GEOMETRIC REALIZATION OF THE SEGAL–SUGAWARA CONSTRUCTION

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To Graeme Segal on his 60th birthday

ABSTRACT. We apply the technique of localization for vertex algebras to the Segal-Sugawara construction of an “internal” action of the Virasoro algebra on affine Kac-Moody algebras. The result is a lifting of twisted differential operators from the moduli of curves to the moduli of curves with bundles, with arbitrary decorations and complex twistings. This construction gives a uniform approach to a collection of phenomena describing the geometry of the moduli spaces of bundles over varying curves: the KZB equations and heat kernels on non-abelian theta functions, their critical level limit giving the quadratic parts of the Beilinson-Drinfeld quantization of the Hitchin system, and their infinite level limit giving a Hamiltonian description of the isomonodromy equations.

1. Introduction.

1.1. Uniformization. Let $G$ be a complex connected simply-connected simple algebraic group with Lie algebra $\mathfrak{g}$, and $X$ a smooth projective curve over $\mathbb{C}$. The geometry of $G$–bundles on $X$ is intimately linked to representation theory of the affine Kac-Moody algebra $\hat{\mathfrak{g}}$, the universal central extension of the loop algebra $L\mathfrak{g} = \mathfrak{g} \otimes \mathcal{K}$, where $\mathcal{K} = \mathbb{C}[[t]]$. More precisely, let $G(\mathcal{O})$, where $\mathcal{O} = \mathbb{C}[[t]]$, denote the positive half of the loop group $G(\mathcal{K})$. There is a principal $G(\mathcal{O})$–bundle over the moduli space (stack) $\text{Bun}_G(X)$ of $G$–bundles on $X$, which carries an infinitesimally simply transitive action of $L\mathfrak{g}$. This provides an infinitesimal “uniformization” of the moduli of $G$–bundles. Moreover, this uniformization lifts to an infinitesimal action of $\hat{\mathfrak{g}}$ on the “determinant” line bundle $\mathcal{L}$ on $\text{Bun}_G(X)$, whose sections give the nonabelian versions of the spaces of theta functions.

The geometry of the moduli space $\mathcal{M}_{g,1}$ of smooth pointed curves of genus $g$ is similarly linked to the Virasoro algebra $Vir$, the universal central extension of the Lie algebra $\text{Der } \mathcal{K}$ of derivations of $\mathcal{K}$ (or vector fields on the punctured disc). Let $\text{Aut } \mathcal{O}$ denote the group of automorphisms of the disc. Then there is a principal $\text{Aut } \mathcal{O}$–bundle over $\mathcal{M}_{g,1}$ which carries an infinitesimally transitive action of $\text{Der } \mathcal{K}$. This uniformization lifts to an infinitesimal action of the Virasoro algebra on the Hodge line bundle $\mathcal{L}$ on $\mathcal{M}_{g,1}$.

These uniformizations can be used, following [BB, BS], to construct localization functors from representations of the Virasoro and Kac-Moody algebras to sheaves on...
the corresponding moduli spaces. Using these localization functors, one can describe sheaves of modules over (twisted) differential operators, as well as sheaves on the corresponding (twisted) cotangent bundles over the moduli spaces. The sheaves of twisted differential operators and twisted symbols (functions on twisted cotangent bundles) themselves are particularly easy to describe in this fashion: they are the localizations of the vacuum modules of the respective Lie algebras.

The Virasoro algebra acts on the Kac-Moody algebra via its natural action by derivations on the space \( \mathcal{K} \) of Laurent series. Therefore we form the semidirect product \( \mathfrak{g} = Vir \rtimes \hat{\mathfrak{g}} \), which uniformizes the moduli stack \( \text{Bun}_{G,g,1} \) classifying \( G \)-bundles over varying pointed curves of genus \( g \). There is now a two-parameter family of line bundles \( \mathcal{L}_{k,c} \) on which the (two-parameter) central extension \( \hat{\mathfrak{g}} \) acts, and the corresponding two-parameter families of sheaves of twisted differential operators and twisted symbols. Thus it is natural to look to the structure of \( \hat{\mathfrak{g}} \) for information about the variation of \( G \)-bundles over the moduli of curves.

1.2. The Segal-Sugawara construction. The interaction between the variation of curves and that of bundles on curves is captured by a remarkable feature of affine Kac-Moody algebras, best seen from the perspective of the theory of vertex algebras.

The Segal-Sugawara construction presents the action of the Virasoro algebra on \( \hat{\mathfrak{g}} \) as an internal action, by identifying infinite quadratic expressions in the Kac-Moody generators which satisfy the Virasoro relations. More precisely, the Virasoro algebra embeds in a certain completion of \( U\hat{\mathfrak{g}} \), and hence acts on all smooth representations of \( \hat{\mathfrak{g}} \) of non-critical levels. From the point of view of vertex algebras, the construction simply involves the identification of a conformal structure, i.e., a certain distinguished vector, in the vacuum module over \( \hat{\mathfrak{g}} \).

Our objective in this paper is to draw out the different geometric consequences of this construction in a simple uniform fashion. Our approach involves four steps:

- We consider the Segal-Sugawara construction as defining a \( G(\mathcal{O}) \)-invariant embedding of the vacuum module of the Virasoro algebra into the vacuum module of the Virasoro–Kac-Moody algebra \( \hat{\mathfrak{g}} \).
- Twists of the vacuum modules form sheaves of algebras on the relevant moduli spaces, and the above construction gives rise to a homomorphism of these sheaves of algebras.
- The sheaves of twisted differential operators on the moduli spaces are quotients of the above algebras, and the vertex algebraic description of the Segal-Sugawara operators guarantees that the homomorphism descends to a homomorphism between sheaves of differential operators.
- All of the constructions vary flatly with respect to the Kac-Moody level and the Virasoro central charge and possess various “classical” limits which may be described in terms of the twisted cotangent bundles to the respective moduli spaces.

Our main result is the following theorem. Let \( \Pi : \text{Bun}_{G,g} \to \mathcal{M}_g \) denote the projection from moduli of curves and bundles to moduli of curves, \( k,c \in \mathbb{C} \) (the level and charge), \( \mu_{\mathfrak{g}} = h^\vee \dim \mathfrak{g} \) and \( c_k = c - \frac{k \dim \mathfrak{g}}{k + h^\vee} \). We denote by \( \mathcal{D}_{k,c}, \mathcal{D}_{k/\mathfrak{m}_g} \), and
$D_{ck}$ respectively the corresponding sheaves of twisted differential operators on $Bun_{G,g}$, relative to $M_g$, and on $M_g$. The sheaves of twisted symbols (functions on twisted cotangent bundles) with twists $\lambda, \mu$ on $M_g$ and $Bun_{G,g}$ are denoted by $\mathcal{O}(T_{*\lambda,\mu}^* Bun_{G,g})$ and $\mathcal{O}(T_{*\lambda,\mu}^* M_g)$, respectively.

1.2.1. **Theorem.**

(1) Let $k, c \in \mathbb{C}$ with $k$ not equal to minus the dual Coxeter number $h^\vee$. There is a canonical homomorphism of sheaves of algebras on $M_g$,

\[ D_{ck} \longrightarrow \Pi_* D_{k,c}, \]

and an isomorphism of twisted $D$–modules on $Bun_{G,g}$

\[ D_{k/M_g} \otimes_{\mathcal{O}} \Pi^* D_{ck} \cong D_{k,c}. \]

(2) Let $c \in \mathbb{C}$, $k = -h^\vee$. There is an algebra homomorphism

\[ \mathcal{O}(T_{*\mu}^* M) \longrightarrow \Pi_* D_{-h^\vee,c}. \]

(3) For every $\lambda, \mu \in \mathbb{C}$ there is a ($\lambda$–Poisson) homomorphism

\[ \mathcal{O}(T_{*\lambda,\mu}^* M_g) \longrightarrow \Pi_* \mathcal{O}(T_{*\lambda,\mu}^* Bun_{G,g}). \]

1.2.2. Moreover, the homomorphisms (2) and (3) are suitably rescaled limits of (1). In fact we prove a stronger result, valid for moduli of curves and bundles with arbitrary “decorations”. Namely, we consider moduli of curves with an arbitrary number of marked points, jets of local coordinates and jets of bundle trivializations at these points. (For the sake of simplicity of notation, we work with the case of a single marked point or none – the multipoint extension is straightforward.)

1.3. **Applications.** The homomorphism (1) allows us to lift vector fields on $M_g$ to differential operators on $Bun_{G,g}$, which are first order along the moduli of curves and second order along moduli of bundles. In the special case of $k \in \mathbb{Z}_+$ and no decorations, this gives a direct algebraic construction of the heat operators acting on non-abelian theta functions (the global sections of $L_{k,c}$ along $Bun_G(X)$) constructed by different means in [ADW, Hi2, BK, Fa1]. Our approach via vertex algebras makes clear the compatibility of this heat equation with the projectively flat connection on the bundle of conformal blocks coming from conformal field theory ([LUM, FrS] — the Knizhnik–Zamolodchikov–Bernard equations, or higher genus generalization of the Knizhnik–Zamolodchikov equations, [KZ]). This compatibility has also been established in [La].

The important feature of our proof which follows immediately from part (1) of the above theorem is that the heat operators depend on two complex parameters $k$ and $c$, allowing us to consider various limits. This answers some of the questions raised by Felder in his study of of the general KZB equations over the moduli space of pointed curves for arbitrary complex levels (see [Fel, § 6]).

The geometric significance of the two classical limit statements (2), (3) comes from the identification of the twisted cotangent bundles of $M_g$ and $Bun_G(X)$ as the moduli spaces $T^*_g M_g = Proj_g$ of algebraic curves with projective structures and $T^*_{Bun_G(X)} = Conn_G(X)$ of $G$–bundles with holomorphic connections, respectively. As the parameter $k$ approaches the critical level $-h^\vee$, the component of the heat kernels along the moduli
of curves drops out, and we obtain commuting global second order operators on the moduli $\text{Bun}_G(X)$ (for a fixed curve $X$) from linear functions on the space of projective structures on $X$. Thus we recover the quadratic part of the Beilinson–Drinfeld quantization of the Hitchin hamiltonians (see [BD1]) in part (2). When we take $k, c$ to infinity in different fashions, we obtain the classical limits in part (3). For $\lambda, \mu = 0$ (i.e., as the symbols of our operators) we recover the quadratic part of the Hitchin integrable system [Hi1], namely, the hamiltonians corresponding to the trace of the square of a Higgs field. For $\lambda \neq 0$ we obtain a canonical (non-affine) decomposition of the twisted cotangent bundle $T^*_{\lambda,\mu}\text{Bun}_G = \mathcal{E}x\text{Conn}_{\lambda,\mu}^G$, the space of extended connections ([BS], [BZB2]), into a product over $\mathfrak{M}_g$ of the spaces of projective structures and flat connections. This recovers a construction of Bloch–Esnault and Beilinson [BE]. (This decomposition is also described from the point of view of kernel functions in [BZB1], where it is related to the Klein construction of projective connections from theta functions and used to give a conjectural coordinate system on the moduli of bundles.)

For $\lambda \neq 0$, the homomorphism (3) gives a Hamiltonian action of the vector fields on $\mathfrak{M}_g$ on the moduli space of flat $G$–bundles. This gives a time-dependent Hamiltonian system (see [M]), or flat symplectic connection, on the moduli $\text{Conn}_{G,g}$ of flat connections, with times given by the moduli of decorated curves. By working with arbitrary level structures, we obtain a similar statement for moduli spaces of connections with arbitrary (regular or irregular) singularities. We show that this is in fact a Hamiltonian description of the equations of isomonodromic deformation of meromorphic connections with respect to variation of the decorated curve. In particular, since all of the maps of the theorem come from a single multi-parametric construction, Theorem 1.2.1 immediately implies the following picture suggested by the work of [LO] (see also [I]), with arrows representing degenerations:

\[
\begin{align*}
\text{Quadratic Hitchin System} & \quad \leftarrow \quad \text{Isomonodromy Equations} \\
& \uparrow \\
\text{Quadratic Beilinson–Drinfeld System} & \quad \leftarrow \quad \text{Heat Operators/KZ Equations}
\end{align*}
\]

Namely, the isospectral flows of the Hitchin system arise as a degeneration of the isomonodromy equations as $\lambda \to 0$ (see also [Kr]), and the isomonodromy Hamiltonians appear as a degeneration of the KZB equations, previously proved for the Schlesinger equations (isomonodromy for regular singularities in genus zero) in [Res] (see also [Har]).

1.3.1. Outline of the paper. We begin in Section 2 with a review of the formalism of localization and the description of twisted differential operators and symbols from vacuum modules following [BB]. We then apply this formalism in the special case when the Lie algebras are the affine Kac-Moody and the Virasoro algebras. First, we review the necessary features of these Lie algebras in Section 3. Theorem 3.3.4 provides the properties of the Segal-Sugawara construction at the level of representations. Next, we describe in Section 4 the spaces on which representations of these Lie algebras localize. These are the moduli stacks of curves and bundles on curves. In Section 5 we describe the implications of the Segal-Sugawara constructions for the sheaves of differential operators on these moduli spaces. Finally, we consider the classical limits of the localization and their applications to the isomonodromy questions in Section 6.
2. Localization.

In this section we review the localization construction for representations of Harish–Chandra pairs as modules over algebras of twisted differential operators or sheaves on twisted cotangent bundles. In particular we emphasize the localization of vacuum representations, which give the sheaves of twisted differential operators or twisted symbols (functions on twisted cotangent bundles) themselves.

2.1. Deformations and limits. We first review the standard pattern of deformations and limits of algebras (see [BB]).

First recall that by the Rees construction, a filtered vector space \( B = \bigcup_i B_i \) has a canonical deformation to its associated graded \( \text{gr} B \) over the affine line \( \mathbb{A}^1 = \text{Spec} \mathbb{C}[\lambda] \), with the fiber at the point \( \lambda \) isomorphic to \( B \) for \( \lambda \neq 0 \), and to \( \text{gr} B \) for \( \lambda = 0 \). The corresponding \( \mathbb{C}[\lambda] \)-module is just the direct sum \( \bigoplus_i B_i \), on which \( \lambda \) acts by mapping each \( B_i \) to \( B_i + 1 \).

Let \( g \) be any Lie algebra. The universal enveloping algebra \( U_g \) has the canonical PBW filtration. The Rees construction then gives us a one-parameter family of algebras \( U_{\lambda} g \), with the associated graded algebra \( U_0 g = \text{Sym} g \) being the symmetric algebra. The \( \lambda \)-deformation rescales the bracket by \( \lambda \), so that taking the \( \lambda \)-linear terms defines the standard Poisson bracket on the \( \text{Sym} g = \mathbb{C}[g^*] \).

Let

\[
0 \rightarrow \mathbb{C} \rightarrow \widehat{g} \rightarrow g \rightarrow 0
\]

be a one–dimensional central extension of \( g \). We define a two–parameter family of algebras

\[
U_{k,\lambda} \widehat{g} = U_{\lambda} \widehat{g} / (1 - k \cdot 1),
\]

which specializes to \( U_{\lambda} g \) when \( k = 0 \). When \( \lambda = 1 \) we obtain the family \( U_k \widehat{g} \) of level \( k \) enveloping algebras of \( \widehat{g} \), whose representations are the same as representations of \( \widehat{g} \) on which \( 1 \) acts by \( k \cdot \text{Id} \). This family may be extended to \( (k, \lambda) \in \mathbb{P}^1 \times \mathbb{A}^1 \), by defining a \( \mathbb{C}[k, k^{-1}] \)-algebra

\[
\widehat{U} = \oplus \left( \frac{\lambda^i}{k^i} \right) (U_{k,\lambda} \widehat{g})_{\leq i}
\]

and setting

\[
U_{\infty,\lambda} \widehat{g} = \widehat{U} / k^{-1} \widehat{U}.
\]

Equivalently, we consider the \( \mathbb{C}[k^{-1}] \)-lattice \( \Lambda \) inside \( U_{k,\lambda} \) (for \( k \in \mathbb{A}^1 \setminus \{0\} \)) generated by \( 1 \) and \( \bar{\tau} = \frac{k}{\lambda} x \) for \( x \in g \). The algebra \( U_{\infty,\lambda} \widehat{g} \) is then identified with \( \Lambda / k^{-1} \Lambda \). (Note that the algebras \( U_{k,\lambda} \widehat{g} \) may also be obtained from the standard \( \lambda \)-deformation of the filtered algebras \( U_k \widehat{g} = U_{k,1} \widehat{g} \).)

Let us choose a vector space splitting \( \widehat{g} \cong g \oplus \mathbb{C} 1 \), so that the bracket in \( \widehat{g} \) is given by a two–cocycle \( \langle \cdot, \cdot \rangle \) on \( g 

\[
[x, y]_\widehat{g} = [x, y]_g + \langle x, y \rangle \cdot 1
\]
for $x, y \in \mathfrak{g} \subset \hat{\mathfrak{g}}$. If we let $\mathfrak{p}$ denote $\frac{\lambda}{k} x$ for $x \in \mathfrak{g} \subset \hat{\mathfrak{g}}$, the algebra $U_{k, \lambda} \hat{\mathfrak{g}}$ may be described as generated by elements $x, y \in \mathfrak{g}$ and 1, with the relations

$$[x, y] = \frac{\lambda}{k} [x, y]_\mathfrak{g} + \lambda \langle x, y \rangle \cdot 1, \quad [1, \mathfrak{p}] = 0.$$ 

In particular, in the limit $k \to \infty$ we obtain a commutative algebra

$$U_{\infty, \lambda} \hat{\mathfrak{g}} \cong \text{Sym}_{\lambda} \hat{\mathfrak{g}} = \text{Sym} \hat{\mathfrak{g}}/(1 - \lambda)$$

with a Poisson structure.

The algebra $\text{Sym}_{\lambda} \hat{\mathfrak{g}}$ is nothing but the algebra of functions $\mathbb{C}[\hat{\mathfrak{g}}^*]$ on the hyperplane in $\hat{\mathfrak{g}}^*$ consisting of functionals, whose value on 1 is $\lambda$. The functions on $\hat{\mathfrak{g}}^*$ form a Poisson algebra, with the Kirillov–Kostant Poisson bracket determined via the Leibniz rule by the Lie bracket on the linear functionals, which are elements of $\hat{\mathfrak{g}}$ itself. Since 1 is a central element, $\text{Sym}_{\lambda} \hat{\mathfrak{g}}$ inherits a Poisson bracket as well.

Thus, we have defined a two–parameter family of “twisted enveloping algebras” $U_{k, \lambda} \hat{\mathfrak{g}}$ for $(k, \lambda) \in \mathbb{P}^1 \times \mathbb{A}^1$ specializing to $U_{k, \lambda} \hat{\mathfrak{g}}$ for $\lambda = 1$ and to the Poisson algebras $\text{Sym}_{\lambda} \hat{\mathfrak{g}}$ for $k = \infty$. The algebras $U_{k, 0} \hat{\mathfrak{g}}$ with $\lambda = 0$ are all isomorphic (as Poisson algebras) to the symmetric algebra $\text{Sym} \mathfrak{g}$.

2.1.1. Harish-Chandra pairs. We now suppose that the Lie algebra $\mathfrak{g}$ is part of a Harish-Chandra pair $(\mathfrak{g}, K)$. Thus $K$ is an algebraic group, and we are given an embedding $\mathfrak{k} = \text{Lie} K \subset \mathfrak{g}$ and an action of $K$ on $\mathfrak{g}$ compatible with the adjoint action of $K$ on $\mathfrak{k}$ and the action of $\mathfrak{k}$ on $\mathfrak{g}$. Suppose further that we have a central extension $\hat{\mathfrak{g}}$, split over $\mathfrak{k}$. Then the action of $K$ lifts to $\hat{\mathfrak{g}}$ and $(\hat{\mathfrak{g}}, K)$ is also a Harish-Chandra pair.

It follows that the subalgebra of $U_{k, \lambda} \hat{\mathfrak{g}}$ generated by the elements $\mathfrak{p}$ for $x \in \mathfrak{k}$ is isomorphic to $U_{\lambda/k} \mathfrak{k}$, degenerating to $\text{Sym} \mathfrak{k}$ for $\lambda = 0$ or $\lambda = \infty$. Moreover, the action of $\mathfrak{k}$ on $U_{k, \lambda} \hat{\mathfrak{g}}$ given by bracket with $x = \frac{\lambda}{k} \mathfrak{p}$ preserves the lattice $\Lambda$ and thus is well-defined in the limit $k = \infty$. (In this limit the action is generated by Poisson bracket with $\mathfrak{k} \subset \text{Sym} \mathfrak{k} \subset \text{Sym}_{\lambda} \hat{\mathfrak{g}}$. ) In particular $\mathfrak{k}$ acts on $U_{k, \lambda} \hat{\mathfrak{g}}$ for all $(k, \lambda) \in \mathbb{P}^1 \times \mathbb{A}^1$, and so it makes sense to speak of $(U_{k, \lambda} \hat{\mathfrak{g}}, K)$–modules.

