THE VIRASORO GROUP AND DIFFERENTIAL COHOMOLOGY

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Summary:
- The Virasoro group is a particular central extension of Diff\(^+(S^1)\) by \(\mathbb{T}\).
- A theorem of Segal [Seg81, Corollary 7.5] proves that

\[
\text{Cent}_{\mathbb{T}}(\text{Diff}^+(S^1)) \xrightarrow{\sim} \text{Cent}_{\mathbb{T}}(\text{PSL}_2(\mathbb{R})) \times \text{Cent}_{\mathbb{R}}(\mathfrak{w}),
\]

where \(\mathfrak{w} = \text{Lie}(\text{Diff}^+(S^1))\) is the Witt algebra. The map is: restrict the central extension to \(\text{PSL}_2(\mathbb{R}) \subset \text{Diff}^+(S^1)\) for the first component, and differentiate for the second component.
- The (isomorphism classes of) central extensions created from differential lifts of \(p_1\) are expected to be trivial when restricted to \(\text{PSL}_2(\mathbb{R})\). Proving this might be a good warm-up to studying \(\text{Diff}^+(S^1)\).
- The Virasoro extension is also trivial when restricted to \(\text{PSL}_2(\mathbb{R})\). Therefore, to identify which differential lift \(\tilde{p}_1\) of \(p_1\) induces the Virasoro central extension, it suffices to look at the induced central extension of Lie algebras of \(\mathfrak{w}\). There is an \(\mathbb{R}\) worth of differential lifts of \(p_1\), and \(\text{Cent}_{\mathbb{R}}(\mathfrak{w}) \cong \mathbb{R}\).

1. Review of central extensions

Definition 1.1. Let \(G\) be a group and \(A\) be an abelian group. A central extension of \(G\) by \(A\) is a short exact sequence of groups

\[
1 \longrightarrow A \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1,
\]

such that \(A \subset Z(\tilde{G})\). An equivalence of central extensions is a map of short exact sequences which is the identity on \(G\) and on \(A\). These form an abelian group we denote \(\text{Cent}_A(G)\).

When \(G\) and \(A\) have additional structure, we will ask that central extensions respect that structure: for example, when both are Lie groups (possibly infinite-dimensional), we want (1.2) to be a short exact sequence of Lie groups.

For discrete \(G\) and \(A\), central extensions are classified by \(H^2(G; A)\). Explicitly, given a cocycle \(b: G \times G \rightarrow A\), we build the central extension by setting \(\tilde{G} = G \times A\) as sets, with the twisted multiplication

\[
(g_1, a_1) \cdot (g_2, a_2) := (g_1g_2, a_1 + a_2 + b(g_1, g_2)).
\]

Associativity follows from the cocycle condition; if two cocycles are related by a coboundary, their induced central extensions are equivalent.

Generalizing this to Lie groups is not straightforward — you can’t just use smooth cochains unless \(A\) is a topological vector space. We are interested in central extensions by \(\mathbb{T}\), so we’ll have to be craftier. The fix is due to Segal [Seg70], and was later rediscovered by Brylinski [Bry00], following Blanc [Bla85]. We rephrase it in language familiar to this seminar.

Let \(A\) be an abelian Lie group. Throughout today’s talk, \(A\) denotes the simplicial sheaf on \(\text{Man}\) whose value on a test manifold \(M\) is the space of smooth maps \(M \rightarrow A\).

Theorem 1.4 (Segal [Seg70], Brylinski [Bry00]). Let \(G\) and \(A\) be abelian Lie groups. Then, equivalences classes of central extensions in which \(\tilde{G} \rightarrow G\) is a principal \(A\)-bundle are classified by \(H^2(B_{\bullet}G; A)\).

The idea of the characterization is that \(B_{\bullet}G\) admits a simplicial resolution

\[
B_{\bullet}G \simeq \left( \ast \xrightarrow{\ldots} G \xrightarrow{\ldots} G \times G \xrightarrow{\ldots} G \times G \times G \times \cdots \right),
\]

\(^{\text{1By contrast, the simplicial sheaf just denoted \("A\"\) treats \(A\) as having the discrete topology. This is a little bit counterintuitive but is standard notation.}\)
which is the content of the bar construction, and we want to compute $\pi_0$ of the simplicial set of maps

\[
\begin{array}{c}
\ast \\
\downarrow \\
G \\
\downarrow \\
g \times g \\
\downarrow \\
A \\
\downarrow \\
\ast
\end{array}
\] (1.6)

The blue map corresponds to the 2-cocycle for the extension in ordinary group cohomology. (Note: I wasn’t able to figure out the proof in time to include it in this talk.)

