified as in (6). See the following picture for $k = n = 2$: 
Figure 1: An example of a 2-morphism in $\text{Bord}_2^G$ with collars and the associated collapse map to the stratified space $\mathbb{R}^2$. 

$\mathbb{R}^0 < \mathbb{R}^4 < \mathbb{R}^2$