2.2. Twisted differential operators and twisted cotangent bundles. We review briefly the notions of sheaves of twisted differential operators and of twisted symbols (functions on twisted cotangent bundles) following [BB].

Let $M$ be a smooth variety equipped with a line bundle $L$. The Atiyah sequence of $L$ is an extension of Lie algebras

$$0 \to \mathcal{O}_M \to \mathcal{J}_L \xrightarrow{a} \Theta_M \to 0,$$

where $\mathcal{J}_L$ is the Lie algebroid of infinitesimal automorphisms of $L$, $\Theta_M$ is the tangent sheaf, and the anchor map $a$ describes the action of $\mathcal{J}_L$ on its subsheaf $\mathcal{O}_M$.

We can generalize the construction of the family of algebras $U_{k, \lambda} \hat{\mathfrak{g}}$ from Section 2.1 to the Lie algebroid $\mathcal{J}_L$. We first introduce the sheaf of unital associative algebras $\mathcal{D}_k$ generated by the Lie algebra $\mathcal{J}_L$, with $1 \in \mathcal{O}_M \subset \mathcal{J}_L$ identified with $k$ times the unit. Due to this identification, $\mathcal{D}_k$ is naturally a filtered algebra. Thus we have the Rees $\lambda$–deformation $\mathcal{D}_{k, \lambda}$ from $\mathcal{D}_k$ to its associated graded algebra at $\lambda = 0$. Specializing to $\lambda = 0$ we obtain the symmetric algebra $\text{Sym} \Theta_M$ with its usual Poisson structure,
independently of \( k \). We may also define a limit of the algebras \( \mathcal{D}_{k,\lambda} \) as \( k \) goes to infinity. In order to do so we introduce a \( \mathbb{C}[k, k^{-1}] \)-algebra \( \mathcal{D}_{\lambda,k} \) as
\[
\mathcal{D}_{\lambda,k} = \bigoplus_{i=0}^{\infty} \left( \frac{\lambda}{k} \right)^i \cdot \mathcal{D}_{\leq i}(\mathcal{L}^\otimes k),
\]
and define the limit algebra
\[
\mathcal{D}_{\infty,\lambda} = \mathcal{D}_{\lambda,k}/k^{-1} \cdot \mathcal{D}_{\lambda,k}.
\]
This algebra is commutative, and hence inherits a Poisson structure from the deformation process.

2.2.1. Twisted differential operators. Consider the sheaf \( \mathcal{D}(\mathcal{L}) \) of differential operators acting on sections of \( \mathcal{L} \). This is a sheaf of filtered associative algebras, and the associated graded sheaf is identified with the sheaf of symbols (functions on the cotangent bundle) as Poisson algebras. Similarly, the sheaf of differential operators \( \mathcal{D}(\mathcal{L}^\otimes k) \) for any integer \( k \) is also a sheaf of filtered associative unital \( \mathcal{O} \)-algebras, whose associated graded is commutative and isomorphic (as a Poisson algebra) to the sheaf of functions on \( T^*M \). The sheaf \( \mathcal{D}_k \) defined above for an arbitrary \( k \in \mathbb{C} \) shares these properties, which describe it as a sheaf of twisted differential operators. For \( k \in \mathbb{Z} \) there is a canonical isomorphism of sheaves of twisted differential operators \( \mathcal{D}_k \cong \mathcal{D}(\mathcal{L}^\otimes k) \). In particular, the subsheaf \( \mathcal{D}(\mathcal{L}^\otimes k)_{\leq 1} \) of differential operators of order at most one is identified with the subsheaf \( (\mathcal{D}_k)_{\leq 1} \), which is nothing but \( \mathcal{T}_\mathcal{L} \) with the extension rescaled by \( k \). The sheaf \( \mathcal{D}_k \) for general \( k \) may also be described as a subquotient of the sheaf of differential operators on the total space of \( \mathcal{L}^\times \), the complement of the zero section in \( \mathcal{L} \) (namely, as a quantum reduction by the natural action of \( \mathbb{C}^\times \)).

2.2.2. Twisted cotangent bundles. We may also associate to \( \mathcal{L} \) a twisted cotangent bundle of \( M \). This is the affine bundle \( \text{Conn} \mathcal{L} \) of connections on \( \mathcal{L} \), which is a torsor for the cotangent bundle \( T^*M \). For \( \lambda \in \mathbb{C} \) we have a family of twisted cotangent bundles \( T^*_\lambda M \), which for \( \lambda \in \mathbb{Z} \) are identified with the \( T^*M \)-torsors \( \text{Conn} \mathcal{L}^\otimes \lambda \) of connections on \( \mathcal{L}^\otimes \lambda \). Let \( \mathcal{L}^\times \) denote the principal \( \mathbb{C}^\times \)-bundle associated to the line bundle \( \mathcal{L} \). Then the space \( T^*_\lambda M \) is identified with the Hamiltonian reduction (at \( \lambda \in \mathbb{C} = (\text{Lie} \mathbb{C}^\times)^* \)) of the cotangent bundle \( T^*\mathcal{L}^\times \) by the action of \( \mathbb{C}^\times \). In particular, \( T^*_\lambda M \) carries a canonical symplectic structure for any \( \lambda \).

Thus the twisted symbols, i.e., functions on \( T^*_\lambda M \) (pushed forward to \( M \)) form a sheaf of \( \mathcal{O}_M \)-Poisson algebras \( \mathcal{O}(T^*_\lambda M) \). Its subsheaf of affine functions \( \mathcal{O}(T^*_\lambda M)_{\leq 1} \) on the affine bundle \( T^*_\lambda M \) forms a Lie algebra under the Poisson bracket. It is easy to check that there is an isomorphism of sheaves of Poisson algebras \( \mathcal{D}_{\infty,\lambda} \cong \mathcal{O}(T^*_\lambda M) \), identifying \( \mathcal{O}(T^*_\lambda M)_{\leq 1} \) with \( \mathcal{T}_\mathcal{L} \) (with the bracket rescaled by \( \lambda \)).

Thus to a line bundle we have associated a two–parameter family of algebras \( \mathcal{D}_{k,\lambda} \) for \( (k, \lambda) \in \mathbb{P}^1 \times \mathbb{A}^1 \). This specializes for \( k \in \mathbb{Z}, \lambda = 1 \) to the differential operators \( \mathcal{D}(\mathcal{L}^\otimes k) \), for \( \lambda = 0 \) and \( k \) arbitrary to symbols \( \mathcal{O}(T^*M) \), and for \( k = \infty, \lambda \in \mathbb{Z} \) to the Poisson algebra of twisted symbols, \( \mathcal{O}(\text{Conn} \mathcal{L}^\otimes \lambda) \).
2.3. Localization. In this section we combine the algebraic picture of Section 2.1 with the geometric picture of Section 2.2 following [BB, BD1] (see also [FB], Ch. 16).

Let \( g, \hat{g}, K \) be as in Section 2.1. We will consider the following geometric situation: \( \hat{M} \) is a smooth scheme equipped with an action of \((g, K)\), in other words, an action of the Lie algebra \( g \) on \( \hat{M} \) which integrates to an algebraic action of the algebraic group \( K \).

Let \( M = \hat{M}/K \) be the quotient, which is a smooth algebraic stack, so that \( \pi : \hat{M} \to M \) is a \( K \)-torsor. Thus if the \( g \) action on \( \hat{M} \) is infinitesimally transitive, then given \( \hat{x} \in \hat{M} \) and \( x \in M \) its \( K \)-orbit, the formal completion of \( M \) at \( x \) is identified with the double quotient of the formal group \( \exp(g) \) of \( g \): by the formal group \( \exp(\mathfrak{t}) \) of \( \mathfrak{t} \) on one side, and the formal stabilizer of \( \hat{x} \) on the other.

Let \( \mathcal{L} \) denote a \( K \)-equivariant line bundle on \( \hat{M} \) and the corresponding line bundle on \( M \). We assume that the action of \( g \) on \( \hat{M} \) lifts to an action of \( \hat{g} \) on \( \mathcal{L} \), so that \( 1 \) acts as the identity. For \( k, \lambda \in \mathbb{P}^1 \times \mathbb{A}^1 \), we have an algebra \( U_{k,\lambda} \hat{g} \) from Section 2.1 and a sheaf of algebras \( D_{k,\lambda} \) on \( \hat{M} \) from Section 2.2. The action of \( \hat{g} \) on \( \mathcal{L} \) matches up these definitions: we have an algebra homomorphism

\[
O_{\hat{M}} \otimes U_{k,\lambda} \hat{g} \to D_{k,\lambda}, \quad (k, \lambda) \in \mathbb{P}^1 \times \mathbb{A}^1.
\]

This may also be explained by reduction. Namely, \( \hat{g} \) acts on the total space \( \mathcal{L}^\times \) of the \( \mathbb{C}^\times \)-bundle associated to \( \mathcal{L} \). Therefore \( U_{\hat{g}} \) maps to differential operators on \( \mathcal{L}^\times \). It also follows that \( \hat{g} \) acts in the Hamiltonian fashion on \( T^*\mathcal{L}^\times \), so that \( \text{Sym} \hat{g} \) maps in the Poisson fashion to symbols on \( \mathcal{L}^\times \). The actions of \( U_{\hat{g}} \to D_k \) and \( \text{Sym}_{\hat{g}} \to \mathcal{O}(T^*_k \hat{M}) \) are then obtained from the quantum (respectively, classical) Hamiltonian reduction with respect to the \( \mathbb{C}^\times \) action on \( \mathcal{L}^\times \) and the moment values \( k \) (respectively, \( \lambda \)).

2.3.1. Localization functor. The action of the Harish-Chandra pair \((\hat{g}, K)\) on \( \hat{M} \) and \( \mathcal{L} \) may be used to construct localization functors from \((\hat{g}, K)\)-modules to sheaves on \( M \). Let \( M \) be a \( U_{k,\lambda} \hat{g} \)-module with a compatible action of \( K \). We then consider the sheaf of \( D_{k,\lambda} \)-modules

\[
\tilde{\Delta}(M) = D_{k,\lambda} \otimes_{U_{k,\lambda} \hat{g}} M
\]

on \( \hat{M} \), whose fibers are the coinvariants of \( M \) under the stabilizers of the \( \hat{g} \)-action on \( \mathcal{L}^\times \). (By our hypotheses, these lift to \( \hat{g} \) the stabilizers of the \( g \)-action on \( \hat{M} \).) The sheaf \( \tilde{\Delta}(M) \) is a \( K \)-equivariant \( D_{k,\lambda} \)-module, and so it descends to a \( D_{k,\lambda} \)-module \( \Delta_{k,\lambda}(M) \) on \( M \), the localization of \( M \):

\[
\Delta_{k,\lambda}(M) = (\pi_*(D_{k,\lambda} \otimes_{U_{k,\lambda} \hat{g}} M))^K.
\]

Equivalently, we may work directly on \( \hat{M} \) by twisting. Let

\[
\hat{M} = \hat{M} \times_K M = \pi_* (O_{\hat{M}} \otimes M)^K
\]

denote the twist of \( M \) by the \( K \)-torsor \( \hat{M} \) over \( M \) (i.e., the vector bundle on \( \hat{M} \) associated to the principal bundle \( \hat{M} \) and representation \( M \)). Then there is a surjection

\[
(2.3.1) \quad \hat{M} \twoheadrightarrow \Delta_{k,\lambda}(M),
\]
and $\Delta_{k,\lambda}(M)$ is the quotient of $M$ by the twist $(\mathfrak{g}_{\text{stab}})_{\mathfrak{g}_{\mathfrak{m}}} / (\mathfrak{g})_{\mathfrak{m}}$ of the sheaf of stabilizers.

2.3.2. Remark. The localization functor $\Delta_{k,\lambda}$ “interpolates” between the localization $\Delta_k : (U_k \hat{\mathfrak{g}}, \mathfrak{K}) \to \mathcal{D}_k$ as modules for twisted differential operators, and the classical localization $\Sigma_\lambda : \left( \text{Sym}_\lambda \hat{\mathfrak{g}}, \mathfrak{K} \right) \to \mathcal{O}(T^*_\lambda \mathfrak{M})$ as quasicoherent sheaves on $T^*_\lambda \mathfrak{M}$. The latter assigns to a module over the commutative ring $\text{Sym}_\lambda \hat{\mathfrak{g}}$ a quasicoherent sheaf on $T^*_\lambda \mathfrak{M}$ via the embedding of $\text{Sym}_\lambda \hat{\mathfrak{g}}$ into global twisted symbols. This $K$–equivariant sheaf descends to $M$, where it becomes a module over the sheaf of twisted symbols $\mathcal{O}(T^*_\lambda \mathfrak{M})$ and therefore we obtain a sheaf on the twisted cotangent bundle.

2.4. Localizing the vacuum module. The fundamental example of a $(\mathfrak{g}, \mathfrak{K})$–module is the vacuum module $V_{\mathfrak{g},\mathfrak{K}} = \text{Ind}_{U_k \hat{\mathfrak{g}}}^U \mathbb{C} = U \mathfrak{g}/(U \mathfrak{g} \cdot \mathfrak{k})$ (we will denote it simply by $V$ when the relevant Harish-Chandra pair $(\mathfrak{g}, \mathfrak{K})$ is clear). The vector in $V_{\mathfrak{g},\mathfrak{K}}$ corresponding to $1 \in \mathbb{C}$ is denoted by $\left| 0 \right\rangle$ and referred to as the vacuum vector. It is a cyclic vector for the action of $\mathfrak{g}$ on $V_{\mathfrak{g},\mathfrak{K}}$. The vacuum representation has the following universality property: given a $k$–invariant vector $m \in M$ in a representation of $\mathfrak{g}$ on $V_{\mathfrak{g},\mathfrak{K}}$. There exists a unique $\mathfrak{g}$–homomorphism $m^* : V_{\mathfrak{g},\mathfrak{K}} \to M$ with $m^*(\left| 0 \right\rangle) = m$.

An important feature of the localization construction for Harish-Chandra modules is that the sheaf of twisted differential operators $\mathcal{D}_k$ itself arises as the localization $\Delta(V_{\mathfrak{g},\mathfrak{K}})$ of the vacuum module. This is a generalization of the description of differential operators on a homogeneous space $G/K$ as sections of the twist $\nabla_{\mathfrak{g},\mathfrak{K}} = \mathcal{P}_K \times V_{\mathfrak{g},\mathfrak{K}}$ by the $K$–torsor $\mathcal{P}_K = G$ over $G/K$. Informally, the general statement follows by applying reduction from $G/K$ to $\mathfrak{M}$, which is (formally) a quotient of the formal homogeneous space $\exp(\mathfrak{g})/\exp(\mathfrak{k})$.

Recall that $\pi : \mathfrak{M} \to \mathfrak{M}$ denotes a $K$–torsor with compatible $\mathfrak{g}$–action. We assume $K$ is connected and affine. In particular, the functor $\pi_*$ is exact.

2.4.1. Proposition.

1. The twist $\nabla = \mathfrak{M} \times V$, its subsheaf of invariants $\nabla^K = \mathcal{O}_{\mathfrak{M}} \otimes V^K$, and the localization $\Delta(\nabla)$ are sheaves of algebras on $\mathfrak{M}$.

2. The localization $\Delta(\nabla)$ is naturally identified with the sheaf $\mathcal{D}$ of differential operators on $\mathfrak{M}$.

3. The natural maps

$$\mathcal{O}_{\mathfrak{M}} \otimes V^K \to \nabla \to \Delta(\nabla) = \mathcal{D}_{\mathfrak{M}}$$

are algebra homomorphisms.
(4) For any \((g, K)\)-module \(M\), the sheaves
\[
\mathcal{O}_{\mathfrak{m}} \otimes M^K \rightarrow M \rightarrow \Delta(M)
\]
are modules over the corresponding algebras in (3).

2.4.2. Proof. The vector space \(V^K = [Ug/Ug \cdot \mathfrak{k}]^\mathfrak{k}\) is naturally an algebra, isomorphic to the quotient \(N(I)/I\) of the normalizer \(N(I)\) of the ideal \(I = Ug \cdot \mathfrak{k}\) by \(I\). It is also isomorphic to the algebra \((\text{End}_g(V))^\text{opp}\) (with the opposite multiplication), thanks to the universality property of \(V\), and thus to the algebra of endomorphisms of the functor of \(K\)-invariants (with the opposite multiplication). Hence \(\mathcal{O}_{\mathfrak{m}} \otimes V^K\) is an \(\mathcal{O}_{\mathfrak{m}}\)-algebra.

The sheaf \(\mathcal{O}_{\mathfrak{m}} \otimes Ug\) is a sheaf of algebras, where \(Ug\) acts on \(\mathcal{O}_{\mathfrak{m}}\) as differential operators, and so is \(\mathcal{A} = \pi^* (\mathcal{O}_{\mathfrak{m}} \otimes Ug)\). Let \(\mathcal{K}\) be the sheaf of \(\mathcal{O}_{\mathfrak{m}}\)-Lie algebras \(\pi^*(\mathcal{O}_{\mathfrak{m}} \otimes \mathfrak{k})\) (note that the action of \(\mathfrak{k}\) on \(\mathcal{O}_{\mathfrak{m}}\) is \(\pi^{-1}(\mathcal{O}_{\mathfrak{m}})\)-linear). Then we have
\[
\mathcal{V} = [\pi^*(\mathcal{O}_{\mathfrak{m}} \otimes V)]^\mathcal{K} = [\pi^*(\mathcal{O}_{\mathfrak{m}} \otimes V)]^\mathcal{K} = [\mathcal{A}/\mathcal{A} \cdot \mathcal{K}]^\mathcal{K},
\]
which is thus a sheaf of algebras (again as the normalizer of an ideal modulo that ideal).

On the other hand, the localization \(\Delta(V) = [\pi^*(D_{\mathfrak{m}} \otimes UgV)]^\mathfrak{k}\) since \(\mathcal{K}\) is the sheaf of vertical vector fields. This is in fact the common description of differential operators on a quotient as the quantum hamiltonian (BRST) reduction of differential operators upstairs. The obvious maps are compatible with this definition of the algebra structures. This proves parts (1)–(3) of the proposition.

Finally, for part (4), observe that the \(Ug\)-action on \(M\) descends to a \(\mathcal{V}\)-action on \(M\), since the ideal \(\mathcal{K}\) acts trivially on the \(\mathcal{K}\)-invariants \(M = \pi^*(\mathcal{O}_{\mathfrak{m}} \otimes M)^\mathcal{K}\). Equivalently, the Lie algebroid \(\widehat{\mathfrak{m}} \times \mathfrak{g}\) on \(\mathfrak{m}\) acts on the twist \(M\), and the action descends to the quotient algebroid \(\widehat{\mathfrak{m}} \times \mathfrak{g}/\mathfrak{t}\), which generates \(\mathcal{V}\). The compatibility with the actions of invariants on \(M^K\) and of the localization \(\mathcal{D}\) on the \(\mathcal{D}\)-module \(\Delta(W)\) follow.

2.4.3. Corollary. The space of invariants \(V^K\) is naturally a subalgebra of the algebra \(\Gamma(\mathfrak{m}, \mathcal{D})\) of global differential operators on \(\mathfrak{m}\).

2.4.4. Changing the vacuum. Let \(\overline{K} \subset K\) be a normal subgroup. Then a minor variation of the proof establishes that the twist of \(\overline{K}\)-invariants \(\mathcal{V}^{\overline{K}}\) is a subalgebra. Note moreover that \(\mathcal{V}^{\overline{K}}\) depends only on the induced \(K/\overline{K}\)-torsor \(\widehat{\mathfrak{m}} = (K/\overline{K})_{\mathfrak{m}}\):
\[
\mathcal{V}^{\overline{K}} = \widehat{\mathfrak{m}} \times_{K/\overline{K}} \mathcal{V}^{\overline{K}}.
\]

Now suppose \((\mathfrak{a}, K)\) is a sub–Harish-Chandra pair of \((g, K)\) (i.e. \(\mathfrak{t} \subset \mathfrak{a} \subset \mathfrak{g}\)). Then we have the two vacuum modules \(V = V_{\mathfrak{g}, K} \supset V_{\mathfrak{a}, K}\) and the module \(V_{\mathfrak{g}, \mathfrak{a}} = \text{Ind}_{\mathfrak{g}}^{\mathfrak{a}} \mathbb{C}\),
which is not literally the vacuum module of a Harish-Chandra pair unless \( a \) integrates to an algebraic group.

By Proposition 2.4.1, the twist \( V_{a,K} \) and the localization \( \Delta(V_{a,K}) \) — with respect to the Harish-Chandra pair \( (a, K) \) — are sheaves of algebras, and act on \( V \) and \( \Delta(V) \) respectively. We then have

2.4.5. **Lemma.** The quotients \( V/(V_{a,K} \cdot V) \) and \( \Delta(V)/(\Delta(V_{a,K}) \cdot \Delta(V)) \) are isomorphic, respectively, to the twist \( V_{g,a} \) and the localization \( \Delta(V_{g,a}) \) with respect to the Harish-Chandra pair \( (g, K) \).