**Remark 1.7.** Differentiating a central extension of Lie groups produces a central extension of Lie algebras

\[
0 \longrightarrow \mathfrak{a} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0,
\] (1.8)

which is what you would expect ($\mathfrak{a}$ is an abelian Lie algebra contained in the center of $\tilde{\mathfrak{g}}$).

Central extensions of Lie algebras are classified by second Lie algebra cohomology $H^2(g;\mathfrak{a})$. Cocycles are alternating bilinear maps $\omega: \Lambda^2\mathfrak{g} \rightarrow \mathfrak{a}$ satisfying a version of the Jacobi identity,

\[
\omega(X, [Y,Z]) + \omega(Y, [Z,X]) + \omega(Z, [X,Y]) = 0.
\] (1.9)

From such an $\omega$, we build a central extension which, as a vector space, is $\mathfrak{g} \oplus \mathfrak{a}$, but with Lie bracket

\[
[(X_1, A_1), (X_2, A_2)] := [X_1, X_2] + \omega(X_1, X_2).
\] (1.10)

A 1-cochain is a map $\lambda: \mathfrak{g} \rightarrow \mathfrak{a}$, and its differential is $d\lambda(X,Y) := \Lambda([X,Y])$.

So we have a map $H^2(B\ast G,A) \rightarrow H^2(g;\mathfrak{a})$. The van Est theorem says this is an equivalence in certain nice situations (not ours, unfortunately). \hfill \checkmark

2. **The Virasoro algebra and the Virasoro group**

Let $\Gamma := \text{Diff}^+(S^1)$, the group of orientation-preserving diffeomorphisms of the circle. This is an infinite-dimensional Fréchet Lie group, meaning it is locally modeled on a Fréchet space and has a group structure in which multiplication and inversion are smooth.

**Definition 2.1.** The Witt algebra $\mathfrak{w}$ is the infinite-dimensional real Lie algebra of polynomial vector fields on $S^1$. Explicitly, it is generated by $\xi_n := -x^{n+1} \frac{\partial}{\partial x}$ for $n \in \mathbb{Z}$, with bracket

\[
[\xi_m, \xi_n] = (m-n)\xi_{m+n}.
\] (2.2)

Skating over issues of regularity, the Witt algebra is the Lie algebra of $\Gamma$. \footnote{If we were to treat regularity more carefully, we would allow some infinite linear combinations of the $\xi_n$, corresponding to the Fourier series of a smooth vector field.}

The Virasoro algebra $\mathfrak{v}$ is a central extension of $\mathfrak{w}$ by $\mathbb{R}$. There is also a Virasoro group $\tilde{\Gamma}$, a central extension of $\Gamma$; the Virasoro algebra is its Lie algebra, and is easier to define (since Lie algebra $H^2$ Just Works to produce central extensions, whereas we had to modify group cohomology). Specifically, consider the 2-cocycle $c: \Lambda^2\mathfrak{w} \rightarrow \mathbb{R}$ given by

\[
c(\xi_m, \xi_n) := \frac{1}{12}(m^3 - m)\delta_{m+n,0}c,
\] (2.3)

where $c$ is a chosen basis for $\mathbb{R}$. The 1/12 is not there for any deep reason, just as a normalization constant. Anyways, as in (1.10) this defines for us an extension $1 \rightarrow \mathbb{R} \rightarrow \tilde{\mathfrak{v}} \rightarrow \mathfrak{w} \rightarrow 1$, called the Virasoro algebra. The element $c$ inside $\tilde{\mathfrak{v}}$ is called the central charge.

