Differential cohomology seminar overview

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1 Motivation for differential cohomology

1.1 Observation (Simons–Sullivan [9, §1]). Let $M$ be a manifold. Then we have exact sequences

\[
\begin{align*}
H^{k-1}(M; \mathbb{R}/\mathbb{Z}) & \xrightarrow{-\beta} H^k(M; \mathbb{Z}) \\
\Omega^{k-1}(M) & \xrightarrow{d} \Omega^k_{\text{ct}}(M),
\end{align*}
\]

where the top sequence is the Bockstein sequence associated to the short exact sequence

\[0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0,\]

and we are identifying singular and de Rham cohomology via the de Rham isomorphism $H^*_dR(M) \equiv H^*(M; \mathbb{R})$.

The top sequence is ‘purely homotopy-theoretic’ in nature, while the bottom sequence is ‘purely geometric’ in nature (e.g., the functor $\Omega^k_{\text{ct}}$ is not homotopy-invariant).

1.3 Question. Can we fill (1.2) in with an invariant $\tilde{H}^k(M; \mathbb{Z})$ in red that better blends homotopy theory and geometry, and makes the diagonals exact?

Now let us attempt to provide a satisfactory answer to Question 1.3 when $k = 1$. 
1.4 Attempt (for $k=1$). Let $M$ be a manifold. Consider the abelian group $C^\infty(M, \mathbb{R}/\mathbb{Z})$ of smooth functions to the circle (with the group structure defined pointwise). Recall that the inclusion $C^\infty(M, \mathbb{R}/\mathbb{Z}) \subset \text{Map}(M, \mathbb{R}/\mathbb{Z})$ from the space of smooth maps to the space of all maps is a homotopy equivalence. So since the circle is $1$-truncated, $C^\infty(M, \mathbb{R}/\mathbb{Z})$ is also $1$-truncated.

Since $\mathbb{R}/\mathbb{Z}$ is a $K(\mathbb{Z},1)$, we see that $\pi_0 C^\infty(M, \mathbb{R}/\mathbb{Z}) \cong H_1(M;\mathbb{Z})$.

In particular, we have a surjection $\pi_0 : C^\infty(M, \mathbb{R}/\mathbb{Z}) \twoheadrightarrow H_1(M;\mathbb{Z})$. Also notice that

$$\pi_1 C^\infty(M, \mathbb{R}/\mathbb{Z}) \cong \pi_0 \text{Map}_*(S^1, C^\infty(M, \mathbb{R}/\mathbb{Z})) \cong \pi_0 \text{Map}(M, \Omega(\mathbb{R}/\mathbb{Z})) \cong \pi_0 \text{Map}(M, \Omega(\mathbb{R}/\mathbb{Z})) \cong \pi_0 \text{Map}(M, \Omega(\mathbb{R}/\mathbb{Z})) \cong H^0(M;\mathbb{Z}).$$

1.5 Construction. Define a curvature map $\text{curv} : C^\infty(M, \mathbb{R}/\mathbb{Z}) \to \Omega^1_{cl}(M)$ by

$$\text{curv}(f) := f^*(\text{vol}),$$

where $\text{vol}$ is the standard volume form on $S^1 \cong \mathbb{R}/\mathbb{Z}$.

The kernel of $\text{curv}$ consists of the locally constant maps $M \to \mathbb{R}/\mathbb{Z}$, i.e.,

$$\ker(\text{curv}) \cong H^0(M;\mathbb{R}/\mathbb{Z}).$$

Note that the curvature map is not surjective:

$$\text{im}(\text{curv}) = \{ \alpha \in \Omega^1_{cl}(M) \mid \int_{S^1} \alpha \in \mathbb{Z} \text{ for every embedding } S^1 \hookrightarrow M \}.$$
1.9. These maps give rise to a commutative diagram with exact diagonals

\[
\begin{array}{ccc}
\mathbb{H}^0(M; \mathbb{R}/\mathbb{Z}) & \xrightarrow{-\beta} & \mathbb{H}^1(M; \mathbb{Z}) \\
\mathbb{H}_d^0(M) & \xrightarrow{\pi_0} & \mathbb{C}^{\infty}(M, \mathbb{R}/\mathbb{Z}) \\
\mathbb{H}_d^1(M) & \xrightarrow{\text{curv}} & \Omega^1_c(M) \\
\Omega^0(M) & \xrightarrow{d} & \Omega^1_c(M) \\
0 & & 0
\end{array}
\]

The diagonals become short exact sequences if we replace \(\Omega^0(M)\) by \(\Omega^0(M)/\Omega^0_c(M)\) and \(\Omega^1_c(M)\) by \(\Omega^1_c(M)_\mathbb{Z}\):

\[
\begin{array}{ccc}
0 & \xrightarrow{0} & 0 \\
\mathbb{H}^0(M; \mathbb{R}/\mathbb{Z}) & \xrightarrow{-\beta} & \mathbb{H}^1(M; \mathbb{Z}) \\
\mathbb{H}_d^0(M) & \xrightarrow{\pi_0} & \mathbb{C}^{\infty}(M, \mathbb{R}/\mathbb{Z}) \\
\mathbb{H}_d^1(M) & \xrightarrow{\text{curv}} & \Omega^1_c(M)_\mathbb{Z} \\
\Omega^0(M)/\Omega^0_c(M)_\mathbb{Z} & \xrightarrow{d} & \Omega^1_c(M)_\mathbb{Z} \\
0 & & 0
\end{array}
\]

1.10. The takeaway is that in Question 1.3, we should really replace \(\Omega^{k-1}(M) / \text{im}(d)\) by \(\Omega^{k-1}(M)/\Omega^0_c(M)_\mathbb{Z}\) and \(\Omega^1_c(M)\) by \(\Omega^1_c(M)_\mathbb{Z}\) and ask for the diagonal sequences to be short exact.