2.4.6. **Twisted and deformed vacua.** In general, we define the twisted, classical and deformed vacuum representations \( V_k(\hat{\mathfrak{g}}) \), \( \nabla_\lambda(\hat{\mathfrak{g}}) \) and \( V_{k,\lambda}(\hat{\mathfrak{g}}, \mathfrak{V}) \), respectively, as the induced representations

\[
V_k = U_k \hat{\mathfrak{g}} \otimes \mathbb{C}
\]

\[
\nabla_\lambda = \text{Sym}_\lambda \hat{\mathfrak{g}} \otimes \mathbb{C}
\]

\[
V_{k,\lambda} = U_{k,\lambda} \hat{\mathfrak{g}} \otimes \mathbb{C}
\]

where we use the observation (Section 2.4.1) that \( U_k \mathfrak{f} \) is isomorphic to \( U \mathfrak{f} \) for \( k \neq \infty \) and to \( \text{Sym} \mathfrak{f} \) for \( k = \infty \). All of these representations carry compatible \( K \)-actions.

We then have a general localization principle, describing twisted differential operators and twisted symbols on \( M \) in terms of the corresponding vacuum representations. The proof of Proposition 2.4.1 generalizes immediately to the family \( V_{k,\lambda} \) over \( \mathbb{P}^1 \times \mathbb{A}^1 \). This is deduced formally from the analogous statement for homogeneous spaces \( G/K \), twisted by an equivariant line bundle. In that case the algebra of functions on the cotangent space \( T^*G/K |_K = (g/\mathfrak{f})^* \) is identified with

\[
\nabla_0 = \text{Sym}(g/\mathfrak{f}) = \text{Sym} g \otimes \mathbb{C},
\]

and sections of the twist of \( \nabla_0 \) give the sheaf of functions on \( T^*G/K \). Note that \( \nabla_\lambda \) is a (commutative) algebra; however, it only becomes Poisson after twisting or localization. To summarize, we obtain the following

2.4.7. **Proposition.** The localization \( \Delta_{k,\lambda}(V_{k,\lambda}) \) is canonically isomorphic to the sheaf of algebras \( \mathcal{D}_{k,\lambda} \), for \( (k, \lambda) \in \mathbb{P}^1 \times \mathbb{A}^1 \). In particular \( \Delta_k(V_k) \cong \mathcal{D}_k \), and \( \overline{\Delta}_\lambda(\nabla_\lambda) \cong \mathcal{O}(T^*_\lambda \mathfrak{M}) \) as Poisson algebras.

3. **Virasoro–Kac-Moody Algebras.**

In this section we introduce the Lie algebras to which we wish to apply the formalism of localization outlined in the previous sections. These are the affine Kac-Moody algebras, the Virasoro algebra and their semi-direct product. We describe the Segal-Sugawara construction which expresses the action of the Virasoro algebra on an affine algebra as an “internal” action. We interpret this construction in terms of a homomorphism between vacuum representations of the Virasoro and Kac-Moody algebras,
and identify the critical and classical limits of these homomorphisms. In the subsequent sections we will use the localization of this construction to describe sheaves of differential operators on the moduli spaces of curves and bundles on curves.

3.1. Virasoro and Kac-Moody algebras. Let \( g \) be a simple Lie algebra. It carries an invariant bilinear form \((\cdot, \cdot)\) normalized in the standard way so that the square length of the maximal root is equal to 2. We choose bases \( \{ J^a \} \), \( \{ J_a \} \) dual with respect to this bilinear form. The affine Kac-Moody Lie algebra \( \hat{g} \) is a central extension
\[
0 \to \mathbb{C}K \to \hat{g} \to Lg \to 0
\]
of the loop algebra \( Lg = g \otimes \mathbb{K} \) of \( g \), with (topological) generators \( \{ J^a_n = J^a \otimes t^n \}_{n \in \mathbb{Z}} \) and relations
\[
[J^a_n, J^b_m] = [J^a, J^b]_{n+m} + (J^a, J^b)\delta_{n,-m}K.
\]
The central extension splits over the Lie subalgebra \( g(\mathcal{O}) \subset g(\mathbb{K}) \) (topologically spanned by \( J^a_n, n \geq 0 \)), so that the affine proalgebraic group \( G(\mathcal{O}) \) acts on \( \hat{g} \). Thus we have a Harish-Chandra pair \((\hat{g}, G(\mathcal{O}))\). We denote by
\[
V_k = V_k(\hat{g}) = U_k\hat{g} \otimes \mathbb{C}_{Ug(\mathcal{O})}
\]
the corresponding vacuum module (see Section 2.4). More generally, let \( m \subset \mathcal{O} \) denote the maximal ideal \( t\mathbb{C}[[t]] \). Let \( g_n(\mathcal{O}) = g \otimes m^n \subset g(\mathcal{O}) \) denote the congruence subalgebra of level \( n \), and \( G_n(\mathcal{O}) \subset G(\mathcal{O}) \) the corresponding algebraic group, consisting of loops which equal the identity to order \( n \). We denote by
\[
V^c_k = V^c_k(\hat{g}) = U_k\hat{g} \otimes \mathbb{C}_{Ug_n(\mathcal{O})}
\]
the vacuum module for \((\hat{g}, G_n(\mathcal{O}))\).

Let \( \text{Vir} \) denote the Virasoro Lie algebra. This is a central extension
\[
0 \to \mathbb{C} \to \text{Vir} \to \text{Der} \mathbb{K} \to 0
\]
of the Lie algebra of derivations of the field \( \mathbb{K} \) of Laurent series. It has (topological) generators \( L_n = -t^{n+1}\partial_t, n \in \mathbb{Z} \), and the central element \( C \), with relations
\[
[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}(n^3 - n)\delta_{n,-m}K.
\]
The central extension splits over the Lie subalgebra \( \text{Der} \mathcal{O} \subset \text{Der} \mathbb{K} \), topologically spanned by \( L_n, n \geq -1 \). Consider the module
\[
\text{Vir}^c = U_c\text{Vir} \otimes \mathbb{C}_{U\text{Der} \mathcal{O}}.
\]
Since \( \text{Der} \mathcal{O} \) is not the Lie algebra of an affine group scheme, \( \text{Vir}^c \) is not strictly speaking a vacuum module of a Harish–Chandra pair.

The affine group scheme \( \text{Aut} \mathcal{O} \) of all changes of coordinates on the disc fixes the closed point \( t = 0 \), so that \( L_{-1} = -\partial_t \) is not in its Lie algebra, which we denote by \( \text{Der}_1 \mathcal{O} \). More generally, we set
\[
\text{Der}_n(\mathcal{O}) = m^n\text{Der} \mathcal{O} \cong t^n\mathbb{C}[[t]]\partial_t
\]
and let \( \text{Aut}_n(0) \subset \text{Aut}_0(0) = \text{Aut} \emptyset \) be the corresponding algebraic subgroups (for \( n \geq 0 \)). Then we have Harish-Chandra pairs \((\text{Vir}, \text{Aut}_n(0))\) and the corresponding vacuum modules \( \text{Vir}_c^n \) (thus we may denote \( \text{Vir}_c = \text{Vir}_c^0 \)).

The Virasoro algebra acts on \( \hat{g} \), via its action as derivations of \( \mathcal{K} \). In terms of our chosen bases this action is written as follows:

\[
[L_n, J_m^a] = -mJ_{n+m}^a.
\]

Let \( \hat{g} \) be the resulting semidirect product \( \hat{\mathfrak{g}} = \text{Der} \emptyset \ltimes \mathfrak{g}(\emptyset) \). We let

\[
\text{Ind}_{\hat{\mathfrak{g}}_+ \oplus \mathbb{C}1}^{\mathfrak{g}_+ \oplus \mathbb{C}1} \mathcal{C}_{k,c} = U_{\hat{\mathfrak{g}}} \mathcal{C}_{k,c} \mathbb{C},
\]

where \( \mathcal{C}_{k,c} \) is the one-dimensional representation of \( \mathfrak{g}_+ \oplus \mathbb{C}K \oplus \mathbb{C}C \subset \hat{\mathfrak{g}} \) on which \( \mathfrak{g}_+ \) acts by zero and \( K, C \) act by \( k, c \). Let

\[
\hat{g}_n(\emptyset) = \text{Der}_{2n}(0) \ltimes g_n(\emptyset) = m^{2n} \text{Der} \emptyset \ltimes m^n g(\emptyset).
\]

3.1.1. Remark. This normalization above, by which we pair \( g_n(\emptyset) \) with \( \text{Der}_{2n}(0) \), is motivated by the Segal-Sugawara construction, cf. Proposition 3.3.1: the Virasoro and Kac-Moody central extensions, giving rise to two families of algebras which we denote \( g_{k,c} \) and \( \mathfrak{g}_+ \mathbb{C} \). Thus in geometric applications of the Segal–Sugawara operators the orders of trivializations or poles along the Virasoro (moduli of curves) directions will turn out to be double those along the Kac–Moody (moduli of bundles) directions (for example the quadratic Hitchin hamiltonians double the order of pole of a Higgs field).

3.1.2. The corresponding representation

\[
V^n_{k,c} = U_{k,c} \hat{\mathfrak{g}} \mathbb{C}
\]

agrees with \( V_{k,c} \) for \( n = 0 \), while for \( n > 0 \) it is identified as the vacuum module for the Harish-Chandra pair \( (\hat{\mathfrak{g}}, \hat{G}_n(\emptyset)) \) where \( \hat{G}_n(\emptyset) = \text{Aut}_{2n} \emptyset \ltimes G_n(\emptyset) \). We will also denote by \( \hat{G}(\emptyset) \) the semidirect product \( \text{Aut} \emptyset \ltimes G(\emptyset) \).

The pattern of deformations and limits from Section 2.1 applies to the Kac-Moody and Virasoro central extensions, giving rise to two families of algebras which we denote \( U_{k,\lambda} \hat{\mathfrak{g}} \) and \( U_{c,\mu} \text{Vir} \). Thus for \( \lambda = \mu = 1 \) and \( k = c = 0 \), \( U_{k, \lambda} \hat{\mathfrak{g}} = U \mathfrak{g}(\mathcal{K}) \) and \( U_{c, \mu} \text{Vir} = U \text{Der} \mathcal{K} \), etc. For \( k = c = \infty \) we obtain the Poisson algebras \( \text{Sym}_\lambda \hat{\mathfrak{g}} \) and \( \text{Sym}_\mu \text{Vir} \).

We also have classical vacuum modules \( \nabla_{\lambda}(\hat{\mathfrak{g}}) \) for \( (\text{Sym} \hat{\mathfrak{g}}, \hat{G}(\emptyset)) \) and \( \nabla_{\mu}(\text{Vir}) \) for \( (\text{Sym} \text{Vir}, \text{Aut} \emptyset) \) and the interpolating families \( V_{k,\lambda}(\hat{\mathfrak{g}}) \) and \( V_{c,\mu}(\text{Vir}) \).

3.2. Vertex algebras. The vacuum representations \( V_k, \text{Vir}_c \) and \( V_{k,c} \) have natural structures of vertex algebras, and \( V^n_k, \text{Vir}_c^n \) and \( V^n_{k,c} \) are modules over the respective vertex algebras (see [102] for the definition of vertex algebras and in particular the Virasoro and Kac-Moody vertex algebras).

The vacuum vector \( \langle 0 \rangle \) plays the role of the unit for these vertex algebras. Moreover, all three vacuum modules carry natural Harish-Chandra actions of \( (\text{Der} \emptyset, \text{Aut} \emptyset) \): these
are defined by the natural action of \(\text{Der } O\) on \(\hat{\mathfrak{g}}, \text{Vir}\) and \(\tilde{\mathfrak{g}}\), preserving the positive halves and hence giving rise to actions on the corresponding representations. These actions give rise to a grading operator \(L_0\) and a translation operator \(T = L_{-1} = -\partial_z \in \text{Der } O\). The vertex algebra \(V_k\) is generated by the fields

\[
J^a(z) = Y(J^a_{-1}|0\rangle, z) = \sum_{n \in \mathbb{Z}} J^a_n z^{-n-1}
\]

associated to the vectors \(J^a_{-1}|0\rangle \in V_k\), which satisfy the operator product expansions

\[
J^a(z)J^b(w) = \frac{k(J^a, J^b)}{(z-w)^2} + \frac{[J^a, J^b](w)}{z-w} + \cdots
\]

(where the ellipses denote regular terms). This may be seen as a shorthand form for the defining relations (3.1.1). This structure extends to the family \(V_{k,\lambda}\). Explicitly, introduce the \(\mathbb{C}[\lambda, k^{-1}]\)–lattice \(\Lambda\) in \(V_{k,\lambda}\) generated by monomials in \(J^a_n = \lambda^k J^a_n\). These satisfy relations

\[
[J^a_n, J^b_m] = \frac{\lambda}{k}([J^a, J^b]_{n+m} + \lambda(J^a, J^b))\delta_{n,-m},
\]

or in shorthand

\[
[J, J] = \frac{\lambda}{k}(J + \lambda(\cdots))
\]

The vertex algebra structure is defined by replacing the \(J\) operators by the rescaled versions \(\mathcal{J}\). In particular we see that we recover the commutative vertex algebra \(\mathcal{V}_\lambda(\mathfrak{g})\) when \(k = \infty\). Recall from [FB], Ch. 15, that a commutative vertex algebra is essentially the same as a unital commutative algebra with derivation and grading. Thus the classical vacuum representations \(\mathcal{V}_\lambda(\mathfrak{g}), \mathcal{V}_{\text{Vir}}\mu\) and \(\mathcal{V}_{\lambda,\mu}(\mathfrak{g})\) are naturally commutative vertex algebras.

Moreover, the description as a degeneration endows \(\mathcal{V}_\lambda(\mathfrak{g})\) with a vertex Poisson algebra structure. The Poisson structure comes with the following relations

\[
\{\mathcal{J}_n, \mathcal{J}_m\} = [J^a, J^b]_{n+m} + \lambda(J^a, J^b))\delta_{n,-m}.
\]

Likewise, the vertex algebra \(\text{Vir}_c\) is generated by the field

\[
T(z) = Y(L_{-2}|0\rangle, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},
\]

satisfying the operator product expansion

\[
T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \cdots
\]

encapsulating the relations (3.1.2). The \(\mu\)–deformation \(\text{Vir}_{c,\mu}\) is straightforward. The commutation relations for the generators \(\mathcal{T}_n = \frac{\mu}{c}L_n\) read

\[
[\mathcal{T}_n, \mathcal{T}_m] = \frac{\mu}{c}(n-m)\mathcal{T}_{n+m} + \frac{\mu^2}{c} \frac{1}{12} (n^3 - n)\delta_{n,-m},
\]

or simply

\[
[\mathcal{T}, \mathcal{T}] = \frac{\mu}{c}(\mathcal{T} + \mu(\cdots)).
\]
In the limit $c = \infty$ we obtain a vertex Poisson algebra structure on $V^\mu_\nu(g)$. (As a commutative algebra with derivation it is freely generated by the single vector $\overline{\mathcal{L}}_{-2|0}$.) The Poisson operators $\overline{\mathcal{L}}$ have relations

$$\{\overline{\mathcal{L}}, \overline{\mathcal{L}}\} = \overline{\mathcal{L}} + \mu(\cdots).$$

The space $V_{k,c}$ also carries a natural vertex algebra structure, so that $\text{Vir}_c \subset V_{k,c}(g)$ is a vertex subalgebra, complementary to $V_{k}(\hat{g}) \subset V_{k,c}(g)$, which is a vertex algebra ideal. In particular, the vertex algebra is generated by the fields $J^a(z)$ and $T(z)$, with the additional relation

$$T(z)J^a(w) = J^a(w)T(z) - \frac{\partial_w J^a(w)}{z - w} + \cdots$$

We may combine the above deformations into a two–parameter deformation of $V_{k,c}$. We introduce the $C[k^{-1}, c^{-1}]$–lattice $\Lambda$ in $V_{k,c}$ generated by monomials in $J^a_n = \lambda^k J^a_n$ and $\mathcal{L}_n = \mu^c \mathcal{L}_n$. These satisfy relations

$$[\mathcal{J}, \mathcal{J}] = \frac{\lambda}{k} (\mathcal{J} + \lambda), \quad [\overline{\mathcal{L}}, \mathcal{L}] = \frac{\mu}{c} (\mathcal{L} + \mu), \quad [\overline{\mathcal{L}}, \mathcal{J}] = \frac{\mu}{c} \mathcal{J}.$$  

If we impose $\lambda = \mu = 0$, we obtain the deformation of the enveloping vertex algebra $V_{k,c}$ to the symmetric vertex Poisson algebra associated to $\tilde{g}$. We will denote this limit vertex Poisson algebra by $V_{0,0}(g)$. We may also specialize $\mu$ to 0, making the $\overline{\mathcal{L}}$ generators central. Hence the Kac-Moody part does not acquire a vertex Poisson structure, and we do not obtain an action of the Kac-Moody vertex Poisson algebra on $V_{k,c}$ in this limit.

3.3. The Segal-Sugawara vector. To any vertex algebra $V$ we associate a Lie algebra $\mathfrak{U}(V)$ topologically spanned by the Fourier coefficients of vertex operators from $V$ (see [FB]). This Lie algebra acts on any $V$–module. In the case of the Kac-Moody vertex algebra $V_k$, the Lie algebra $\mathfrak{U}(V_k)$ belongs to a completion of the enveloping algebra $U_{k\hat{g}}$. An important fact is that it contains inside it a copy of the Virasoro algebra (if
$k \neq -h^\vee$). In vertex algebra terminology, this means that the Kac-Moody vacuum modules are *conformal* vertex algebras.

The conformal structure for $V_{k,c}$ is automatic. Let $\omega_V = L_{-2}|0\rangle \in V_{k,c}(\mathfrak{g})$. This is a conformal vector for the vertex algebra $V_{k,c}(\mathfrak{g})$: the field $T(z)$ it generates satisfies the operator product expansion (3.2.2), and hence its Fourier coefficients $L_n$ (see (3.2.1)) give rise to a Virasoro action. The operator $L_0$ is the grading operator and $L_{-1}$ is the translation operator $T$ on $V_{k,c}$. The action of $\text{Der} \, \mathcal{O}$ induced by the $L_n$ ($n \geq -1$) preserves the Kac-Moody part $V_k \subset V_{k,c}$, but the negative $L_n$’s take us out of this subspace.

For level $k$ not equal to minus the dual Coxeter number $h^\vee$ of $\mathfrak{g}$, the vertex algebra $V_k$ itself carries a Virasoro action, and in fact a conformal structure, given by the Segal-Sugawara vector. This means that we have a conformal vector $\omega_S \in V_k(\hat{\mathfrak{g}}) \subset V_{k,c}(\mathfrak{g})$ such that the corresponding field satisfies the Virasoro operator product expansion (3.2.2). This conformal vector is given by

\begin{equation}
\omega_S = \frac{S_{-2}}{k + h^\vee}, \quad S_{-2} = \frac{1}{2} \sum_a J^a_1 J^a_{-1}.
\end{equation}

The corresponding Virasoro algebra has central charge $\frac{k \dim \mathfrak{g}}{k + h^\vee}$. In other words,

\begin{equation}
Y(\omega_S, z) = \sum_{n \in \mathbb{Z}} L^S_n z^{-n - 2} \quad (L^S_n \in \text{End} \, V_{k,c}(\mathfrak{g}))
\end{equation}

and we have

\begin{equation}
[L^S_n, J^a_m] = -m J^a_{n+m},
\end{equation}

\begin{equation}
[L^S_n, L^S_m] = (n - m) L^S_{n+m} + \frac{1}{12} \frac{k \dim \mathfrak{g}}{k + h^\vee} (n^3 - n) \delta_{n,m}.
\end{equation}

We will continue to take the vector $\omega_V$ as the conformal vector for $V_{k,c}$, and use the notation $L_n$ for the coefficients of the corresponding field $T(z)$. Due to the above commutation relations, $\omega_S$ is *not* a conformal vector for $V_{k,c}$, because $L^S_0$ and $L^S_{-1}$ do not act as the grading and translation operators on the Virasoro generators.

Note that the commutator $[L_n, J^a_m]$ coincides with the right hand side of formula (3.3.2), and therefore $[L_n, L^S_m]$ is given by the right hand side of formula (3.3.3).

