DISCUSSION SECTION 11
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ABSTRACT. Below you’ll find a collection of questions. You should think of these questions less as a homework assignment and more as a playground to run around in and spark your imagination. It’s more fun playing on the swings with friends, so come to discussion section to talk with others about anything you find confusing or exciting!

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1. BASICS OF STRATIFIED SPACES

At the most basic level, a topological space $T$ stratified by a linearly ordered poset $\{0 < \cdots < n\}$ is given by the data of a filtration

$$F_0(T) \subset \cdots \subset F_{n-1}(T) \subset T$$

of $T$ by closed subsets. The following definition gives a convenient way of working with topological spaces that are stratified by more general posets.

**Definition 1.** Let $P$ be a poset (or more generally, a preorder). The *Alexandroff topology* on $P$ is the topology on the underlying set of $P$ in which a subset $U \subset P$ is open if and only if $U$ is *upwards closed* in the following sense: $p \in U$ and $q \geq p$ implies that $q \in U$. We write $\text{Alex}(P) \in \text{Top}$ for the set $P$ equipped with the Alexandroff topology.

The category of $P$-stratified topological spaces is the overcategory $\text{Top}/\text{Alex}(P)$. Given a stratified space $s : T \to \text{Alex}(P)$, for each $p \in P$ we call $T_p := s^{-1}(p)$ the *$p$-th stratum* of $T$.

**Question 2.** Do you know another name for the topological space $\text{Alex}(\{0 < 1\})$?

**Question 3.** Let $T$ be a topological space. Provide a natural bijection between stratifications of $T$ by $[n] := \{0 < \cdots < n\}$ in the sense of Definition 1 and filtrations

$$F_0(T) \subset \cdots \subset F_{n-1}(T) \subset T$$

of $T$ by closed subsets.

**Question 4.** Prove that the functor $\text{Alex} : \text{Preord} \to \text{Top}$ is fully faithful. Does $\text{Alex}$ admit a left or right adjoint?

**Question 5.** The point of this problem is to work through some examples.
(1) Show that every CW complex has a natural stratification by the poset $\mathbb{Z}_{\geq 0}$.
(2) Let $V$ be a variety over the complex numbers. Prove that the complex points $V(\mathbb{C})$ with the analytic topology admits a stratification where all of the strata are (the complex points of) smooth $\mathbb{C}$-varieties.
(3) Come up with some examples of topological spaces stratified by posets that are not linearly ordered.

**Question 6.** For each integer $n \geq 0$, there is a stratification of the topological $n$-simplex $\Delta^n := \{(t_0, \ldots, t_n) \in [0, 1]^{n+1} \mid t_0 + \cdots + t_n = 1\}$ by $[n]$ defined by
$$(t_0, \ldots, t_n) \mapsto \max\{i \in [n] \mid t_i \neq 0\}.$$ Use these stratifications to show that every poset $P$ admits a natural stratification $s_P : |N(P)| \to \text{Alex}(P)$.

**Question 7.** Let $P$ be a poset.
(1) Can you write down a convenient basis of open subsets of $P$?
(2) Prove that for each $p \in P$, the one-point set $\{p\} \subset \text{Alex}(P)$ is locally closed (i.e., the intersection of an open and a closed).
(3) Provide a natural equivalence of categories
$$\text{Sh}(\text{Alex}(P); \text{Set}) \equiv \text{Fun}(P, \text{Set})$$.

In the expression ‘Fun($P, \text{Set}$)’, we’re thinking of the poset $P$ as a category. **Hint:** use (1).

## 2. Singularity Data

The following is [3, Rmk. 4.3.6].

**Question 8.** Let $\bar{X}$ be an $n$-dimensional singularity datum. Say
$$\bar{X} = (\{X_i\}, \{\zeta_i\}, \{p_i : E_i \to X_i\}).$$

Show that a $\bar{X}$-manifold of dimension $m$ determines a stratified space
$$M_n \subset M_{n-1} \subset \cdots \subset M_0 = M,$$
so that each open stratum $M_k - M_{k-1}$ is a smooth $(m-k)$-manifold with $(X_k, \zeta_k)$-structure. What conditions are there on how the strata are glued together?