The Virasoro group $\tilde{\Gamma}$ is the extension of $\Gamma$ by $\mathbb{T}$ which is, as a space, $\mathbb{T} \times \mathbb{R}$, with multiplication

\[
(z_1, f) \cdot (z_2, g) := (z_1 + z_2 + B(f,g), f \circ g),
\] (2.4)

where $B: \Gamma \times \Gamma \rightarrow \mathbb{T}$ is the Bott cocycle

\[
B(f,g) := \int_{S^1} \log(f \circ g)' d(log g)'.
\] (2.5)

**Remark 2.6.** The identification $S^1 \cong \mathbb{R}/\mathbb{Z}$ embeds $\text{PGL}_2^+(\mathbb{R}) = \text{PSL}_2(\mathbb{R}) \subset \Gamma$ as the real fractional linear transformations; hence also $\mathfrak{psl}_2(\mathbb{R}) = \mathfrak{sl}_2(\mathbb{R}) \subset \mathfrak{w}$, as the Lie algebra generated by $\xi_{-1}$, $\xi_0$, and $\xi_1$. Restricted to $\text{PSL}_2(\mathbb{R})$, the Virasoro central extension is trivializable, which will be useful later. \hfill \checkmark
Some authors’ definitions will differ, e.g. defining the Witt and Virasoro algebras as complex Lie algebras, or defining the Virasoro group as the universal cover of ours. In particular, I used a few different sources when preparing this talk, and though I think they were consistent, it’s possible that I missed something and they used different conventions.

Remark 2.8 (Applications). The Virasoro group and algebra appear in two-dimensional conformal field theory (CFT). Usually, in quantum field theory, one specifies a (Riemannian or Lorentzian) metric on spacetime, and the information in the theory depends on the metric. A conformal field theory is a quantum field theory in which all information only depends on the conformal class of the metric. Two-dimensional CFTs in particular connect to many areas of mathematics and physics.

- The mathematical formalization of 2d CFT, using vertex algebras, has connections to representation theory, and, famously, to monstrous moonshine.
- One way to think of string theory is as a 2d CFT on the worldsheet, one of whose fields is a map into (10- or 26-dimensional) spacetime.
- In condensed-matter physics, Wess-Zumino-Witten models (particular 2d CFTs) are used in modeling the quantum Hall effect.
- Maybe closest to the hearts of the attendees of this seminar: the Stolz-Teichner conjecture suggests that cocycles for TMF on a space $X$ are given by families of 2d supersymmetric quantum field theories parametrized by $X$. Superconformal field theories are particularly nice examples of these, and have been used to shine light on this conjecture.

So how does the Virasoro appear in CFT? Let’s suppose we’re on a Riemann surface $\Sigma$ in a local holomorphic coordinate $z$. If you write out commutators for the Lie algebra $\mathfrak{c}$ of infinitesimal conformal transformations, you might notice they look like those for the Witt algebra — in fact, if you complexify it, you obtain precisely $\mathfrak{w}_C \oplus \mathfrak{w}_C$. So this acts on the system as a symmetry; you can think of it as two different Witt group symmetries.

The fact that we obtain a central extension is standard lore from quantum mechanics. The state space in a quantum system is a complex Hilbert space, but if $\lambda \in \mathbb{C}^\times$, the states $|\psi\rangle$ and $\lambda |\psi\rangle$ are thought of as the same, in that measurements cannot distinguish them. Nonetheless, the formalism of quantum mechanics uses the Hilbert space structure.

The takeaway, though, is that a symmetry of the system, as in acting on the states and all that, only has to be a projective representation on the state space! So to describe an honest Lie group or Lie algebra acting on the state space, we need to take a central extension of the symmetry group or Lie algebra. This leads us to the (complexified) Virasoro algebra and Virasoro group. Thus, the symmetry algebra of conformal field theory is (at least) a product of two copies of the Virasoro algebra, and the space of states is a representation of the Virasoro algebra.

3. Constructing the central extension with differential cohomology

The key fact bridging differential cohomology and central extensions is:

Lemma 3.1. There is an equivalence of simplicial sheaves $\mathbb{Z}(1) \simeq \Sigma^{-1} \mathbb{T}$.

Proof. By definition, $\mathbb{Z}(1)$ is the sheaf $0 \to \mathbb{Z} \to \Omega^0 \to 0$, and $\Omega^0 = \mathbb{R}$. The chain map

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \text{mod } \mathbb{Z} \\
0 & \longrightarrow & 0
\end{array}
$$

is a quasi-isomorphism.