The above construction of the Virasoro algebra action on $V_k$ is nontrivial from the Lie algebra point of view. Indeed, the operators $L^S_n$ are given by infinite sums of quadratic expressions in the $J^a_n$, which are nonetheless well-defined as operators on $V_k$:

\begin{equation}
Y(\omega_S, z) = \frac{1}{2(k + h^\vee)} \sum_a : J^a(z) J^a(z) : ,
\end{equation}
so that
\[ L_n^S = \frac{1}{2(k + h)} \sum_a \sum_m : J^a_{m} J_{a,n-m} : = \frac{1}{2(k + h)} \sum_a \left( \sum_{m < 0} J^a_m J_{a,n-m} + \sum_{m \geq 0} J_{a,n-m} J^a_m \right). \]

In fact, since this action is given by a vertex operator in \( V_k \), it follows immediately that any module over the vertex algebra \( V_k \), \( k \neq -h \), carries a compatible action of the Virasoro algebra. This may also be expressed using completions of the enveloping algebras of \( \hat{g} \) and \( \tilde{g} \). These are the completions which act on all smooth representations, i.e., those in which every vector is stabilized by a deep enough congruence subgroup \( G_n(\mathcal{O}) \), \( n \geq 0 \). Namely, we define a completion of \( \hat{U}_{k,c} \tilde{g} \) as the inverse limit
\[ \hat{U}_{k,c} \tilde{g} = \lim_{\leftarrow \infty} U_{k,c} \tilde{g}/(U_{k,c} \tilde{g} \cdot \tilde{g}_{n}(\mathcal{O})). \]

This is a complete topological algebra, since the multiplication on \( U_{k,c} \tilde{g} \) is continuous in the topology defined by declaring the left ideals generated by \( \tilde{g}_{n}(\mathcal{O}) \) to be base of open neighborhoods of 0. We define a completion of \( U_{k,c} \tilde{g} \) in the same way. These completions contain the Lie algebras \( U(V_{k,c}) \) and \( U(V_k) \), and in particular for \( k \neq -h \) they contain the Virasoro algebra generated by the Segal-Sugawara operators \( L_n^S \), \( n \in \mathbb{Z} \). Hence any smooth representation of \( \tilde{g} \) or \( \hat{g} \) of level \( k \neq -h \) inherits a Virasoro action. In particular, the algebra \( \hat{U}_{k,c} \tilde{g} \) acts on the vacuum modules \( V_{k,c}^n \).

3.3.1. Proposition. For any \( k \neq -h \), the Segal-Sugawara operators \( L_m^S \), \( m \in \mathbb{Z} \), define \( (\text{Vir}, \text{Aut}_{2n} \mathcal{O}) \)-action of central charge \( c = \frac{k}{k + h} \) on the vacuum modules \( V_{k,c}^n \) and \( V_{k,c}^n \). Together with the action of \( (\hat{g}, G_n(\mathcal{O})) \), this \( (\text{Vir}, \text{Aut}_{2n} \mathcal{O}) \)-action combines into a \( (\tilde{g}, \tilde{G}_n(\mathcal{O})) \)-action on \( V_{k,c}^n \).

3.3.2. Proof. The action of the \( L_m^S \) given by the Segal-Sugawara operators is well-defined on \( V_k^n \) because \( V_k^n \) is a smooth \( \hat{g} \)-module. Next, we claim \( L_m^S \cdot |0\rangle_n = 0 \) for \( m \geq 2n - 1 \). For (precisely) such \( m \), for each term : \( J^a_{m-i} J_{a,i} \) : at least one of the two factors lies in \( \tilde{g}_{n}(\mathcal{O}) \), either immediately annihilating \( |0\rangle_n \) or first passing through the other factor to annihilate \( |0\rangle_n \), leaving a commutator of degree \( m \) which also annihilates \( |0\rangle_n \).

Observe that \( g_{n}(\mathcal{O}) \) acts locally nilpotently on \( V_k^n \), \( \text{Der}_{2n} \mathcal{O} \) acts locally nilpotently on \( \hat{g}/g_{n}(\mathcal{O}) \) and \( \text{Der}_{2n} \mathcal{O} \) annihilates \( |0\rangle_n \). This shows that \( \text{Der}_{2n} \mathcal{O} \) acts locally nilpotently on \( V_k^n \). It follows that this action may be exponentiated to the pro-unipotent group \( \text{Aut}_{2n}(\mathcal{O}) \). The arguments for \( V_{k,c}^n \) are identical.

The fact that the \( \text{Vir} \)-action on \( V_k^n \) defined by the Segal-Sugawara operators is compatible with the action of \( \tilde{g} \) follows from commutation relations [3.3.2].

3.3.3. The Segal-Sugawara singular vector. We now have two Virasoro actions on the representations \( V_{k,c}^n \) : one given by the operators \( L_m \) and one given by the operators \( L_m^S \). Moreover, both actions have the same commutation relations with the Kac-Moody
generators $J^a_n$. Their difference now defines a third Virasoro action, which has the crucial feature that it commutes with $\hat{g}$. Indeed, setting $S_m = L_m - L^S_m$, we have
\begin{align*}
[S_l, S_m] &= [L_l, L_m] + ([L^S_l, L^S_m] - [L^S_l, L^S_m] - [L^S_l, R_m]) \\
&= [L_l, L_m] - [L^S_l, L^S_m] \\
&= (l - m)S_{l-m} + \frac{c_k}{12}(l^3 - l)\delta_{l,-m}
\end{align*}
and
\[ [S_m, J^a_l] = 0, \]
where we have introduced the notation
\begin{equation}
(3.3.4) \quad c_k = c - \frac{k \dim g}{k + h^\vee}
\end{equation}
for the central charge of the $S_m$.

These operators are also defined from the action of a vertex operator. Define the Segal-Sugawara singular vector $S \in V_{k,c}$ as the difference $S = \omega_V - \omega_S$, for $k \neq -h^\vee$. The corresponding field
\[ S(z) = \sum_{m \in \mathbb{Z}} S_m z^{-m-2} \]
generates the action of the $S_m$’s on $V^n_{k,c}$. The crucial property of $S$ is that it is a singular vector for the Kac-Moody action, i.e., it is $g(\mathcal{O})$–invariant:
\[ J^a_n \cdot S = 0, \quad n \geq 0. \]
In what follows we consider $\mathfrak{Vir}^n_{2c}$ as a $\tilde{g}$–module, where $\hat{g} \subset \tilde{g}$ acts by zero. By Proposition 3.3.1, $V^n_k$ is also a $\tilde{g}$–module for $k \neq -h^\vee$. Therefore their tensor product is a $\tilde{g}$–module.

3.3.4. **Proposition.** Let $k, c \in \mathbb{C}$ with $k \neq -h^\vee$.

1. The action of the $S_m$ on $V^n_{k,c}$ defines an embedding $S^n_{k,c} : \mathfrak{Vir}^n_{2c} \to V^n_{k,c}$ of $\mathfrak{Vir}$–modules.
2. $S^n_{k,c}$ is a homomorphism of $\tilde{G}_n(\mathcal{O})$–modules with respect to the standard action of $\tilde{G}_n(\mathcal{O})$ on $V^n_{k,c}$ and the trivial action of $\tilde{G}_n(\mathcal{O})$ on $\mathfrak{Vir}^n_{2c}$.
3. There is an isomorphism of $(\tilde{g}, \tilde{G}_n(\mathcal{O}))$–modules
\[ s_{k,c} : V^n_k \otimes \mathfrak{Vir}^n_{2c} \to V^n_k \]
such that $s_{k,c}(0) \otimes v) = S^n_{k,c}(v)$.

3.3.5. **Proof.** The morphism $S^n_{k,c}$ is uniquely determined by the requirement that it intertwine the Virasoro action on $\mathfrak{Vir}^n_{2c}$ with the action of the Virasoro algebra generated by the $S_m$’s on $V^n_{k,c}$, which annihilate the vacuum vector for $m \geq 2n - 1$. (In particular, for $n = 0$, $\omega_V$ is sent to the Segal-Sugawara singular vector $S = S_{-2} \cdot |0\rangle_n$.) To show that this is an embedding, note that for $m < 2n - 1$ the operators $L_m$ act freely on $V^n_{k,c}$, hence so do the $S_m$’s, which have the same leading term with respect to the filtration on $V^n_{k,c}$ by the order in the $L_m$ operators.
The Segal-Sugawara operators commute with the action of \( \hat{g} \). It follows that the subspace \( S^k(\text{Vir}_{ck}) \) of \( V^n_{k,c} \) generated by the action of the \( S_m \)'s on \(|0\rangle_n \) is \( g(0) \)-invariant, and hence \( G(0) \)-invariant. Therefore any vector \( v \in S^n_{k,c}(\text{Vir}_{ck}) \subset V^n_{k,c} \) determines a unique \( \hat{g} \)-homomorphism \( v_\ast : V^n_k \to V^n_{k,c} \) with \( v_\ast(|0\rangle_n) = v \). Hence we have a natural embedding of \( S^n_{k,c}(\text{Vir}_{ck}) \) in \( \text{Hom}_g(V^n_k, V^n_{k,c}) \), and therefore a \( \hat{g} \)-homomorphism

\[
s^n_{k,c} : V^n_k \otimes \text{Vir}_{ck} \to V^n_{k,c}.
\]

The fact that this is a homomorphism of \( \text{Vir} \)-modules, and therefore of \( \hat{g} \)-modules, of this map is immediate from the formula \( S_n = L_n - \frac{c}{2} \mathbb{1} \) for the Virasoro action, where \( L_n \) and \( n \) denote the Virasoro actions on \( V^n_k \) and \( V^n_{k,c} \), respectively. In particular, we see that the map \( S^n_{k,c} \), identified with the inclusion \( \text{Vir}_{ck} \to |0\rangle \otimes \text{Vir}_{ck} \), followed by \( s^n_{k,c} \) is a homomorphism of \( \tilde{G}_n(0) \)-modules, since the actions of \( \tilde{G}_n(0) \) fixes the vector \(|0\rangle \subset V^n_k \).

We claim that the map \( s^n_{k,c} \) in (3.3.5) is an isomorphism. By the Poincaré–Birkhoff–Witt theorem, \( V^n_{k,c} \) has a basis of monomials of the form

\[
J_{m_1}^{n_1} \cdots J_{m_k}^{n_k} L_{n_1} \cdots L_{n_l} |0\rangle_n \quad (m_i < n, \; n_j < 2n - 1),
\]

where we choose an ordering on the Kac-Moody and Virasoro generators. It follows that the same holds with the \( L_{n_j} \) replaced by \( S_{m_j} \). The map \( s^n_{k,c} \) acts as follows:

\[
(J_{m_1}^{n_1} \cdots J_{m_k}^{n_k} |0\rangle) \otimes (L_{n_1} \cdots L_{n_l} |0\rangle) \mapsto J_{m_1}^{n_1} \cdots J_{m_k}^{n_k} S_{m_1} \cdots S_{m_l} |0\rangle_n.
\]

Hence it is indeed an isomorphism as claimed.

### 3.4. Limits of Segal-Sugawara

We would like to describe the behavior of the homomorphisms \( S^n_{k,c} \), or equivalently, of the vector \( S \in V^n_{k,c} \) which generates them, as we vary the parameters \( k,c \). In order to describe the different limits, it is convenient to introduce the parameters \( \lambda, \mu \) and consider the full four–parameter family \( V^n_{k,c,\lambda,\mu} \) of vertex algebras. The vector \( S = \omega_V - \omega_S \) is a well-defined element of \( V^n_{k,c,\lambda,\mu} \) for \( k \in \mathbb{C} \setminus -h^\vee \), \( c \in \mathbb{C} \) and \( \lambda, \mu \in \mathbb{C} \setminus 0 \). It has a first order pole when \( k = -h^\vee \), since the Segal-Sugawara central charge \( c_k = c - \frac{k \dim \mathfrak{g}}{k + h^\vee} \) does. It also has a second order pole along \( \lambda = 0 \) since it contains a term quadratic in the \( J \) generators, a first order pole along \( \mu = 0 \) and \( c = \infty \) since it is first order in the \( L \) generators, and a first order pole along \( k = \infty \) since it is quadratic in the \( J \)'s but divided by \( k + h^\vee \). Thus in these limits it is necessary to normalize \( S \) by its leading term to obtain well-defined, non-zero limits of the map \( S_{k,c} \).

#### 3.4.1. Critical level

We would like to specialize the vector \( S \) and morphism \( S_{k,c} \) to the critical level \( k = -h^\vee \). Introduce the rescaled operators

\[
\overline{S}_m = (k + h^\vee) S_m = (k + h^\vee)(L_m - L_m^S),
\]

which are generated by the vertex operator \( \overline{S}(z) \) associated to

\[
\overline{S} = (k + h^\vee) S = (k + h^\vee) L_{-2} - S_{-2}.
\]
Thus when \( k = -h^\vee \), the vector \( S \) is well-defined and equal to \(-S_{-2}\). The operators \( S_m \) satisfy

\[
[S_l, S_m] = (k + h^\vee)(l - m)S_{l+m} + \frac{(k + h^\vee)(k \dim \mathfrak{g})}{12}(l^3 - l)\delta_{l,m}.
\]

Let us introduce the notation

\[
\mu_{\mathfrak{g}} = h^\vee \dim \mathfrak{g}.
\]

We see that as \( k \) approaches the critical level \(-h^\vee\), as the central charge of the Virasoro action of the \( S_m \) becomes infinite, the renormalized operators \( S_m \) become commuting elements, and moreover satisfy the Poisson relations of the classical Virasoro algebra

\[
U_{\infty, \mu_{\mathfrak{g}}} \Vir = \Sym_{\mu_{\mathfrak{g}}} \Vir.
\]

3.4.2. Proposition. The action of the operators \( \overline{S}_m \) defines a \( G(\mathcal{O}) \)-invariant homomorphism \( \overline{S}_{-h^\vee,c} : \overline{\Vir}_{\mu_{\mathfrak{g}}}^{2n} \to \overline{V}_{-h^\vee,c}^n \) of \( \Sym_{\mu_{\mathfrak{g}}} \Vir \)-modules. For \( n = 0 \), \( \overline{S}_{-h^\vee,c} \) defines a homomorphism of vertex Poisson algebras \( \overline{\Vir}_{\mu_{\mathfrak{g}}} \to \mathbb{Z}(V_{-h^\vee,c}(\mathfrak{g})) \) to the center of the Virasoro–Kac-Moody vertex algebra.

3.4.3. Proof. The morphism \( \overline{S}_{-h^\vee,c} \) is defined by the universal mapping property of the vacuum module \( \overline{\Vir}_{\mu_{\mathfrak{g}}}^{2n} \) of \( \Sym_{\mu_{\mathfrak{g}}} \Vir \). The centrality of \( S \) (and hence the map) for \( k = -h^\vee \) follows from the fact that the commutators of \( \overline{S}_n = (k + h^\vee)S_n \) (for \( k \neq -h^\vee \)) with the \( L_n \) and \( J_n^a \) are divisible by \( k + h^\vee \). That this is a morphism of vertex Poisson algebras is immediate from the commutation relation (3.4.1).

3.4.4. Infinite limit. Now we would like to study the “generic” classical limit of the Segal-Sugawara construction in \( V_{\lambda,\mu} \). In order to do so we approach the plane \( c = k = \infty \) along the direction \( \frac{\lambda}{k} = \frac{\mu}{c} \), which is the direction we used to define the vertex Poisson structure on \( V_{\lambda,\mu}(\mathfrak{g}) \). The Segal-Sugawara operators are rescaled as follows: \( \overline{S}_m = \frac{\lambda^2}{k}S_m \). These are the Fourier modes of the vertex operator \( \overline{S}(z) \) associated to the vector

\[
\overline{S} = \frac{\lambda^2}{k}S \in V_{k,c,\lambda,\mu},
\]

which is regular for \( k \neq 0, -h^\vee \) and \( c, \lambda, \mu \) arbitrary. In terms of the regular elements \( \overline{L} \) and \( \overline{J} \), we have

\[
\overline{S} = \lambda \overline{L}_{-2} - \frac{1}{2(k + h^\vee)} \sum_a \overline{J}_a^{a_1} \overline{J}_{a_2} + \cdots.
\]

Thus when \( \lambda = 0 \) the linear term drops out and we recover the symbol, \( -\frac{1}{2(k + h^\vee)} \overline{S}_{-2} \).

The commutation relations for the \( \overline{S}_m \) are as follows:

\[
[\overline{S}_l, \overline{S}_m] = \frac{\lambda^4}{k^2} [S_l, S_m]
\]

\[
= \frac{\lambda}{k} ((l - m) \overline{S}_{l-m} + \frac{\lambda \mu}{12} (l^3 - l) \delta_{l,-m} + \cdots).
\]
We have used the relation \( \mu = \frac{\lambda c}{k} \), and that
\[
\frac{\lambda^2 k \dim g}{k + h^\vee} = \frac{\lambda^2 \dim g}{k + h^\vee}
\]
vanishes in the limit \( k \to \infty \). Therefore in this limit the \( \mathbb{S}_m \)'s satisfy the relations of the Virasoro Poisson algebra \( \text{Sym}_\lambda \mu \text{Vir} \), with the bracket rescaled by \( \lambda \). We will refer to a morphism which is a homomorphism after rescaling by \( \lambda \) as a \( \lambda \)-homomorphism. We therefore obtain the following analogue of Proposition 3.4.2:

3.4.5. **Proposition.** The action of the Segal-Sugawara operators \( \mathbb{S}_m \) defines a \( \lambda \)-homomorphism \( \mathbb{S}_n^{\lambda, \mu} : \text{Vir}_{2n}^{\lambda, \mu} \to \text{Sym}_{\lambda, \mu} \text{Vir} \)-modules. The image is \( G(\mathcal{O}) \)-invariant, and \( \mathbb{S}_m^{\lambda, \mu} \) is a \( \lambda \)-homomorphism of vertex Poisson algebras.

4. **Moduli Spaces.**

In this section we describe the spaces on which representations of the Virasoro and Kac-Moody algebras and their semi-direct product localize. These are the moduli spaces of curves, of bundles on a fixed curve and bundles on varying curves, respectively. We will consider these localization functors following the general formalism outlined in Section 2. Note that the above moduli spaces are not algebraic varieties, but algebraic stacks. However, as explained in [BB, BD1], the localization formalism is applicable to them because they are “good” stacks, i.e., the dimensions of their cotangent stacks are equal to the twice their respective dimensions.

4.1. **Moduli of bundles.** Let \( X \) be a smooth projective curve over \( \mathbb{C} \). Denote by \( \text{Bun}_G(X) \) the moduli stack of principal \( G \)-bundles on \( X \), and \( \mathcal{P} \) the tautological \( G \)-bundle on the product \( X \times \text{Bun}_G(X) \). (Its restriction to \( X \times \{ \mathcal{P} \} \), for \( \mathcal{P} \in \text{Bun}_G(X) \), is identified with \( \mathcal{P} \).

Given \( x \in X \), we denote by \( \text{Bun}_G(X, x, n) \) the moduli stack of \( G \)-bundles with an \( n \)th order jet of trivialization at \( x \), and by \( \widetilde{\text{Bun}}_G(X, x) \) the moduli stack of \( G \)-bundles with trivializations on the formal neighborhood of \( x \) (the latter moduli stack is in fact a scheme of infinite type). For now we fix a formal coordinate \( t \) at \( x \), so that the complete local ring \( \mathcal{O}_x \) is identified with \( \mathcal{O} = \mathbb{C}[[t]] \). Later we will vary this coordinate by the \( \text{Aut} \mathcal{O} \)-action. The group scheme \( G(\mathcal{O}) \) acts on \( \widetilde{\text{Bun}}_G(X, x) \) by changing the formal trivialization, making \( \widetilde{\text{Bun}}_G(X, x) \to \text{Bun}_G(X) \) into a \( G(\mathcal{O}) \)-torsor. More generally \( \widetilde{\text{Bun}}_G(X, x) \to \text{Bun}_G(X, x, n) \) is a \( G_n(\mathcal{O}) \)-torsor.

4.1.1. **Theorem.** (Kac-Moody Uniformization.)

1. The \( G_n(\mathcal{O}) \) action on the moduli space \( \widetilde{\text{Bun}}_G(X, x) \) extends to a formally transitive action of the Harish-Chandra pair \( (g(\mathcal{K}), G_n(\mathcal{O})) \).

2. ([BL1, BL2, DSI]) For \( G \) semisimple, the action of the ind-group \( G(\mathcal{K}) \) on \( \widetilde{\text{Bun}}_G(X, x) \) is transitive, and there are isomorphisms
\[
\widetilde{\text{Bun}}_G(X, x) \simeq G(\mathcal{K})_\text{out} \backslash G(\mathcal{K}), \quad \text{Bun}_G(X) \simeq G(\mathcal{K})_\text{out} \backslash G(\mathcal{K})/G(\mathcal{O}).
\]
4.1.2. Line bundles on \( \text{Bun}_G(X) \). We refer to [10] for a detailed discussion of the line bundles on \( \text{Bun}_G(X) \). They are classified by integral invariant forms on \( \mathfrak{g} \), which also label the Kac-Moody central extensions of \( LG \). The action of \( \hat{\mathfrak{g}} \) on \( \text{Bun}_G(X,x) \) lifts with level one to the line bundle \( \mathcal{L} \) given by the corresponding invariant form. This line bundle may be defined by using Theorem 4.1.1 from the action of the Kac-Moody group \( \hat{G}(\mathcal{X}) \) (the central extension splits over \( G(X \setminus x) \) and hence gives rise to a line bundle on \( G(X \setminus x) \backslash \hat{G}(\mathcal{X}) = \text{Bun}_G(X,x) \), which descends to \( \text{Bun}_G(X) \)).