**Question 9.** If I wanted the stratification
$$M_n \subset M_{n-1} \subset \cdots \subset M_0 = M$$
to be of the form
$$M_k = M_{k-1} \cup N_k$$
for all $k$ and some $(m-k)$-manifolds $N_k$, what conditions does the singularity datum $\bar{X}$ have to satisfy?

**Question 10.** Is it possible to define an $n$-dimensional singularity datum $\bar{X}$ (you choose the $n$) so that $\mathbb{R}^m$ with it’s stratification
$$\mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^m$$
is a $\bar{X}$-manifold?
3. REGULARITY CONDITIONS ON STRATIFIED SPACES

Recall the definition of a Whitney stratified space. You can do this by checking out the Wikipedia page on Whitney conditions or these notes from a course David Nadler taught.

**Question 11.** Are the stratified spaces coming from $\bar{X}$-manifolds Whitney stratified spaces?

Recall the definition of conically stratified spaces; see Definition 1.2 of Peter’s notes or [2, Defn. A.5.5].

**Question 12.** Are the stratified spaces coming from $\bar{X}$-manifolds conically stratified spaces? Are Whitney stratified spaces conically stratified?

If you’re feeling really bold, recall the definition of conically smooth stratified spaces \(^1\) from [1].

**Question 13.** Are the stratified spaces coming from $\bar{X}$-manifolds conically smooth stratified spaces?

**Question 14.** Do you think every Whitney stratified space can be given a conically smooth structure? (This is actually an open problem!)

**Question 15.** Write out in detail what a 1-dimensional singularity datum is. What is the data of a closed $\bar{X}$-manifold?

4. FEYNMAN DIAGRAMS & STRATIFIED SPACES

**Question 16.** Let $\bar{X}$ be a 1-dimensional singularity datum,

$$\bar{X} = (\{X_i\}, \{\z_i\}, \{p_i : E_i \to X_i\})$$

Let $\bar{X}_0$ be the double cover associated to $\z_0$. Assume $\pi_j \bar{X}_0 = 0$ for $j > 0$.

Show that a closed $\bar{X}$-manifold determines a Feynman diagram.

If you don’t have a definition of “Feynman diagram” in mind, draw pictures of the data of a closed $\bar{X}$-manifold and compare your pictures to the types of pictures that come up when you google image search “Feynman diagram.”

5. EXIT-PATH $\infty$-CATEGORIES & CONSTRUCTIBLE SHEAVES

**Question 17.** Read Lurie’s definition of the exit-path simplicial set of a stratified space [2, Definition A.6.2 & Remark A.6.5]. Come up with an example of a non-conically stratified space for which the exit-path simplicial set is not a quasicategory.

**Definition 18.** Let $P$ be a poset and let $T \to \text{Alex}(P)$ be a $P$-stratified topological space. A sheaf $\mathcal{F}$ on $T$ is constructible if for each $p \in P$, the restriction $\mathcal{F}|_{T, p}$ is locally constant.

Write $\text{Cons}_P(T) \subset \text{Sh}(T; \text{Spc})$ for the full subcategory spanned by the constructible sheaves.

**Question 19.** Read through the first few pages of [2, §A.9] about Lurie’s “exodromy equivalence”

$$\text{Cons}_P(T) \cong \text{Fun}(\text{Exit}_P(T), \text{Spc})$$

between constructible sheaves and representations of the exit-path $\infty$-category of a nice conically stratified space $T$. What is this theorem saying when $T = \mathbb{R}$ with the stratification $\{0\} \subset \mathbb{R}$?

**Question 20.** Describe the exit-path $\infty$-category of the open cone $\text{Cone}(\mathbb{R}P^\infty)$ on $\mathbb{R}P^\infty$. (Here $\mathbb{R}P^\infty$ is given the trivial stratification.) What’s another name for a constructible sheaf on $\text{Cone}(\mathbb{R}P^\infty)$?

\(^1\)Note that a ‘conically smooth stratified space’ isn’t just a stratified space satisfying a property; it is an extra structure on a stratified space!
REFERENCES