Corollary 3.3. For any Lie group $G$, possibly infinite-dimensional, $H^2(B\mathcal{G};\mathbb{T}) \cong H^3(B\mathcal{G};\mathbb{Z}(1))$; in particular, this classifies central extensions of $G$ by $\mathbb{T}$ which are principal $\mathbb{T}$-bundles over $G$.

\[3\]The appearances of SCFTs, rather than just CFTs, in superstring theory and in the Stolz-Teichner conjecture aren’t as related to the Virasoro group and algebra; they have a larger symmetry algebra, though it’s closely related.
Thus, we would like to construct the Virasoro central extension via a differential cohomology class in $H^4(B_\ast \Gamma; \mathbb{Z}(1))$. This builds on the hard work of the previous few talks. Mike Hopkins described how $H^4(B_\ast \text{GL}_n(\mathbb{R}); \mathbb{Z}(2))$ fits into a pullback square

$$
\begin{array}{ccc}
H^4(B_\ast \text{GL}_n(\mathbb{R}); \mathbb{Z}(2)) & \longrightarrow & H^4(B\text{GL}_n(\mathbb{R}); \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^4(B_\ast \text{GL}_n(\mathbb{R}); \mathbb{R}(2)) & \longrightarrow & H^4(B\text{GL}_n(\mathbb{R}); \mathbb{R}).
\end{array}
$$

(3.4)

**Definition 3.5.** A differential lift of $p_1$ is a class $\tilde{p}_1 \in H^4(B_\ast \text{GL}_n(\mathbb{R}); \mathbb{Z}(2))$ whose image under the blue map is the usual $p_1 \in H^4(B\text{GL}_n(\mathbb{R}); \mathbb{Z})$.

Mike Hopkins also showed how to prove that $H^4(B_\ast G; \mathbb{R}(2)) \cong \text{Sym}^2(\mathfrak{g}^*)^G$; for $\text{GL}_n(\mathbb{R})$, this is an $\mathbb{R}^2$, spanned by the invariant polynomials $\text{tr}(A)^2$ and $\text{tr}(A)^2$, which we call $c_1^2$ and $c_2$, respectively. And $H^4(BG; \mathbb{R})$ can be dispatched with ordinary Chern-Weil theory: we repeat the same story, but retracting $G$ onto its maximal compact. Here, we get $H^4(BO(n); \mathbb{R}) \cong \mathbb{R}$, spanned by $\text{tr}(A^2)$, as $\text{tr}(A)^2 = 0$. Accordingly, the red map in (3.4) is a rank-1 map $\mathbb{R}^2 \to \mathbb{R}$. Since (3.4) is a pullback square, there is an $\mathbb{R}$ worth of differential lifts of $p_1$: explicitly, $\lambda \in \mathbb{R}$ gives you the lift of $p_1$ which maps in the lower left to $(1/2)(\lambda^2 - 2c_2)$. However, if you want $\tilde{p}_1(E_1 + E_2) = \tilde{p}_1(E_1) + \tilde{p}_1(E_2)$, you force $\lambda = 1$, which is a quick calculation with the Whitney formula. (All this was in Araminta’s talk.)

Last time, Araminta also discussed the fiber integration map for an $\tilde{H}$-oriented fiber bundle $F \to E \to B$, which has the form $H^k(E; \mathbb{Z}(\ell)) \to H^{k-\dim F}(B; \mathbb{Z}(\ell - \dim F))$. Combining all this, consider the universal oriented sphere bundle $E_\ast \Gamma \times_\Gamma S^1 \to B_\ast \Gamma$, which is an $\tilde{H}$-oriented fiber bundle with fiber $S^1$. Therefore, given a differential lift of $p_1$, we can apply it to the vertical tangent bundle $V \to E_\ast \Gamma \times_\Gamma S^1$, and get a class $\tilde{p}_1(V) \in H^4(E_\ast \Gamma \times_\Gamma S^1; \mathbb{Z}(2))$. Then we can push it forward to a class in $H^4(B_\ast \Gamma; \mathbb{Z}(1))$, which determines an isomorphism class of central extensions of $\Gamma$ as above. Our goal is to determine the choice of $\lambda$ such that this central extension gives the Virasoro group. I was unable to figure this out in time for the talk; in the remainder of the talk, I’ll suggest some ways forward.

The first thing we need is a way to get a handle on the group of extensions of $\Gamma$. Recall that $\text{PSL}_2(\mathbb{R}) \subset \Gamma$ as the real fractional linear transformations of $\mathbb{R}P^1 = S^1$; hence a central extension of $\Gamma$ restricts to a central extension of $\text{PSL}_2(\mathbb{R})$.