For example, if \( G = SL_n \), the line bundle \( \mathcal{L} \) may be identified with the determinant of the cohomology of the universal vector bundle \( \mathcal{E} = \mathcal{P} \times \mathbb{C}^n \) over \( X \times \text{Bun}_{SL_n}(X) \), \( \det R\pi_2^\ast \mathcal{E} \) (where \( \pi_2 : X \times \text{Bun}_G(X) \to \text{Bun}_G(X) \)). This identification however is not canonical (it is not valid for bundles over varying curves, see Section 4.3). More generally, for any simple algebraic group \( G \) powers of \( \mathcal{L} \) can be defined as determinant line bundles associated to representations of \( G \).

4.1.3. Localization. For every \( n \) the triple
\[
(\mathcal{M}, \mathfrak{M}, \mathcal{L}) = (\text{Bun}_G(X,x), \text{Bun}_G(X,x,n), \mathcal{L})
\]
defined above carries a transitive Harish–Chandra action of \( (\hat{\mathfrak{g}}, G_n(0)) \) as in Section 2.3. Therefore we have localization functors from \( (\hat{\mathfrak{g}}, G_n(0)) \)-modules to twisted \( \mathcal{D} \)-modules on \( \text{Bun}_G(X,x,n) \) (we denote these functors by \( \Delta \) as before). In particular, according to Proposition 2.4.1 (2), for the vacuum module \( V^\lambda_k \), the sheaf \( \Delta(V^\lambda_k) \) on \( \text{Bun}_G(X,x,n) \) is just the corresponding sheaf of twisted differential operators, which we denote uniformly by \( \mathcal{D}_k \). Furthermore, the twist \( \mathcal{V}^\lambda_k = V_{\hat{\mathfrak{g}}, G_n(0)}^\lambda \) is a sheaf of algebras on \( \text{Bun}_G(X,x,n) \), and we have a surjective homomorphism \( \mathcal{V}^\lambda_k \to \mathcal{D}_k \).

The corresponding classical vacuum representations \( \mathcal{V}^\lambda_k \) localize to give the Poisson sheaves of functions on the twisted cotangent bundles \( T^\lambda_\mathfrak{g} \text{Bun}_G(X,x,n) = \text{Conn} \mathcal{C}^\lambda \) corresponding to \( \mathcal{L} \). Recall that the cotangent space \( T^\lambda_\mathfrak{g} \text{Bun}_G(X) \) at a bundle \( \mathcal{P} \) is the space of \( \mathfrak{g}_\mathfrak{P} \)-valued differentials \( H^0(X, \mathfrak{g}_\mathfrak{P} \otimes \Omega) \). Therefore we obtain

4.1.4. Proposition. ([10], [11], [12]). The twisted cotangent bundle \( T^\lambda_\mathfrak{g} \text{Bun}_G(X,x,n) \) is canonically identified, as a torsor over \( T^\lambda_\mathfrak{g} \text{Bun}_G(X,x,n) \), with the moduli stack \( \text{Conn}_G(X,x,n) \) of bundles with connections with a pole of order at most \( n \) at \( x \).

4.2. Moduli of curves. Let \( \mathcal{M}_g \) denote the moduli stack of smooth projective curves of genus \( g \), and \( \pi : \mathcal{X}_g \to \mathcal{M}_g \) the universal curve. The stack \( \mathcal{X}_g \) is identified with the moduli stack \( \mathcal{M}_{g,1} \) of pointed genus \( g \) curves. More generally, we denote by \( \mathcal{M}_{g,1,n} \) the moduli stack of pointed curves with an \( n \)th order jet of coordinate at the marked point, and \( \mathcal{M}_{g,1} = \mathcal{M}_{g,1,\infty} \) the moduli scheme of curves with a marked point and formal coordinate (i.e. classifying triples \( (X,x,z) \), where \( (X,x) \in \mathcal{M}_{g,1} \) and \( z \) is a formal coordinate on \( X \) at \( x \)). The group scheme \( \text{Aut} \mathcal{O} \) acts on \( \mathcal{M}_{g,1} \) by changing the coordinate \( z \), making \( \mathcal{M}_{g,1} \) into an \( \text{Aut} \mathcal{O} \)-torsor over \( \mathcal{M}_{g,1} \) and an \( \text{Aut}_n \mathcal{O} \)-torsor over \( \mathcal{M}_{g,1,n} \).

For any family of curves \( \pi : X \to S \), we have the Hodge line bundle \( \mathcal{H} \) on \( S \), defined by
\[
\mathcal{H} = \det R\pi_\ast \omega_{X/S},
\]
the determinant of the cohomology of the canonical line bundle of $X$ over $S$. By abuse of notation we will denote by $\mathcal{H}$ the Hodge line bundle of an arbitrary family of curves. Over the moduli stack $\mathcal{M}$, Mumford has shown that (for $g > 1$) $\mathcal{H}$ generates the Picard group $\text{Pic} \mathcal{M} \cong \mathbb{Z} \cdot \mathcal{H}$.

4.2.1. **Theorem.** ([BS] [TUY] [ADKP] [K] (Virasoro Uniformization.) The $\text{Aut} \mathcal{O}$ action on the moduli space $\widehat{\mathcal{M}}_{g,1}$ of pointed, coordinatized curves extends to a formally transitive action of the Harish-Chandra pair $(\text{Vir}, \text{Aut} \mathcal{O})$ (of level $0$). The action of $\text{Vir}$ lifts to an action with central charge $-2$ on the line bundle $\mathcal{H}$.

4.2.2. **Localization.** The pattern of localization from Section 2.3 applies directly to the $\text{Aut}_n \mathcal{O}$–bundle $\widehat{\mathcal{M}}_{g,1} \to \mathcal{M}_{g,1,n}$ and the Harish–Chandra pairs $(\text{Vir}, \text{Aut}_n \mathcal{O})$. Therefore we have localization functors from $(\text{Vir}, \text{Aut}_n \mathcal{O})$–modules to twisted $\mathcal{D}$–modules on $\mathcal{M}_{g,1,n}$. In particular, the localization of the vacuum module $\text{Vir}_c^n$ gives us the sheaf of twisted differential operators on $\mathcal{M}_{g,1,n}$, denoted by $\mathcal{D}_c$, and their classical versions give us Poisson algebras of functions on twisted cotangent bundles. Furthermore, the twist $\text{Vir}_c^n = \mathcal{V}_{\text{Vir}, \text{Aut}_n(\mathcal{O})}$ is a sheaf of algebras on $\mathcal{M}_{g,1,n}$, and we have a surjective homomorphism $\text{Vir}_c^n \to \mathcal{D}_c$.

A similar picture holds for the fibration $\widehat{\mathcal{M}}_{g,1} \to \mathcal{M}_g$ and the pair $(\text{Vir}, \text{Der} \mathcal{O})$, except that $\text{Der} \mathcal{O}$ does not integrate to a group (only to an ind-group, see [FB], Ch. 5), and $\widehat{\mathcal{M}}_{g,1} \to \mathcal{M}_g$ is not a principal bundle, so that the definition of localization does not carry over directly. Nevertheless, we obtain the desired description of differential operators on $\mathcal{M}_g$ by first localizing $\text{Vir}_c$ on $\mathcal{M}_{g,1}$ (or $\mathcal{M}_{g,1,n}$), using the fact that the corresponding $\mathcal{D}$–modules descend along the projection $\pi : \mathcal{M}_{g,1} \to \mathcal{M}_g$.

4.2.3. **Proposition.** The localization of the Virasoro module $\text{Vir}_c$ on $\mathcal{M}_{g,1}$ is isomorphic to the pullback $\pi^* \mathcal{D}_c$ of the sheaf of twisted differential operators on $\mathcal{M}_g$.

4.2.4. **Proof.** By Lemma 2.4.5, the localization $\Delta(\text{Vir}_c)$ is the quotient of $\Delta(\text{Vir}_c^0) = \mathcal{D}_c$ by the action of the partial vacuum representation $\Delta(V_{\text{Der} \mathcal{O}, \text{Der} \mathcal{O}})$. However, the latter is readily identified as the sheaf of relative differential operators, $\Delta(V_{\text{Der} \mathcal{O}, \text{Der} \mathcal{O}}) = \mathcal{D}_c/\mathcal{M}_g$. Indeed the action of $\text{Der} \mathcal{O}$ on $\widehat{\mathcal{M}}_g$ is free and generates the relative vector fields on the universal curve – the $\widehat{\mathcal{M}}_g$ twist $(\text{Der} \mathcal{O}/\text{Der} \mathcal{O})_{\widehat{\mathcal{M}}_g}$ is precisely the relative tangent sheaf of $\mathcal{M}_{g,1}$ over $\mathcal{M}_g$. But the quotient of the sheaf of differential operators by the ideal generated by vertical vector fields is the pullback of the sheaf of differential operators downstairs, whence the proposition.

4.2.5. Recall (see, e.g., [FB]) that the space of projective structures on $X$ is a torsor over the quadratic differentials $H^0(X, \Omega^{0,2})$, which is the cotangent fiber of $\mathcal{M}_g$ at $x$. The Virasoro uniformization of $\mathcal{M}_g$, together with the canonical identification of projective structures on the punctured disc with a hyperplane in $\text{Vir}^*$, gives us the following:

4.2.6. **Corollary.** ([BS]) There is a canonical identification of the twisted cotangent bundles $T^*_\lambda \mathcal{M}_g = \text{Proj} \, g$ (for $\lambda = 12$).
4.2.7. Similarly the twisted cotangents to the moduli $\mathcal{M}_{g,1,n}$ of curves with marked points and level structures are identified with the moduli $\Proj^{\lambda}_{g,1,n}$ of $\lambda$–projective structures with poles at the corresponding points.

4.3. Moduli of curves and bundles. The discussions of the moduli spaces $\mathcal{M}_g$ and $\text{Bun}_G(X)$ above may be generalized to the situation where we vary both the curve and the bundle on it. Let $\text{Bun}_{G,g}$ denote the moduli stack of pairs $(X, \mathcal{P})$, where $X$ is a smooth projective curve of genus $g$, and $\mathcal{P}$ is a principal $G$–bundle on $X$. Thus we have a projection

$$\Pi : \text{Bun}_{G,g} \longrightarrow \mathcal{M}_g$$

with fiber over $X$ being the moduli stack $\text{Bun}_G(X)$. Let $\text{Bun}_{G,g,1}$ denote the moduli stack classifying $G$–bundles on pointed curves, i.e., the pullback of the universal curve to $\text{Bun}_{G,g}$:

$$\text{Bun}_{G,g,1} = \mathcal{M}_{g,1} \times \text{Bun}_G \xrightarrow{\pi_G} \text{Bun}_{G,g}$$

We denote by $\Pi^0 : \text{Bun}_{G,g,1} \rightarrow \mathcal{M}_{g,1}$ the map forgetting the bundles. We let $\text{Bun}_{G,g,1,n}$ denote the moduli stack of quintuples $(X, x, \mathcal{P}, z, \tau)$ consisting of a $G$–bundle $\mathcal{P}$ with an $n$th order jet of trivialization $\tau$ on a pointed curve $(X, x)$ with $2n$th order jet of coordinate $z$. We let $\Pi^n : \text{Bun}_{G,g,1,n} \rightarrow \mathcal{M}_{g,1,2n}$ denote the map forgetting $(\mathcal{P}, \tau)$. For $n = \infty$ we obtain the moduli scheme $\tilde{\text{Bun}}_{G,g}$ of bundles with formal trivialization on pointed curves with formal coordinates.

There is a natural two–parameter family of line bundles on $\text{Bun}_{G,g}$. Namely, there is a Hodge bundle $\mathcal{H}$ associated to the family of curves $\pi_G : \text{Bun}_{G,g,1} \rightarrow \text{Bun}_{G,g}$ (which is the $\Pi$ pullback of the Hodge bundle on $\mathcal{M}_g$). The universal principal $G$–bundle $\mathcal{P}$ lives on this universal curve $\text{Bun}_{G,g,1}$, so we may also consider the line bundle $\mathcal{C}$ associated to the principal bundle $\mathcal{P}$. The bundle $\mathcal{C}$ is trivial on the section $\text{triv} : \mathcal{M}_g \rightarrow \text{Bun}_{G,g}$ sending a curve to the trivial $G$–bundle. Let $\mathcal{L}_{k,c} = \mathcal{C}^\otimes k \otimes \mathcal{H}^\otimes c$. For simply-connected $G$ this assignment gives an identification $\text{Pic}(\text{Bun}_{G,g}) \cong \mathbb{Z} \otimes \mathbb{Z}$ (see [La]).

For $G = \text{SL}_n$, we may consider the determinant bundle $\det R\pi_2_*\mathcal{E}$ as before, where $\pi_2$ is the projection from the universal curve and $\mathcal{E} = \mathcal{P} \times_{\text{SL}_n} \mathbb{C}^n$ is the universal vector bundle. The determinant of the cohomology of the trivial rank $n$ bundle gives the $n$th power of the Hodge line bundle, while for fixed curve the determinant bundle may be identified with $\mathcal{C}$. Thus we have the Riemann-Roch identification

$$\det R\pi_2_*\mathcal{E} \cong \mathcal{C} \otimes \mathcal{H}^\otimes n = \mathcal{L}_{1,n}.$$

(In this case the determinant and Hodge bundles also span the Picard group.) In general we have determinant line bundles $\mathcal{L}_\rho$ for any representation $\rho : G \rightarrow \text{SL}_n$.

4.3.1. Extended connections. For each $\lambda, \mu \in \mathbb{C}$, we have the corresponding twisted cotangent bundle $T^*_{\lambda,\mu} \text{Bun}_{G,g}$, which is $\text{Conn}(\mathcal{L}_{\lambda,\mu})$ when $\lambda, \mu$ are integral. Following [BZB2, BZB1], we will refer to points of $\mathcal{E}_x \text{Conn}_{G,g} = T^*_{\lambda,\mu} \text{Bun}_{G,g}$ as $(\lambda, \mu)$–extended connections. Using the short exact sequence of cotangent bundles

$$0 \rightarrow \Pi^* T^* \mathcal{M}_g \rightarrow T^* \text{Bun}_{G,g} \rightarrow T^*_{/\mathcal{M}} \text{Bun}_{G,g} \rightarrow 0$$
and the description (see Proposition 4.1.1) of the relative twisted cotangent bundle of $\text{Bun}_{G,g}$, we obtain an affine projection

$$\mathcal{E}_x \mathcal{C}\text{onn}^\lambda_{G,g} \to \mathcal{C}\text{onn}^\lambda_{G,g},$$

with fibers affine spaces over quadratic differentials. Here $\mathcal{C}\text{onn}^\lambda_{G,g}$ denotes the moduli space of curves equipped with $G$–bundles and $\lambda$–connections, which is identified as the relative twisted cotangent bundle

$$\mathcal{C}\text{onn}^\lambda_{G,g} \cong \mathcal{C}\text{onn} / \mathcal{M}_{L^\lambda,\mu}$$

for any $\mu \in \mathbb{C}$. (Note that $\mathcal{L}_{0,\mu} = \Pi^* \mathcal{H}_\mu$ has a canonical connection relative to $\mathcal{M}$, so that $\mathcal{C}\text{onn} / \mathcal{M}_{L^\lambda,\mu} \cong \mathcal{C}\text{onn} / \mathcal{M}_{L^\lambda,0}$ for any $\mu$.)

4.3.2. Remark: kernel functions. See also [BZB2] where a concrete description is given of the identification of projective structures, connections and extended connections with the twisted cotangent bundles of the moduli of curves and bundles using kernel functions along the diagonal (in particular the Szegö kernel), in the spirit of [BS].

4.3.3. The group schemes $\tilde{G}_n(\mathcal{O})$ act on $\text{Bun}_{G,g}$, changing the coordinate $z$ and trivialization $\tau$. This action makes $\text{Bun}_{G,g}$ into a $\tilde{G}_n(\mathcal{O})$–torsor over $\text{Bun}_{G,g,1,n}$.

4.3.4. Theorem. (Virasoro–Kac-Moody Uniformization.) The $\tilde{G}_n(\mathcal{O})$–action on the moduli stack $\text{Bun}_{G,g}$ extends to a formally transitive action of the Harish-Chandra pair $(\tilde{g}, \tilde{G}_n(\mathcal{O}))$ (of level and central charge 0). The action of $\tilde{g}$ lifts to an action with level $k$ and central charge $c$ on the line bundle $L^k_c$.

4.3.5. Localization. The pattern of localization from Section 2.3 again applies directly to $\text{Bun}_{G,g,1,n}$ and the Harish–Chandra pairs $(\tilde{g}, \tilde{G}_n(\mathcal{O}))$. Therefore we have localization functors from $(\tilde{g}, \tilde{G}_n(\mathcal{O}))$–modules to twisted $\mathcal{D}$–modules on $\text{Bun}_{G,g,1,n}$. In particular, the localization of the vacuum module $V^n_{k,c}$ gives us the sheaf of twisted differential operators on $\text{Bun}_{G,g,1,n}$, denoted by $\mathcal{D}_{k,c}$. As in Section 4.2.3, we would like to show that the corresponding sheaves descend along the projection $\pi_G : \text{Bun}_{G,g,1} \to \text{Bun}_{G,g}$ to the moduli of curves and bundles itself. The argument of Proposition 4.2.3 carries over directly to this setting, and we obtain

4.3.6. Proposition. The localization of the Virasoro–Kac-Moody module $V_{k,c}$ on $\text{Bun}_{G,g,1}$ is isomorphic to the pullback $\pi_G^* \mathcal{D}_{k,c}$ of the sheaf of twisted differential operators on $\text{Bun}_{G,g}$.

Note that we now have a two–parameter family of line bundles $\mathcal{L}_{k,c}$, and hence the pattern of deformations of Section 2.3.1 can be “doubled”, to match up with the picture of the Virasoro–Kac-Moody vertex algebra $V_{k,c}$. Thus, we introduce deformation parameters $\lambda, \mu$ coupled to the level and charge $k, c$. This defines a four–parameter family of algebras $\mathcal{D}_{\lambda,\mu}(\mathcal{L}_{k,c})$, to which the analogous quasi-classical localization statements apply.
5. The Segal-Sugawara Homomorphism.

In this section we apply the techniques of Lie algebra localization and vertex algebra conformal blocks to the Segal-Sugawara construction from Section 3. We interpret the result as a homomorphism between sheaves of twisted differential operators on the moduli stacks introduced in the previous section, and as heat operators on spaces of nonabelian theta functions. Various classical limits of this construction will be considered in the next section.

5.1. Homomorphisms between sheaves of differential operators. Let \( k, c \in \mathbb{C} \) with \( k \neq -hV \). Let \( \mathcal{V}_{k,c}^n = \mathcal{V}_{\hat{\mathfrak{g}} \widehat{G}_n(\mathcal{O})} \) be the sheaf on \( \text{Bun}_{G,g,1,n} \) obtained as the twist of the vacuum module \( V_{k,c}^n \) over \( (\hat{\mathfrak{g}},\widehat{G}_n(\mathcal{O})) \) following the construction of Section 2.4. By Proposition 2.4.1, this is a sheaf of algebras, equipped with a surjective homomorphism to the sheaf of \( (\hat{G}_n(\mathcal{O}))^\ast \)-twisted differential operators \( \Delta(V_{k,c}^n) = \mathcal{D}_{k,c} \).

According to Corollary 2.4.3, the subspace of invariants \( (V_{k,c}^n)^{\widehat{G}_n(\mathcal{O})} \) give rise to global differential operators on \( \text{Bun}_{G,g,1,n} \). Unfortunately, for \( k \neq -hV \) the space \( (V_{k,c}^n)^{\widehat{G}_n(\mathcal{O})} \) is one-dimensional, spanned by the vacuum vector. However, the Segal-Sugawara construction provides us with a large space of invariants for the smaller group \( G_n(\mathcal{O}) \subset \widehat{G}_n(\mathcal{O}) \). Namely, by Proposition 2.4.1, the \( G_n(\mathcal{O}) \)-invariants contain a copy of the Virasoro vacuum module \( S^n : \text{Vir}_{k,c}^{2n} \to (V_{k,c}^n)^{\widehat{G}_n(\mathcal{O})} \). We will use this fact to obtain a homomorphism \( \mathcal{D}_{k,c} \to \Pi^n \mathcal{D}_{k,c} \) of sheaves on \( \mathcal{M}_{g,1,2n} \). The first step is the following assertion.

Consider \( \text{Vir}_{k,c}^{2n} \) as a \( \hat{\mathfrak{g}} \)-module by letting \( \hat{\mathfrak{g}} \) act by zero. Then the twist \( \mathcal{V}_{\text{Vir}_{k,c}^{2n}} = \mathcal{V}_{\hat{\mathfrak{g}} \widehat{G}_n(\mathcal{O})} \) becomes a sheaf of algebras on \( \mathcal{M}_{g,1,2n} \).

5.1.1. Proposition. There is a homomorphism of sheaves of algebras on \( \mathcal{M}_{g,1,2n} \),

\[
S^n_{k,c} : \mathcal{V}_{\text{Vir}_{k,c}^{2n}} \to \Pi^n \mathcal{V}^n_{k,c}.
\]

5.1.2. Proof. Note that the map \( S^n : \text{Vir}_{k,c}^{2n} \to (V_{k,c}^n)^{\widehat{G}_n(\mathcal{O})} \) is a homomorphism of \( \widehat{G}_n(\mathcal{O}) \)-modules (where the subgroup \( G_n(\mathcal{O}) \) of \( \widehat{G}_n(\mathcal{O}) \) acts by zero). Hence its gives rise to a homomorphism of the corresponding twists by the \( \widehat{G}_n(\mathcal{O}) \)-torsor \( \text{Bun}_{G,g,1,n} \to \text{Bun}_{G,g,1,n} \),

\[
\text{Bun}_{G,g,1,n} \times \text{Vir}_{k,c}^{2n} \to \text{Bun}_{G,g,1,n} \times (V_{k,c}^n)^{\widehat{G}_n(\mathcal{O})} \to \mathcal{V}^n_{k,c} = \text{Bun}_{G,g,1,n} \times V_{k,c}^n.
\]

The first two sheaves are pullbacks from \( \mathcal{M}_{g,1,2n} \). Indeed, since the actions of \( \widehat{G}_n(\mathcal{O}) \) on \( \text{Vir}_{k,c}^{2n} \) and \( (V_{k,c}^n)^{\widehat{G}_n(\mathcal{O})} \) factor through \( \text{Aut}_{2n} \mathcal{O} \), their twists depend only on the associated \( \text{Aut}_{2n} \mathcal{O} \)-torsor, which is nothing but the \( \Pi^n \)-pullback of the \( \text{Aut}_{2n} \mathcal{O} \)-torsor \( \pi_{2n} : \mathcal{M}_{g,1} \to \mathcal{M}_{g,1,2n} \). Therefore the maps 5.1.1 may be written as

\[
(\Pi^n)^\ast \mathcal{V}_{\text{Vir}_{k,c}^{2n}} \to (\Pi^n)^\ast (V_{k,c}^n)^{\widehat{G}_n(\mathcal{O})} \to \mathcal{V}_{k,c}^n.
\]
By adjunction, we obtain the following maps on \( \mathcal{M}_{g,1,2n} \):

\[
\mathcal{V}ir_{c_k}^{2n} \rightarrow (\mathcal{V}^{n}_{k,c})^{G_n(\mathcal{O})} \rightarrow \Pi^*_n\mathcal{V}^{n}_{k,c}.
\]

We claim that these sheaves are algebras and these maps are algebra homomorphisms. The third term is the pushforward of an algebra, hence an algebra. The first term is the twist of the vacuum module for \((\mathcal{V}ir, \text{Aut}_{2n}\mathcal{O})\), so its algebra structure comes from Proposition 2.4.1. Next, note that the Harish-Chandra pair \((\mathfrak{g}, G_n(\mathcal{O}))\) acts on \(\mathcal{M}_{g,1}\) through the quotient \((\mathcal{V}ir, \text{Aut}_{2n}\mathcal{O})\), so that the \(\text{Aut}_{2n}\mathcal{O}\)–twist of the \(G_n(\mathcal{O})\)–invariants in \(\mathcal{V}^{n}_{k,c}\) may be rewritten in terms of the semidirect product \(\tilde{G}_n(\mathcal{O})\):

\[
(\mathcal{V}^{n}_{k,c})^{G_n(\mathcal{O})} = \left[\pi_n(\mathcal{O}_{\mathfrak{g}_{k,1}^n} \otimes \mathcal{V}^{n}_{k,c})\right]^{\tilde{G}_n(\mathcal{O})}.
\]

Thus as in Proposition 2.4.1 the sheaves \((\mathcal{V}^{n}_{k,c})^{G_n(\mathcal{O})}\) on \(\mathcal{M}_{g,1,2n}\) or \(\text{Bun}_{G,g,1,n}\) are naturally sheaves of algebras. This structure is clearly compatible with that on \(\Pi^*_n\mathcal{V}^{n}_{k,c}\).

Finally, the maps \(\mathcal{V}ir_{c_k}^{2n} \rightarrow \mathcal{V}^{n}_{k,c}\) are induced from the homomorphism \(U(\mathcal{V}ir_{c_k}) \rightarrow \tilde{U}_{k,c}\) into the completion of the enveloping algebra of \(\mathfrak{g}\). This homomorphism maps the subalgebra \(U(\text{Der}_{2n}\mathcal{O})\) to the left ideal generated by the Lie subalgebra \(\mathfrak{g}_n(\mathcal{O})\). Hence we obtain a homomorphism of the corresponding vacuum modules, because \(\mathcal{V}^{n}_{k,c}\) can be defined as the quotient of the completed algebra \(\tilde{U}_{k,c}\) by the left ideal generated by the Lie subalgebra \(\mathfrak{g}_n(\mathcal{O})\). It follows that the map \(\mathcal{V}ir_{c_k}^{2n} \rightarrow (\mathcal{V}^{n}_{k,c})^{G_n(\mathcal{O})}\) above respects algebra structures. More precisely, this map comes from the above homomorphism of enveloping algebras by passing to the normalizers of ideals on both sides, hence it remains a homomorphism.

5.1.3. By Proposition 5.1.1 we have a diagram of algebra homomorphisms on \(\mathcal{M}_{g,1,2n}\):

\[
\begin{array}{ccc}
\mathcal{V}ir_{c_k}^{n} & \xrightarrow{\mathcal{S}^n_{k,c}} & \Pi^*_n\mathcal{V}^{n}_{k,c} \\
\downarrow & & \downarrow \\
\mathcal{D}_{c_k} & \xrightarrow{\Pi^*_n\mathcal{D}_{k,c}} & \Pi^*_n\mathcal{D}_{k,c}
\end{array}
\]

We wish to show that the homomorphism \(\mathcal{S}^n_{k,c}\) descends to the sheaves of twisted differential operators. It will then automatically be a homomorphism of algebras of differential operators.

5.1.4. \textbf{Theorem.} The homomorphism \(\mathcal{S}^n_{k,c}\) of descends to a homomorphism of algebras

\[
\mathcal{S}^n_{k,c} : \mathcal{D}_{c_k} \rightarrow \Pi^*_n\mathcal{D}_{k,c}.
\]

5.1.5. \textit{Spaces of coinvariants.} Equivalently, we need to show that up on \(\text{Bun}_{G,g,1,n}\), the morphism \((\Pi^n)^*\mathcal{V}ir_{c_k}^{n} \rightarrow \mathcal{V}^{n}_{k,c}\) descends to \((\Pi^n)^*\mathcal{D}_{c_k} \rightarrow \mathcal{D}_{k,c}\), since the morphism on \(\mathcal{M}_{g,1,2n}\) is obtained from the former by adjunction.

If we apply the Virasoro–Kac-Moody localization functor \(\Delta\) on \(\text{Bun}_{G,g,1,n}\) to the \((\mathfrak{g}, \tilde{G}_n(\mathcal{O}))\)–modules \(\mathcal{V}ir_{c_k}^{2n}\) and \(\mathcal{V}^{n}_{k,c}\), we obtain the desired sheaves \((\Pi^n)^*\mathcal{D}_{c_k}\) and \(\mathcal{D}_{k,c}\). However, the embedding \(\mathcal{V}ir_{c_k}^{2n} \rightarrow \mathcal{V}^{n}_{k,c}\) of Proposition 3.3.1 is \textit{not} a homomorphism of \(\mathfrak{g}\)–modules: it intertwines the Virasoro action on \(\mathcal{V}ir_{c_k}^{n}\) with the action on \(\mathcal{V}^{n}_{k,c}\) of
the Virasoro algebra generated by the $S_n$'s, not the $L_n$'s. Because of that, it is not immediately clear that the map $Vir_{\mathfrak{g}} \to V_{k,c}$ gives rise to a morphism of sheaves $(\Pi^\times)^*D_{ce} \to D_{k,c}$. In order to prove that, we must use Theorem 5.1.6 below to pass from Lie algebra coinvariants to vertex algebra coinvariants.

Let us recall some results from [FB] on the spaces of (twisted) coinvariants of vertex algebras. Let $V$ be a conformal vertex algebra with a compatible $\tilde{\mathfrak{g}}$-structure (see [FB], Section 6.1.3). This means that $V$ carries an action of the Harish-Chandra pair $(\tilde{\mathfrak{g}}, \tilde{G}(\varnothing))$, such that the action of the Lie algebra $\tilde{\mathfrak{g}}$ is generated by Fourier coefficients of vertex operators. In particular, our vacuum modules $V_{k,c}$ and $Vir_c$ are examples of such vertex algebras.

Given a vertex algebra of this type, we define its space of twisted coinvariants as in [FB], Section 8.5.3. Namely, let $R$ be a local $C$-algebra and $(X, x, \mathcal{P})$ an $R$-point of $\text{Bun}_{G, \mathfrak{g}}$, i.e., a pointed curve $(X, x)$ and a $G$-bundle on $X$, all defined over Spec $R$. Then $X$ carries a natural $\tilde{G}(\varnothing)$-bundle $\tilde{\mathcal{P}}$, whose fiber over $y \in X$ consists of pairs $(z, t)$, where $z$ is a formal coordinate at $y$ and $t$ is a trivialization of $\mathcal{P}$ over the formal disc around $y$.

Set
\[ \mathcal{V}^\mathcal{P} = \tilde{\mathcal{P}} \times_{\tilde{G}(\varnothing)} V. \]
This vector bundle carries a flat connection, and we define the sheaf $h(\mathcal{V}^\mathcal{P} \otimes \Omega)$ as the sheaf of zeroth de Rham cohomology of $\mathcal{V}^\mathcal{P} \otimes \Omega$. The vertex algebra structure on $V$ makes this sheaf into a sheaf of Lie algebras. In particular, the Lie algebra $U^\mathcal{P}(\mathcal{V}_x)$ of sections of $h(\mathcal{V}^\mathcal{P} \otimes \Omega)$ over the punctured disc $D^X_x$ is isomorphic to the Lie algebra $U(V)$ topologically spanned by the Fourier coefficients of all vertex operators from $V$. It acts on $\mathcal{V}^\mathcal{P}_x$, the fiber of $\mathcal{V}^\mathcal{P}$ at $x$.

Let $U^\mathcal{P}_{X \setminus x}(\mathcal{V}_x)$ be the image of the Lie algebra $\Gamma(X \setminus x, h(\mathcal{V}^\mathcal{P} \otimes \Omega))$ in $U^\mathcal{P}(\mathcal{V}_x) = \Gamma(D^X_x, h(\mathcal{V}^\mathcal{P} \otimes \Omega))$. The space $H^\mathcal{P}(X, x, V)$ of twisted coinvariants of $V$ is by definition the quotient of $\mathcal{V}^\mathcal{P}_x$ by the action of $U^\mathcal{P}_{X \setminus x}(\mathcal{V}_x)$.

Let $\mathcal{A}^\mathcal{P}$ be the Atiyah algebroid of infinitesimal symmetries of $\mathcal{P}$. We have the exact sequence
\[ 0 \to \mathfrak{g}^\mathcal{P} \to \mathcal{A}^\mathcal{P} \to \Theta_X \to 0, \]
where $\mathfrak{g}^\mathcal{P}$ is the sheaf of sections of the vector bundle $\mathcal{P} \times \mathfrak{g}$.

When $V = V_{k,c}$, the Lie algebra $U^\mathcal{P}(\mathcal{V}_{k,c,x})$ contains as a Lie subalgebra a canonical central extension of the Lie algebra $\Gamma(D^X_x, \mathcal{A}^\mathcal{P})$ (it becomes isomorphic to $\mathfrak{g}$ if we choose a formal coordinate at $x$ and a trivialization of $\mathcal{P}|_{D^X_x}$). Also, the Lie algebra $U^\mathcal{P}_{X \setminus x}(\mathcal{V}_{k,c,x})$ contains
\[ \tilde{\mathfrak{g}}^\mathcal{P}_{\text{out}} = \Gamma(X \setminus x, \mathcal{A}^\mathcal{P}) \]
as a Lie subalgebra (it is isomorphic to $\text{Vect}(X \setminus x) \ltimes \mathfrak{g} \otimes \mathbb{C}[X \setminus x]$).

Likewise, in the case when $V = Vir_c$, $U^\mathcal{P}(Vir_{c,x})$ contains the Virasoro algebra, and $U^\mathcal{P}_{X \setminus x}(Vir_{c,x})$ contains the Lie algebra $\text{Vect}(X \setminus x)$ of vector fields on $X \setminus x$. The homomorphism $Vir_{ck} \to V_{k,c}$ of vertex algebra induces injective homomorphisms of Lie
algebras
\[ U^p(\text{Vir}_{c,k}) \hookrightarrow U^p(\mathcal{V}_{k,c,x}) \]
\[ U^p_X(\text{Vir}_{c,k}) \hookrightarrow U^p_X(\mathcal{V}_{k,c,x}) \]

(since the action of \( \tilde{\mathfrak{g}} \) on \( \text{Vir}_c \) is trivial, the Lie algebras on the left do not depend on \( P \)). Note that though the Segal-Sugawara vertex operator is quadratic in the generating fields of \( V_{k,c} \), the elements of \( U^p_X(\text{Vir}_{c,k}) \) (and other elements of \( U^p_X(\mathcal{V}_x) \)) cannot be expressed in terms of the Lie subalgebra \( \tilde{\mathfrak{g}}^{p}_{\text{out}} \) of \( U^p_X(\mathcal{V}_{k,c,x}) \). Nevertheless, we have the following result which is proved in [FB], Theorem 8.3.3 (see also Remark 8.3.10).

5.1.6. Proposition. For any smooth \( \tilde{\mathfrak{g}}^{p}_{\text{out}} \)-module \( M \) (which is then automatically a \( U^p(\mathcal{V}_{k,c,x}) \)-module), the space of coinvariants of \( M \) by the action of \( U^p_X(\mathcal{V}_{k,c,x}) \) is equal to the space of coinvariants of \( M \) by the action of \( \tilde{\mathfrak{g}}^{p}_{\text{out}} \).

5.1.7. Proof of Theorem 5.1.4 Fix an \( R \)-point \( (X, x, z, t, \mathcal{P}) \) of \( \text{Bun}_{G,1,n} \), i.e., a pointed curve \( (X, x) \), a 2n-jet of coordinate \( z \) at \( x \), a \( G \)-bundle \( \mathcal{P} \) and an \( n \)-jet of trivialization \( t \) of \( \mathcal{P} \) at \( x \), all defined over the spectrum of some local \( \mathbb{C} \)-algebra \( R \). Let \( v \) be a vector in the fiber of the sheaf \( \text{Vir}_{c,k}^{2n} \) over the \( R \)-point \( (X, x, z) \) of \( \mathfrak{m}_{g,1,2n} \), which lies in the kernel of the surjection \( \text{Vir}_{c,k}^{2n}|_R \to \mathcal{D}_{c,k}|_R \). In order to prove the theorem, it is sufficient to show that the image of \( v \) in the fiber of the sheaf \( \mathcal{V}_{k,c}^{n} \) over the \( R \)-point \( (X, x, z, \mathcal{P}, t) \) of \( \text{Bun}_{G,1,n} \) belongs to the kernel of the surjection \( \mathcal{V}_{k,c}^{n}|_R \to \mathcal{D}_{k,c}|_R \).

But according to [FB], Lemmas 16.2.9 and 16.3.6, the kernel of the map \( \text{Vir}_{c,k}^{2n}|_R \to \mathcal{D}_{c,k}|_R \) is spanned by all vectors of the form \( s \cdot A \), where \( A \in \mathcal{V}_{c,k}^{2n}|_R \) and \( s \in \text{Vect}(X|x) \) (so that \( \mathcal{D}_{c,k}|_R \) is the space of coinvariants of \( \text{Vir}_{c,k}^{2n}|_R \) with respect to \( \text{Vect}(X|x) \)). Likewise, the kernel of the map \( \mathcal{V}_{k,c}^{n}|_R \to \mathcal{D}_{k,c}|_R \) is spanned by all vectors of the form \( f \cdot B \), where \( B \in \mathcal{V}_{k,c}^{n}|_R \) and \( f \in \tilde{\mathfrak{g}}^{p}_{\text{out}} \) (so that \( \mathcal{D}_{k,c}|_R \) is the space of coinvariants of \( \mathcal{V}_{k,c}^{n}|_R \) with respect to \( \tilde{\mathfrak{g}}^{p}_{\text{out}} \)).

So we need to show that the image of a vector of the above form \( s \cdot A \) in \( \mathcal{V}_{k,c}^{n}|_R \) is in the image of the Lie algebra \( \tilde{\mathfrak{g}}^{p}_{\text{out}} \). But by Proposition 5.1.6 the space of coinvariants of \( \mathcal{V}_{k,c}^{n}|_R \) with respect to \( \tilde{\mathfrak{g}}^{p}_{\text{out}} \) is equal to the space of coinvariants of \( \mathcal{V}_{k,c}^{n}|_R \) with respect to \( U^p_X(\mathcal{V}_{k,c,x}) \). This implies the statement of the theorem, because according to the discussion of Section 5.1.5 \( U^p_X(\mathcal{V}_{k,c,x}) \) contains the image of \( \text{Vect}(X|x) \subset U^p_X(\text{Vir}_{c,k}) \) as a Lie subalgebra.

5.1.8. Descent to \( \mathfrak{m}_g \). The homomorphism of Theorem 5.1.4 may be used to describe differential operators on the moduli of unmarked curves and bundles, \( \Pi : \text{Bun}_{G,1} \to \mathfrak{m}_g \). To do so we first localize the corresponding “vacuum” representations \( \text{Vir}_{c,k} \) and \( \mathcal{V}_{k,c} \) on \( \text{Bun}_{G,1} \) and \( \mathfrak{m}_g \) and then descend.

5.1.9. Corollary. There is a canonical homomorphism \( \mathcal{D}_{c,k} \to \Pi_\ast \mathcal{D}_{k,c} \) of sheaves of algebras on \( \mathfrak{m}_g \).
5.1.10. Proof. We will construct a morphism of sheaves $\Pi^*\mathcal{D}_{ck} \to \mathcal{D}_{k,c}$ on $\text{Bun}_{G,g}$, which gives rise to the desired morphism on $\mathcal{M}_g$ by adjunction. This map furthermore is constructed by descent from $\text{Bun}_{G,g,1}$. By Proposition 4.2.8 and Proposition 4.3.6, the sheaves $\Delta_{ck}(\text{Vir}_{ck})$ and $\Delta_{k,c}(V_{k,c})$ on $\mathcal{M}_g,1$ and $\text{Bun}_{G,g,1}$ are identified with $\pi^*\mathcal{D}_{ck}$ and $\pi^*\mathcal{D}_{k,c}$. Moreover these identifications are horizontal with respect to the relative connections inherited by the pullback sheaves. We now need to construct a map between the pullback of $\pi^*\mathcal{D}_{ck}$ to $\text{Bun}_{G,g,1}$ and $\pi^*\mathcal{D}_{k,c}$ which is flat relative to $\pi_G$ and hence descends to $\text{Bun}_{G,g}$.

This morphism $S_{k,c}$ may be constructed directly following Theorem 5.1.4 or by applying the change of vacuum isomorphism Lemma 2.4.5 to the homomorphism $\tilde{\mathfrak{s}}$ of Proposition 3.3.4. Twisting by the $\mathcal{D}_{k,c}$–isomorphism $\pi_{k,c}$ we obtain an isomorphism $\mathcal{D}_{k,c} \to \mathcal{D}_{k,c}$, which gives rise to the desired morphism on $\text{Bun}_{G,g}$.

5.2. Tensor product decomposition. By applying localization to the isomorphism of $(\mathfrak{g},\tilde{G}(\mathcal{O}))$–modules

$$S_{k,c} : V_k \otimes \text{Vir}^{2n}_{ck} \sim V_{k,c}$$

of Proposition 3.3.4, we obtain an isomorphism

$$\Delta_{k,c} : \Delta(V_k \otimes \text{Vir}^{2n}_{ck}) \sim \Delta(V_{k,c}).$$

However, the functors of coinvariants and localization are not tensor functors. Since $V_k$ is a $\mathfrak{g}$–submodule of $V_{k,c}$, we obtain a natural map $\Delta(V_k) \to \Delta(V_{k,c})$. The localization $\Delta(V_k)$ is naturally identified with the sheaf $\mathcal{D}_{k/c} \subset \mathcal{D}_{k,c}$ of relative differential operators. We now use the homomorphism $S_{k,c}$ to lift the Virasoro operators to $\mathcal{D}_{k,c}$:

5.2.1. Theorem. There is an isomorphism of sheaves on $\mathcal{M}_{g,1,2n}$,

$$\Pi^n_\mathcal{D}_{k/c} \otimes \mathcal{O}_{\mathcal{O}} \mathcal{D}_{ck} \cong \Pi^n_\mathcal{D}_{k,c},$$

compatible with the inclusions of the two factors as subalgebras of $\Pi^n_\mathcal{D}_{k,c}$.