**Theorem 3.6 (Segal [Seg81, Corollary 7.5]).** A central extension of $\Gamma$ by $T$ is determined by the pair of (1) its restriction to $\text{PSL}_2(\mathbb{R})$ and (2) the induced Lie algebra central extension of $\mathfrak{g}$ by $\mathfrak{t}$. Said differently, there is an isomorphism of abelian groups $\text{Cent}_T(\Gamma) \to \text{Cent}_T(\text{PSL}_2(\mathbb{R})) \times \text{Cent}_T(\mathfrak{g})$

We can identify both of these groups. First, $\pi_1(\text{PSL}_2(\mathbb{R})) \cong \mathbb{Z}$, and the universal cover $\tilde{\text{SL}}_2(\mathbb{R}) \to \text{PSL}_2(\mathbb{R})$ is the universal central extension of $\text{PSL}_2(\mathbb{R})$: for any abelian group $A$, central extensions of $\text{PSL}_2(\mathbb{R})$ by $A$ are in bijection with maps $\varphi: \mathbb{Z} \to A$, given by

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\text{SL}}_2(\mathbb{R}) & \longrightarrow & \text{PSL}_2(\mathbb{R}) & \longrightarrow & 0 \\
& & & & \varphi & & \downarrow & & & \\
0 & \longrightarrow & A & \longrightarrow & \tilde{\text{SL}}_2(\mathbb{R})_\varphi & \longrightarrow & \text{PSL}_2(\mathbb{R}) & \longrightarrow & 0.
\end{array}
$$

(3.7)

So $\text{Cent}_T(\text{PSL}_2(\mathbb{R})) \cong \text{Hom}(\mathbb{Z}, T) = T$.

The computation that $H^2(\mathfrak{g}; \mathbb{R}) \cong \mathbb{R}$ is standard, e.g. [Obl17, §6.2.1].

Thus the map from differential lifts of $p_1$ to central extensions of $\Gamma$ is a map $\mathbb{R} \to \mathbb{R} \times T$. The Virasoro central extension is in the first factor of $\mathbb{R} \times T$ — it induces the Virasoro algebra central extension, hence is nontrivial on the first factor, and is trivial when restricted to $\text{PSL}_2(\mathbb{R})$.

The $\tilde{p}_1$-induced central extensions should also land purely in the $\mathbb{R}$ factor, which Dan Freed suggested to me. His proof idea involved trivializing this characteristic class over $B_\Gamma \text{PSL}_2(\mathbb{R})$. Alternatively, this amounts to showing that for any differential lift $\tilde{p}_1$ of $p_1$, under the fiber integration map

$$H^4(E_\ast \text{PSL}_2(\mathbb{R}) \times_{\text{PSL}_2(\mathbb{R})} S^1; \mathbb{Z}(2)) \longrightarrow H^3(B_\ast \text{PSL}_2(\mathbb{R}); \mathbb{Z}(1)),$$

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4The notation is because it’s also the universal cover of $\text{SL}_2(\mathbb{R})$, which is the connected double cover of $\text{PSL}_2(\mathbb{R})$. 

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\( \tilde{p}_1(V) \mapsto 0 \). We’re going to need to understand this map for \( B_\bullet \Gamma \) anyways, so studying this one might make for a good baby case to figure out first. After that, it will hopefully be clearer how to study the map for \( B_\bullet \Gamma \).

We have a map between two real lines, though we can identify both of them with \( \mathbb{R} \): the line of differential lifts of \( p_1 \), through \( \lambda \); the line of central extensions of \( \Gamma \) trivialized on \( \text{PSL}_2(\mathbb{R}) \), through the Virasoro extension (either of the group or the Lie algebra).

**Remark 3.9.** Dan Freed suggested another, alternate way to obtain “the right” differential lift of \( p_1 \), though he hasn’t thought about it in much detail: \( H^4(B_\bullet \text{GL}_n(\mathbb{R}); \mathbb{Z}(4)) \cong \mathbb{Z} \), with generator given by a different lift of \( p_1 \) to differential cohomology, namely the one given by Chern-Weil theory. The truncation map \( \mathbb{Z}(4) \to \mathbb{Z}(2) \) induces a map \( H^4(B_\bullet \text{GL}_n(\mathbb{R})\mathbb{Z}(4)) \to H^4(B_\bullet \text{GL}_n(\mathbb{R}); \mathbb{Z}(2)) \), and this map is not surjective; it might send the generating lift of \( p_1 \) to something helpful. 

**References**