5.2.2. Proof. Consider the isomorphism of $(\mathfrak{g},\tilde{G}(\mathcal{O}))$–modules

$$S_{k,c} : V^n_k \otimes \text{Vir}^{2n}_{ck} \longrightarrow V^n_{k,c}$$

of Proposition 3.3.4. Twisting by the $\tilde{G}(\mathcal{O})$–torsor $\text{Bun}_{G,G} \rightarrow \text{Bun}_{G,g,1,2n}$ we obtain an isomorphism

$$V^n_k \otimes \text{Vir}^{2n}_{ck} \longrightarrow V^n_{k,c}$$

of sheaves. This morphism restricts to $S^n_{k,c}$ on $|0) \otimes \text{Vir}^{2n}_{ck}$, and hence descends to an isomorphism

$$\Pi^n_\mathcal{V}_k \otimes \text{Vir}^{2n}_{ck} \longrightarrow \Pi^n_\mathcal{V}_{k,c}.$$
The inclusion \( V_k \to V_{k,c} \) of \( \hat{g} \)-modules gives rise to a natural algebra inclusion

\[
\Delta(V^n_k) = D_{k_{\overline{2n}}} \to \Delta(V^n_{k,c}) = D_{k,c}
\]

from the sheaf of twisted differential operators relative to the moduli of curves \( \mathcal{M}_{g,1,2n} \) to the full sheaf of twisted differential operators. We define the map

\[
\Pi^* \mathcal{D}_{k_{\overline{2n}}} \otimes \mathcal{D}_{ck} \to \Pi^* \mathcal{D}_{k,c}
\]

as one generated by the homomorphisms from \( \Pi^* \mathcal{D}_{k_{\overline{2n}}} \) and \( \mathcal{D}_{ck} \) to \( \Pi^* \mathcal{D}_{k,c} \).

This map is surjective because it comes from the composition

\[
\mathcal{V}_k^n \otimes \mathcal{V}_{ir_{ck}}^{2n} \to \mathcal{V}_{k,c}^n \to \Delta(V^n_{k,c}),
\]

which is surjective because the first map is an isomorphism by Proposition 5.1.4 and the second map is surjective by definition. It remains to check that this map is injective. We have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{V}_k^n \otimes \mathcal{V}_{ir_{ck}}^{2n} & \xrightarrow{\sim} & \mathcal{V}_{k,c}^n \\
\downarrow & & \downarrow \\
\Delta(V^n_k) \otimes \Delta(V_{ir_{ck}}^{2n}) & \to & \Delta(V^n_{k,c})
\end{array}
\]

For an \( R \)-point of \( \text{Bun}_{G,G,1,n} \) as in the proof of Theorem 5.1.4, the kernel of the right vertical map over Spec \( R \) is the image of the action of the Lie algebra \( \mathfrak{g}_{\text{out}} \) in \( \mathcal{V}_{k,c}^n \vert_R \approx \mathcal{V}_k \vert_R \otimes \mathcal{V}_{ir_{ck}}^{2n} \vert_R \). But

\[
\mathfrak{g}_{\text{out}} \cap (\mathcal{V}_k \vert_R \otimes \mathcal{V}_{ir_{ck}}^{2n} \vert_R) \subset (\mathfrak{g}_{\text{out}} \cap \mathcal{V}_k \vert_R) \otimes \mathcal{V}_{ir_{ck}}^{2n} \vert_R + \mathcal{V}_k \vert_R \otimes (\mathfrak{g}_{\text{out}} \cap \mathcal{V}_{ir_{ck}}^{2n} \vert_R),
\]

and so the right hand side is in the kernel of the projection \( \mathcal{V}_k \vert_R \otimes \mathcal{V}_{ir_{ck}}^{2n} \vert_R \to \Delta(V^n_k) \otimes \Delta(V_{ir_{ck}}^{2n}) \vert_R \). This proves that our map is injective and hence an isomorphism.

5.2.3. Remark. The theorem may be applied to describe differential operators on the moduli of curves and bundles without level structure, following Theorem 5.1.4. However, it is known (BD1) that for \( k \neq h^0 \), there are no non-constant global twisted differential operators on \( \text{Bun}_G(X) \), in other words, \( \Pi^* \mathcal{D}_{k_{\overline{2n}}} = \mathcal{O}_{\mathcal{M}_g} \). Thus in this case the pushforward of twisted differential operators \( \mathcal{D}_{k,c} \) to \( \mathcal{M}_g \) is simply identified with the differential operators \( \mathcal{D}_{ck} \) on \( \mathcal{M}_g \).

5.3. Heat operators and projectively flat connections. Let \( \pi : M \to S \) be a smooth projective morphism with connected fibers and \( \mathcal{L} \to M \) a line bundle. Let \( \mathcal{D}_L \) be the sheaf of twisted differential operators on \( \mathcal{L} \).

5.3.1. Definition. A heat operator on \( \mathcal{L} \) relative to \( \pi \) is a lifting of the identity map \( \text{id} : \Theta_S \to \Theta_S \) to a sheaf homomorphism \( \mathcal{H} : \Theta_S \to \pi_* \mathcal{D}_L \leq 1_S \) to differential operators, which are of order one along \( S \), such that the corresponding map \( \Theta_S \to \pi_* \mathcal{D}_L \leq 1_S / \mathcal{O}_S \) is a Lie algebra homomorphism.

Suppose that the sheaf \( \pi_* \mathcal{L} \) is locally free, i.e., is a sheaf of sections of a vector bundle on \( S \). Then a heat operator gives rise to a projectively flat connection on this vector bundle (see [W, BK, Fa1]).
Consider the morphism \( \Pi : \text{Bun}_{G,g} \to \mathcal{M}_g \). Recall that \( \mathcal{E} = \mathcal{L}_{k,0} \) is the line bundle on \( \text{Bun}_{G,g} \) whose restriction to each \( \text{Bun}_G(X) \) is the ample generator of the Picard group of \( \text{Bun}_G(X) \). For any \( k \in \mathbb{Z}_+ \) the sheaf \( \Pi_k \mathcal{E}^k \) is locally free, and its fiber at a curve \( X \in \mathcal{M}_g \) is the vector space \( H^0(\text{Bun}_G(X), \mathcal{E}^k) \) of non-abelian theta functions of weight \( k \). It is well-known that the corresponding vector bundle (which we will also denote by \( \Pi_k \mathcal{E}^k \)), possesses a projectively flat connection \([\Pi_2\text {Fal}][\text{BK}]\). This connection may be constructed using a heat operator on \( \mathcal{E}^k \).

5.3.2. **Theorem.** \([\text{BK}]\) For any \( k \geq 0 \) the sheaf \( \Pi_k \mathcal{E}^k \) on \( \mathcal{M}_g \) possesses a unique flat projective connection given by a heat operator on \( \mathcal{E}^k \).

5.3.3. The existence and uniqueness of this projective connection are deduced for \( \mathcal{L} \) satisfying the vanishing of the composition

\[
\Theta_S \to R^1\pi_*D_\mathcal{L} \to R^1\pi_*D_{\mathcal{L}/S}
\]

and the identification \( \pi_*D_{\mathcal{L}/S} = \Theta_S \). The connection was also constructed by Hitchin \([\Pi_2]\) and Faltings \([\text{Fal}]\) by different means.

The connection on non-abelian theta functions has been explicitly identified in \([\text{La}]\) with the connection on the (dual of) the space of conformal blocks for the basic integrable representation \( L_k(\mathfrak{g}) \) of \( \mathfrak{g} \) at level \( k \in \mathbb{Z}_+ \). The latter connection, known as the KZB or WZW connection (and specializing to the Knizhnik-Zamolodchikov connection in genus zero), is defined following the general procedure of \([\text{FB}]\), Ch. 16. Namely, for any conformal vertex algebra \( V \) we defined a twisted \( \mathcal{D} \)-module of coinvariants on \( \mathcal{M}_g \). Its fibers are the spaces of coinvariants \( H(X, x, V) \) (see Section 5.1.3), and the action of the sheaf \( \mathcal{D}_c \) of differential operators (where \( c \) is the central charge of \( V \)) comes from the action of the Virasoro algebra on \( V \). In our case, we take as \( V \) the module \( L_k(\mathfrak{g}) \).

For any positive integer \( k \) this module is a conformal vertex algebra with central charge \( c(k) = k \dim \mathfrak{g}/(k + h^+) \), with the conformal structure defined by the Segal-Sugawara vector. In this case the sheaf of coinvariants is locally free as an \( \mathcal{O} \)-module, and so it is the sheaf of sections of a vector bundle with a projectively flat connection. The sheaf of sections of its dual vector bundle (whose fibers are the spaces of conformal blocks of \( L_k(\mathfrak{g}) \)) is therefore also a twisted \( \mathcal{D} \)-module, more precisely, a \( \mathcal{D}_{-c(k)} \)-module. The corresponding projectively flat connection on the bundle of conformal blocks is the KZB connection.

On the other hand, in Theorem 5.1.4 we produced homomorphisms \( S^n_{k,0} : \mathcal{D}_{-c(k)} \to \Pi^a_{k,0} \). Their restrictions \( S_{k,0}|_{\leq 1} \) to \( (\mathcal{D}_{-c(k)})_{\leq 1} \) give us heat operators on \( \mathcal{E}^k = \mathcal{L}_{k,0} \). But the sheaf \( \Pi_k \mathcal{E}^k \) is isomorphic to the sheaf of conformal blocks on \( \mathcal{M}_{g,1} \) corresponding to \( L_k(\mathfrak{g}) \) \([\text{BL}][\text{KNR}][\text{Fa2}][\text{Lc}]\). Under this identification, the projectively flat connection on \( \Pi_k \mathcal{E}^k \) obtained from the heat operators \( S_{k,0}|_{\leq 1} \) tautologically coincides with the KZB connection, because both connections are constructed by applying the Segal-Sugawara construction. Thus, we obtain

5.3.4. **Proposition.**

1. For every \( k \in \mathbb{Z}_+ \), the heat operator defining the projectively flat connection on the sheaf \( \Pi_k \mathcal{E}^k \) of non-abelian theta functions over \( \mathcal{M}_g \) is given by the restriction of the Segal-Sugawara map \( S_{k,0}|_{\leq 1} : (\mathcal{D}_{-c(k)})_{\leq 1} \to \Pi^a_{k,0} \).
(2) For any $n$, $S_{k,0}^{n} \leq 1$ gives heat operators defining a projectively flat connection on $\Pi_{k,c(k)}^{c} L$ over $\mathcal{M}_{g,1,2n}$.

(3) Under the identification between $\Pi^{c} C_{k}$ and the sheaf of conformal blocks on $\mathcal{M}_{g,1}$ corresponding to the integrable representation of $\hat{g}$ of level $k$ and highest weight $0$, the KZB connection on the sheaf of conformal blocks is given by the heat operators $S_{k,0}^{c} \leq 1$ on $C_{k}$.

In the same way we obtain heat operators for any level structure (where the morphisms $\Pi^{n}$ are no longer projective). Note also that we can replace $C_{k}$ by $L_{k,c}$ for any integer $c$, which have isomorphic restrictions to $Bun_{G}(X)$ for any $X$, and obtain analogous projectively flat connections.

6. Classical Limits.

In this section we describe the limits of the (suitably rescaled) Segal-Sugawara operators at the critical level $k = -h^{\vee}$, and in the classical limit $k, c \to \infty$. In the former case the algebra of twisted differential operators $D_{c,k}$ degenerates into the Poisson algebra of functions on the space of curves with projective structure, and the construction gives twisted differential operators on $Bun_{G,G}$ which are vertical (i.e. preserve $Bun_{G}(X)$ for fixed $X$) and commute. We identify these operators with the quadratic part of the Beilinson–Drinfeld quantization of the Hitchin system. When the level and charge become infinite, both sides of the construction become commutative (Poisson) algebras, and the Segal–Sugawara construction is interpreted as a map from the moduli of extended connections to the moduli of projective structures or quadratic differentials. We interpret this map as defining a symplectic connection over the moduli of pointed curves on the moduli spaces of connections with arbitrary poles. We also sketch the interpretation of this connection as a new Hamiltonian form of the equations of isomonodromic deformation.

6.1. The Critical level. Recall that we assume throughout that the group $G$ is simply-connected.

By the general formalism of localization (Proposition 2.4.1), we have an algebra homomorphism $(V^{n}_{k})^{G(0)} \to \Gamma(Bun_{G}(X,x,n),D_{k})$ from the $G(0)$–invariants of the vacuum (which are the endomorphisms of the vacuum representation) to global differential operators on the moduli of bundles. For general $k$ this gives only scalars, but at the critical level $k = -h^{\vee}$ the space $V^{G(0)}_{k}$ becomes very large. In [FF] (see also [Fr1, Fr2]), Feigin and Frenkel identify the algebra $V^{G(0)}_{-h^{\vee}}$ canonically with the ring of functions $\mathbb{C} [\mathcal{O} G_{G_{\mu}} (D)]$ on the space of opers on the disc, for the Langlands dual group $G_{\mu}$ of $G$. Opers, introduced in [BD] by Beilinson and Drinfeld, are $G_{\mu}$–bundles equipped with a Borel reduction and a connection, which satisfies a strict form of Griffiths transversality. For $G_{\mu} = PSL_{2}$ (so $G = SL_{2}$), opers are identified with projective structures $\mathcal{O} proj (X)$.

In fact for arbitrary $G$ there is a natural projection $\mathcal{O} G_{\mu} (X) \to \mathcal{O} proj (X)$, which identifies the space of opers with the affine space for the vector space Hitch$G(X)$ induced from the affine space $\mathcal{O} proj (X)$ for quadratic differentials $H^{0}(X, \Omega^{2}_{X}) \subset Hitch_{G}(X)$.
(Similar considerations apply to any level vacuum representations $V^n_k$, replacing regular opers by opers with singularities, which form an affine space for the meromorphic version of the Hitchin space.)

Thus there is a homomorphism
$$\mathbb{C}[\mathcal{O}_{PG^\vee}(D)] \longrightarrow \Gamma(\text{Bun}_G(X), \mathcal{D}_{-h^\vee}).$$

Beilinson and Drinfeld show that this homomorphism factors through functions on global opers $\mathbb{C}[\mathcal{O}_{PG^\vee}(X)]$, is independent of the choice of $x \in X$ used in localization, and gives rise to an isomorphism
$$\mathbb{C}[\mathcal{O}_{PG^\vee}(X)] \cong \Gamma(\text{Bun}_G(X), \mathcal{D}_{-h^\vee}).$$

We wish to compare the restriction of this homomorphism to the subalgebra $\mathbb{C}[\text{Proj}(X)]$ of $\mathbb{C}[\mathcal{O}_{PG^\vee}(X)]$ with the critical level limit of the Segal-Sugawara construction. Let $\mathcal{O}_{PG^\vee} \rightarrow \mathfrak{M}_g$ denote the moduli stack of curves with $G^\vee$–opers.

Recall the notation $\mu_g = h^\vee \dim \mathfrak{g}$.

6.1.1. Theorem. (1) For $c \in \mathbb{C}$, the homomorphism $(k + h^\vee)S^n_{k,c}$ is regular at $k = -h^\vee$, defining an algebra homomorphism
$$S^n_{-h^\vee,c} : \mathcal{O}(T^*_{\mu_g} \mathfrak{M}_{g,1,2n}) \longrightarrow \Pi^n \mathcal{D}_{-h^\vee,c}.$$

(2) The homomorphism $S^n_{-h^\vee,c}$ is the restriction of the Beilinson–Drinfeld homomorphism to projective structures: we have a commutative diagram
$$\begin{array}{ccc}
\mathcal{O}(\text{Proj}_{\mu_g}^\mu) & \longrightarrow & \Pi^* \mathcal{D}_{-h^\vee,c} \\
\downarrow & & \uparrow \\
\mathcal{O}(\mathcal{O}_{PG^\vee}) & \rightarrow & \Pi^* \mathcal{D}_{-h^\vee/\mathfrak{M}_g}
\end{array}$$

6.1.2. Proof. As described in Section 3.4.1 (see Proposition 3.4.2), the rescaled Segal–Sugawara operators $\mathfrak{S}_m = (k + h^\vee)\mathfrak{S}_m$ are regular at $k = -h^\vee$, and define a homomorphism of vertex algebras. We may now repeat the constructions of Section 5 leading to Theorem 5.1.4 for the rescaled Sugawara operators. Note that the classical vacuum representations $\mathfrak{Vir}_{\mu_g}$ are (commutative) vertex algebras – the vertex Poisson structure is not used in the definition of coinvariants – and for commutative vertex algebras, the comparison of coinvariants for a generating set and the algebra it generates is obvious. Identifying the localization of $\mathfrak{Vir}_{\mu_g}^{2n}$ with $\mathcal{O}(T^*_{\mu_g} \mathfrak{M}_{g,1,2n}) = \mathcal{O}(\text{Proj}_{\mu_g}^{\mu})$, we obtain the first assertion.

Moreover, by Section 3.3.4 the critical Segal-Sugawara homomorphism factors as follows:
$$\mathfrak{S}_{-h^\vee,c} : \mathfrak{Vir}_{\mu_g}^{2n} \hookrightarrow (V^n_{-h^\vee})^{G_\eta(0)} \hookrightarrow (V^n_{-h^\vee,c})^{G_\eta(0)}$$

(since $\mathfrak{S} = S_{-2}$ lands inside $V_{-h^\vee} \subset V_{-h^\vee,c}$). It follows that the localized map on twisted differential operators also factors through the localization of $V_{-h^\vee}$, which is the sheaf of relative differential operators. Thus the critical Segal-Sugawara construction...
is part of the localization of the $G(O)$–invariants in the vacuum representation, giving the Beilinson–Drinfeld operators.

6.2. **Infinite level.** Recall from Section 4 (see Proposition 4.1.4) that the $\mathcal{C}$–twisted cotangent bundle of the moduli stack of bundles $\text{Bun}_G(X)$ is isomorphic to the moduli stack $\mathcal{C}onn_G(X)$ of bundles with regular connections on $X$, while the twisted cotangent bundle of the moduli of bundles with $n$–jet of trivialization $\text{Bun}_G(X, x, n)$ is the moduli $\mathcal{C}onn_G(X, x, n)$ of connections having at most $n$–th order poles at $x$. Similarly, we may consider moduli $\text{Bun}_G(X, x_1, \ldots, x_m, n_1, \ldots, n_m)$ of bundles with several marked points and jets of coordinates, whose twisted cotangent bundles are identified with moduli of connections with poles of the corresponding orders. As elsewhere, we restrict to the one–point case for notational simplicity though all constructions carry over to the multipoint case in a straightforward fashion.

By virtue of their identification as twisted cotangent bundles, the moduli of meromorphic connections on a fixed curve carry canonical (holomorphic) symplectic structures. As we vary $X$ and $x$, the moduli stack $\mathcal{C}onn_{G,1,n}$ forms a relative twisted cotangent bundle to $\text{Bun}_{G,g,1,n}$ over $\mathcal{M}_{g,1,2n}$. To obtain a symplectic variety, we consider the absolute twisted cotangent bundles of $\text{Bun}_{G,g,1,n}$. The twisted cotangent bundle corresponding to the line bundle $\mathcal{L}_{\lambda,\mu}$ is the moduli of extended connections $\mathcal{C}onn_{\text{Bun}_{G,g,1,n}}(\mathcal{L}_{\lambda,\mu}) = \mathcal{C}onn_{G,g,1,n}$ defined in Section 4.3. In the limit of infinite level the sheaves of twisted differential operators on $\text{Bun}_{G,g,1,n}$ and $\mathcal{M}_{g,1,2n}$ degenerate to the commutative (and hence Poisson) algebra of functions on the moduli of extended connections and projective structures, respectively. Therefore it is convenient to reinterpret the infinite limit of the Segal-Sugawara homomorphism as a morphism between these moduli spaces (rather than a homomorphism between the corresponding algebras of functions), and examine its Poisson properties.

As in Section 3.4.4 we let $k, c \to \infty$ by introducing auxiliary parameters $\lambda, \mu$ with $\frac{\lambda}{k} = \frac{\mu}{c}$. Then the homomorphism $\frac{\lambda^2}{k} \mathcal{C}_k \mathcal{C}_k$ is regular as $k, c \to \infty$, as is (for $\lambda \neq 0$) the isomorphism $\mathcal{E}_k^n : V_k^n \otimes \mathcal{V}ir_k^n \to V_k^n$. We thus obtain classical limits of Theorem 5.1.4 and Theorem 5.2.1.

Let $\text{Quad}_{g,1,n} = T^*\mathcal{M}_{g,1,1}$ and $\text{Higgs}_{G,g,1,n} = T^*_{/\mathcal{M}_{g,1,2n}} \text{Bun}_{G,g,1,n}$ denote the moduli stacks of curves equipped with a quadratic differential and bundles with Higgs field, having at most $n$–th order pole at the marked point, respectively. Let $\text{Hitch}_{G,g,1,n}$ denote the target of the Hitchin map for $\text{Higgs}_{G,g,1,n}$, i.e.,

$$\text{Hitch}_{G,g,1,n} = \bigoplus_{i=1}^{\ell} \Gamma(X, \Omega^2(nx)^{d_i+1}),$$

where $\ell = \text{rank} \mathfrak{g}$ and $d_i$ is the $i$th exponent of $\mathfrak{g}$.

Recall from Section 3.4.4 that a commutative algebra homomorphism of Poisson algebras (and the corresponding morphism of spaces) is called $\lambda$–Poisson if it rescales the Poisson bracket by $\lambda$.

6.2.1. **Theorem.**
(1) For every $\lambda, \mu \in \mathbb{C}$ there is a $\lambda$-Poisson homomorphism

$$\overline{\Sigma}_{\lambda, \mu} : \mathcal{O}(T^*_{\lambda, \mu} \mathfrak{M}_{g,1,1,2n}) \to \Pi^n_{\lambda, \mu} \mathcal{O}(T^*_{\lambda, \mu} \text{Bun}_{G,g,1,n})$$

Equivalently, for every $\lambda, \mu \in \mathbb{C}$ there is a $\lambda$-Poisson map

$$\Pi^n_{\lambda, \mu} : \text{ExConn}_{G,g,1,n}^{\lambda, \mu} \to \text{Proj}_{g,1,2n}^{\lambda, \mu}$$

lifting $\Pi^n : \text{Bun}_{G,g,1,n} \to \mathfrak{M}_{g,1,2n}$.

(2) For $\lambda \neq 0$, there is a canonical (non-affine) product decomposition over the moduli of curves

$$\text{ExConn}_{G,g,1,n}^{\lambda, \mu} \cong \text{Conn}_{G,g,1,n}^{\lambda, \mu} \times \text{Proj}_{g,1,2n}^{\lambda, \mu}.$$ 

(3) For $\lambda = \mu = 0$, $\Pi^n_{\lambda, \mu}$ factors through the quadratic Hitchin map

$$\begin{align*}
T^* \text{Bun}_{G,g,1,n} & \xrightarrow{\Pi^n_{0,0}} \text{Quad}_{g,1,2n} \\
\text{Higgs}_{G,g,1,n} & \xrightarrow{\text{Hitch}} \text{Hitch}_{G,g,1,n}
\end{align*}$$

6.2.2. Proof. As in Theorem 6.1.1 the construction of the classical Segal-Sugawara homomorphism $\overline{\Sigma}_{\lambda, \mu}$ is identical to the proof of Theorem 5.1.4 appealing to Proposition 3.4.5 for the description of the rescaled vertex algebra homomorphism. The localization of $\overline{V}^2_{\nu, \mu}^{\lambda, \mu}$ is $\mathcal{O}(T^*_{\mu} \mathfrak{M}_{g,1,2n})$ and that of $\overline{V}^2_{\lambda, \mu}^{\lambda, \mu}$ is $\mathcal{O}(T^*_{\lambda, \mu} \text{Bun}_{G,g,1,2n})$, while the Poisson bracket is rescaled by $\lambda$, by Proposition 3.4.5. The homomorphism $\overline{\Sigma}_{\lambda, \mu}$ of $\mathcal{O}_{\text{Bun}_{G,g}}$-algebras defines, upon taking Spec over the moduli of curves, a morphism $T^*_{\lambda, \mu} \text{Bun}_{G,g,1,n} \to T^*_{\lambda, \mu} \mathfrak{M}_{g,1,2n}$ relative to $\mathfrak{M}_{g,1,2n}$, hence the geometric reformulation.

The morphism in (2) is given in components by the natural projection (4.3.1) from extended connections to connections and the map $\Pi^n_{\lambda, \mu}$. The spaces $\text{ExConn}_{G,g,1,n}^{\lambda, \mu}$ and $\text{Proj}_{g,1,2n}^{\lambda, \mu} \times \text{Conn}_{G,g,1,n}^{\lambda, \mu}$ are both torsors for quadratic differentials over $\text{Conn}_{G,g,1,n}$.

It follows from the explicit form of the classical Segal-Sugawara vector that the resulting map of torsors over the space of connections is an isomorphism (up to rescaling by $\lambda$). Equivalently, the assertion (2) is the classical limit of Theorem 5.2.1 and may be proved identically, replacing $s_{b,c}$ by its rescaled version.

For $\lambda = \mu = 0$, the image of $\overline{\Sigma}_{0,0}^{\mu} : \overline{V}^2_{\nu, \mu}^{\lambda, \mu}$ lies strictly in $\overline{V}_{0} \subset \overline{V}_{0,0}^{\nu, \mu}$. It follows that the image of $\mathcal{O}(T^* \mathfrak{M}_{g,1,2n}) = \mathcal{O}(\text{Quad}_{g,1,2n})$ in $\mathcal{O}(T^* \text{Bun}_{G,g,1,n})$ consists of functions which are pulled back from $T^*_{\text{Bun}_{G,g,1,2n}}$. Therefore the map $\Pi^n_{0,0}$ descends to $T^*_{\text{Bun}_{G,g,1,n}}$. Restricting to fibers over a fixed curve $X$ we obtain a map $T^* \text{Bun}_{G}(X, x, n) \to T^* \mathfrak{M}_{g,1,2n}|_X = H^0(X, \Omega^2(2nx))$.

The quadratic Hitchin map is the map $T^* \text{Bun}_{G}(X, x, n) \to H^0(X, \Omega^2(2nx))$ sending a Higgs bundle $(\mathcal{P}, \eta), \eta \in H^0(X, \mathfrak{g}_p \otimes \Omega(nx))$ to $\frac{1}{2} \text{tr} \eta^2$. Using the pairing on $L\mathfrak{g}$, this is identified with the map defined by the Segal-Sugawara vector

$$S_{-2} = \frac{1}{2} \sum_a J_{a,-1}^a J_{a,-1}.$$
as desired.

6.2.3. Remark. In [BZB1], the product decomposition (2) is described concretely in terms of kernel functions along the diagonal (the necessary translation from vertex algebra language to kernel language is described in Chapter 7 of [FB]). Moreover, it is related, for \( \mu \neq 0 \), to the classical constructions of projective structures on Riemann surfaces using theta functions due to Klein and Wirtinger. Composing the projection \( \Pi_{\lambda,\mu} \) with the canonical meromorphic sections of the twisted cotangent bundles over \( \text{Bun}_{G,g} \) given by non-abelian theta functions (see also [BZB2]), this gives a construction of interesting rational maps from the moduli of bundles to the spaces of projective structures, and more generally, opers.

Now we discuss various applications of the above results.

6.2.4. The Bloch–Esnault–Beilinson connection. Let \( \mathcal{X} \rightarrow S \) be a smooth family of curves, and \( E \) a vector bundle on \( \mathcal{X} \). The vector bundle \( E \) defines a line bundle \( \mathcal{C}(E) \) on \( S \), the \( c_2 \)-line bundle defined in [De] (whose Chern class is the pushforward of \( c_2(E) \)) which differs from the determinant of cohomology of \( E \) by the \( n \)th power of the Hodge line bundle on \( S \). In fact, this line bundle is identified canonically with the pullback of \( \mathcal{C} \) to \( S \) from the map \( S \rightarrow \text{Bun}_{G,g} \) classifying \( E \) and \( X \) (see [BK]). Bloch–Esnault and Beilinson [BE] construct a connection on \( \mathcal{C}(E) \) from a regular connection \( \nabla_{\mathcal{X}/S} \) on \( E \) relative to \( S \). It is easy to see that this connection can be recovered from Theorem 6.2.1 (2) (more precisely, its straightforward version for \( G = GL_n \) and unpointed curves).

Let \( \lambda = 1, \mu = 0 \), so that \( \mathcal{L}_{\lambda,\mu} = \mathcal{C} \) and \( \text{Conn}(\mathcal{C}) = \text{ExConn}_{G,g}^{1,0} \) over \( \text{Bun}_{G,g} \). We have a decomposition \( \text{Conn}(\mathcal{C}) = \text{Conn}_{G,g} \times T^*\mathfrak{M}_g \), and hence a canonical lifting from \( \text{Conn}_{G,g} \) to \( \text{Conn}(\mathcal{C}) \), lying over the zero section of \( T^*\mathfrak{M}_g \). When pulled back to \( S \) by the classifying map \( S \rightarrow \text{Conn}_{G,g} \) of \( (E, \nabla_{\mathcal{X}/S}) \), this gives a connection on \( \mathcal{C}(E) \) over \( S \). The compatibility with the construction of Bloch–Esnault–Beilinson follows for example from the uniqueness of the splitting \( \text{Conn}_{G,g} \rightarrow \text{ExConn}_{G,g}^{1,0} \), due to the absence of one–forms on \( \mathfrak{M}_g \).

6.2.5. The Segal-Sugawara symplectic connection. Suppose \( f : N \rightarrow M \) is a smooth Poisson map of symplectic varieties. It then follows that \( N \) carries a flat symplectic connection over \( M \), i.e., a Lie algebra lifting of vector fields on \( M \) to vector fields on \( N \) (defining a foliation on \( N \) transversal to \( f \)), which preserve the symplectic form on fibers (see [GLS] for a discussion of symplectic connections). Namely, we represent local vector fields on \( M \) by Hamiltonian functions, pull back to functions on \( N \) and take the symplectic gradient. The Poisson property of \( f \) guarantees the flatness and symplectic properties of the connection. (In the algebraic category, this defines the structure of \( \mathcal{D} \)-scheme or crystal of schemes on \( N \) over \( M \), compatible with the symplectic structure.)

The morphism \( \Pi_{1,\mu}^n \) of Theorem 6.2.4 is such a Poisson morphism of symplectic varieties, and hence defines a symplectic connection (or crystal structure) on \( \text{ExConn}_{G,g,1,n}^{1,\mu} \) over \( \text{Proj}_{g,1,2n}^\mu \). When \( \mu = 0 \) we may reduce this connection as follows. First we restrict to the zero section \( \mathfrak{M}_{g,1,2n} \hookrightarrow \text{Proj}_{g,1,2n}^{0} \) of the cotangent bundle to obtain a connection on \( \text{ExConn}_{G,g,1,n}^{1,0} \) over \( \mathfrak{M}_{g,1,2n} \). Next, the connection respects the product
decomposition Theorem 6.2.1 (2) relative to $\mathcal{M}_{g,1,2n}$, so that we obtain a flat symplectic connection on the moduli space $\text{Conn}_{G,g,1,n}$ of $G$–bundles with connections over the moduli $\mathcal{M}_{g,1,2n}$ of decorated curves (with respect to the relative symplectic structure):

6.2.6. Corollary. The projection $\Pi_{1,0}$ defines a flat symplectic connection on the moduli stack $\text{Conn}_{G,g,1,n}$ over $\mathcal{M}_{g,1,2n}$.

6.2.7. Time–dependent hamiltonians. The structure of flat symplectic connection on a relatively symplectic $P \to M$ variety can also be encoded in a closed two–form $\Omega$ on $P$, restricting to the symplectic form on the fibers. The connection is then defined by the null–foliation of $\Omega$. If $P$ is locally a product, then this structure can be encoded in the data of Hamiltonian functions on $P$ that are allowed to depend on the “times” $M$. Thus, following [M], we may consider the data of such a form as the general structure of time–dependent (or non–autonomous) Hamiltonian system. In our case $P = \text{Conn}_{G,g,1,n}$, we thus have three equivalent formulations of a non-autonomous Hamiltonian system, with times given by the moduli $\mathcal{M}_{g,1,2n}$ of decorated curves: the Hamiltonian functions $S_{1,0}$ (or more precisely the functions on $\text{Conn}_{G,g,1,n}$ obtained from local vector fields on $\mathcal{M}_{g,1,2n}$, considered as linear functions on $T^*\mathcal{M}_{g,1,2n}$); the symplectic connection induced by $\Pi_{1,0}$; and the two–form $\Omega$ on $\text{Conn}_{G,g,1,n}$ obtained by restricting the symplectic form on $E\times \text{Conn}_{1,0}^{1,0}$ $G,g,1,n$ under the embedding (as in Section 6.2.4) along the zero section of $T^*\mathcal{M}_{g,1,2n}$.

6.3. Isomonodromy. In this final section, we describe the algebraic definition of isomonodromic deformation of arbitrary meromorphic connections over the moduli of decorated curves and sketch its identification with the Segal-Sugawara symplectic connection (as well as the compatibility with the analytic iso–Stokes connections of [JMU, Ma, Bo]). We thus obtain an algebraic time–dependent Hamiltonian description of the isomonodromy equations. It also follows that the isomonodromy Hamiltonians are classical limits of the heat operators defining the KZB equations, and are non–autonomous deformations of the quadratic Hitchin hamiltonians.

6.3.1. Isomonodromic deformation. The moduli spaces of $G$–bundles with regular connection $\text{Conn}_{G,g}$ carry a flat connection (crystal structure) over the moduli of curves, namely the connection of isomonodromic deformation or nonabelian Gauss–Manin connection (see [S]). This connection is a manifestation of the topological description of connections with regular singularities on a Riemann surface as representations of the fundamental group, which does not change under holomorphic deformations. Namely, given a family $X \to S$ of Riemann surfaces and a bundle with (holomorphic, hence flat) connection on one fiber $X$, there is a unique extension of the connection to nearby fibers so that the monodromy representation does not change. The families of connections relative to $S$ one obtains this way are uniquely characterized as those families which admit an absolute flat connection over $X$, the total space of the family. This is captured algebraically in the crystalline interpretation of flat connections: a flat connection on a variety admits a unique flat extension to an arbitrary nilpotent thickening of the variety. Thus algebraically one defines families of flat connections relative to a base to be isomonodromic if they may be extended to an absolute flat connection. It follows
that the moduli spaces of flat connections on varieties carry a crystal structure over the deformation space of the underlying variety – the nonabelian Gauss-Manin connection of Simpson [Si].

More generally, there is an algebraic connection (crystal or $\mathcal{D}$–scheme structure) on the moduli stack of $\text{Conn}_{G,g,1,n}$ of meromorphic connections over curves with formal coordinates, which is the pullback of $\text{Conn}_{G,g,1,n}$ to the moduli of pointed curves with coordinates $\mathfrak{M}_{g,1}$. This connection is defined by “fixing” connections around their poles and deforming them isomonodromically on the complement, combining the crystalline description of isomonodromy and the Virasoro uniformization of the moduli of curves. Namely, given a family $(\tilde{X},\tilde{x},\tilde{t}) \in \mathfrak{M}_{g,1}(S)$ of pointed curves with formal coordinate over $S$, the spectrum of an Artinian local ring, and a connection $(\mathcal{P},\nabla)$ on the special fiber $X$ with pole at $x$ of order at most $n$, we must produce a canonical extension of $(\mathcal{P},\nabla)$ to $\tilde{X}$. As explained in [FB], Ch. 15, p.282, any such deformation $(\tilde{X},\tilde{x})$ of pointed curves is given by “regluing” $X/x$ and the formal disc $D_{x}$ around $x$ using the action of $\text{Aut} \mathcal{K}(S)$.

More precisely, we fix an identification of $X/x$ with $(X/x) \times S$ and “glue” it to $D_{x} \times S$ using an automorphism of the punctured disc over $S$. Now, given $(\mathcal{P},\nabla)$, we define an (absolute) flat connection on $\tilde{X}$ by extending our connection $(\mathcal{P},\nabla)$ trivially (as a product, with the trivial connection along the second factor) onto $(X/x) \times S$ and onto $D_{x} \times S$. Note that since the connection on the nilpotent thickening $\tilde{X} \setminus \tilde{x} \simeq (X/x) \times S$ of $X \setminus x$ is flat, it is uniquely determined by $(\mathcal{P},\nabla)$, independently of the trivialization of the deformation – is the isomonodromic deformation of $(\mathcal{P},\nabla)|_{X/x}$ over $S$ (in particular the stabilizer of $(\tilde{X},\tilde{x})$ in $\text{Aut} \mathcal{K}(S)$ does not change the deformed connection.) This defines the isomonodromy connection on $\mathfrak{M}_{g,1}$.

6.3.2. Proposition. The Segal–Sugawara symplectic connection on $\text{Conn}_{G,g,1,n}$ over $\mathfrak{M}_{g,1,2n}$ pulls back to the isomonodromy connection on $\text{Conn}_{G,g,1,n}$.

6.3.3. Proof. The compatibility between the two connections follows from the local description of the isomonodromy connection through the action of $\text{Aut} \mathcal{K}$ on meromorphic connections, and the description of the classical Segal–Sugawara operators as the corresponding Hamiltonians. Namely, recall from [FB] that the ind–scheme $\text{Conn}_{G}(D^{\circ})$ of connections on the trivial $G$–bundle on the punctured disc is identified with $\widehat{\mathfrak{g}}_{1}^{*}$, the level one hyperplane in the dual to the affine Kac–Moody algebra. Moreover, this identification is equivariant with respect to the natural actions of $G(\mathcal{K})$ and $\text{Aut} \mathcal{K}$, describing the transformation of connections under gauge transformations and changes of coordinates. These actions are Hamiltonian, in an appropriately completed sense. Namely, the Fourier coefficients of fields from the vertex Poisson algebra $\overline{\mathcal{V}}_{1}(\mathfrak{g})$ form the Lie subalgebra of local functionals on $\widehat{\mathfrak{g}}_{1}^{*}$ in the Poisson algebra of all functionals on $\widehat{\mathfrak{g}}_{1}^{*}$. By the Segal–Sugawara construction, this Lie algebra contains as Lie subalgebra $\text{Der} \mathcal{K}$ (as the Fourier coefficients $\overline{L}_{n}$ of the classical limit of the conformal vector), thereby providing Hamiltonians for the action of $\text{Aut} \mathcal{K}$. The construction of the isomonodromy connection on $\mathfrak{M}_{g,1}$ above is expressed in local coordinates (as a flow on meromorphic connections on the disc at $x$) by this action of $\text{Aut} \mathcal{K}$ on connections. Likewise, the
Segal-Sugawara connection is defined by Hamiltonian functions, which are reductions of the classical Segal-Sugawara operators to $\mathcal{C}onn_{G,g,1,n}$. Thus the compatibility of the two connections is a consequence of the local statement.

6.3.4. Remark: analytic isomonodromy and Stokes data. In the complex analytic setting, one can extend to irregular connections the description of regular–singular connections by topological monodromy data, by introducing Stokes data describing the transitions between asymptotic fundamental solutions to the connection in different sectors. One can thereby define an “iso–Stokes” generalization of the isomonodromy equations, i.e., an analytic symplectic connection on moduli of irregular holomorphic connections (see [IMU, Ma] and recently [Bo, Kr]). Thus, one has an isomonodromic deformation with more “times” than just the moduli of pointed curves (one may also vary the most singular term of the connection).

It is easy to see our algebraic isomonodromic connection (with respect to motions of pointed curves) agrees with the analytic one. Since we can choose the gluing transformations in $\text{Aut}\mathcal{K}$ used to describe a given deformation to be convergent rather than formal, it follows that we are not altering the isomorphism type of the connection on a small analytic disc around $x$, and hence all Stokes data are automatically preserved. Thus it follows that the iso–Stokes connection on $\mathcal{C}onn_{G,g,1,n}$ is in fact algebraic.

6.3.5. Conclusion: KZB, isomonodromy and Hitchin. It follows from Proposition 6.3.2 and Theorem 6.2.1 that the moduli spaces of meromorphic connections carry algebraic isomonodromy equations, which form a time–dependent Hamiltonian system. Moreover, the isomonodromy hamiltonians are non–autonomous deformations of the quadratic Hitchin hamiltonians (see also [Kr]). Most significantly, it follows that the heat operators $S^0_{n,k,c}$ defining the KZB connection on the bundles of conformal block quantize the isomonodromy Hamiltonians $S^0_{1,0}$. This makes precise the picture developed in [Il] and [LO] of the non–stationary Schrödinger equations defining the KZB connection on conformal blocks as quantizations of the isomonodromy equations or non–autonomous Hitchin systems. In genus zero we thus generalize the result of [Res, Har] that the KZ connection on spaces of conformal blocks on the $n$–punctured sphere, viewed as a system of multi–time–dependent Schrödinger equations, quantizes the Schlesinger equation, describing isomonodromic deformation of connections with regular singularities on the sphere, which itself is a time–dependent deformation of the Gaudin system (the Hitchin system corresponding to the $n$–punctured sphere). In genus one we obtain a similar relation between KZB equations, the elliptic form of the Painlevé equations and the elliptic Calogero–Moser system as in [LO].

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