2. Differential characters

We now present a unified approach to defining the 'differential cohomology' groups \(\tilde{H}^*(M; \mathbb{Z})\) due to Cheeger–Simons [3]. We follow Bär and Becker's exposition on differential characters [1, Part I, §5].

2.1 Notation. Let \(M\) be a manifold and \(i \geq 0\) an integer. We write \(C^i_\text{sm}(M; \mathbb{Z})\) for the abelian group of smooth (integer-valued) chains on \(M\). We write \(Z^i_\text{sm}(M; \mathbb{Z}) \subset C^i_\text{sm}(M; \mathbb{Z})\) for the subgroup of smooth cycles.

2.2 Definition (Cheeger–Simons [3, §1]). Let \(k \geq 1\) be an integer and \(M\) a manifold. A degree \(k\) differential character on \(M\) is a homomorphism \(\chi: Z^k_\text{sm}(M; \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}\) such that there exists a \(k\)-form \(\omega(\chi) \in \Omega^k(M)\) with the property that

\[
\chi(\partial c) = \int_c \omega(\chi) \mod \mathbb{Z}
\]
for every $c \in C_k^m(M; \mathbb{Z})$. We write

$$\hat{\text{H}}^k(M; \mathbb{Z}) \subset \text{Hom}_\mathbb{Z}(Z_k^m(M; \mathbb{Z}), \mathbb{R}/\mathbb{Z})$$

for the abelian group of degree $k$ differential characters on $M$.

It follows that $\omega(\chi)$ is unique and closed. Moreover, $\omega(\chi)$ has integral periods. The form $\omega(\chi)$ is called the curvature of $\chi$, and we have a curvature map

$$\text{curv} : \hat{\text{H}}^k(M; \mathbb{Z}) \to \Omega^k(M)$$

$$\chi \mapsto \omega(\chi)$$

with image $\Omega^k(M)_\mathbb{Z}$ those closed $k$-forms with integral periods.

2.3 Warning. The indexing convention used here is off by $1$ from the indexing convention in [3, §1]. However, this indexing convention is better and is what was later adopted by Simons–Sullivan [9, §1].

2.4 Remark. When $k = 0$, the diagram (1.2) is quite degenerate, and it will be convenient to define $\hat{\text{H}}^0(M; \mathbb{Z}) = \text{H}^0(M; \mathbb{Z})$.

Now let us construct maps to fill in the ‘differential cohomology’ diagram (1.2).

2.5 Construction. There is a characteristic class map $\text{ch} : \hat{\text{H}}^k(M; \mathbb{Z}) \to \text{H}^k(M; \mathbb{Z})$ defined as follows. Since $Z_k^m(M; \mathbb{Z})$ is a free $\mathbb{Z}$-module and the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is an epimorphism, any homomorphism $\chi : Z_k^m(M; \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ lifts to a homomorphism $\tilde{\chi} : Z_k^m(M; \mathbb{Z}) \to \mathbb{R}$. Now define a homomorphism $I(\tilde{\chi}) : C_k^m(M; \mathbb{Z}) \to \mathbb{Z}$ by the assignment

$$c \mapsto -\tilde{\chi}(\partial c) + \int_c \text{curv}(\chi) .$$

Since $\text{curv}(\chi)$ is closed, $I(\tilde{\chi})$ defines a cocycle. Moreover, $I(\tilde{\chi})$ takes integral values, and the cohomology class $[I(\tilde{\chi})] \in \text{H}^k(M; \mathbb{Z})$ does not depend on the choice of lift $\tilde{\chi}$. We define $\text{ch}$ by the assignment

$$\text{ch} : \hat{\text{H}}^k(M; \mathbb{Z}) \to \text{H}^k(M; \mathbb{Z})$$

$$\chi \mapsto [I(\tilde{\chi})] .$$

2.6 Construction. Consider the universal coefficient sequence

$$0 \to \text{Ext}^j_\mathbb{Z}(\text{H}_{i-1}(M; \mathbb{Z}), \mathbb{R}/\mathbb{Z}) \to \text{H}^i(M; \mathbb{R}/\mathbb{Z}) \xrightarrow{\langle -,- \rangle} \text{Hom}_\mathbb{Z}(\text{H}_i(M; \mathbb{Z}), \mathbb{R}/\mathbb{Z}) \to 0 ,$$

where the morphism $\langle -,- \rangle$ is given by sending a the class of a cocycle $u$ to the homomorphism

$$\langle u,- \rangle : \text{H}_j(M; \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$$

$$[z] \mapsto u(z) .$$

Since the circle $\mathbb{R}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, for any $\mathbb{Z}$-module $A$ and integer $j > 0$, we have $\text{Ext}^j_\mathbb{Z}(A, \mathbb{R}/\mathbb{Z}) = 0$. In particular, $\langle -,- \rangle$ is an isomorphism.
Setting \( i = k - 1 \), precomposition with the quotient map \( Z^m_{k-1}(M; \mathbb{Z}) \to H_{k-1}(M; \mathbb{Z}) \) defines an injection

\[
H^i(M; \mathbb{R}/\mathbb{Z}) \to \text{Hom}_\mathbb{Z}(H_i(M; \mathbb{Z}), \mathbb{R}/\mathbb{Z}) \to \text{Hom}_\mathbb{Z}(Z^m_{k-1}(M; \mathbb{Z}), \mathbb{R}/\mathbb{Z}).
\]

It follows from the definitions that this factors through \( \hat{H}^k(M; \mathbb{Z}) \). We simply denote this composite by \( \langle -, - \rangle : H^{k-1}(M; \mathbb{R}/\mathbb{Z}) \to \hat{H}^k(M; \mathbb{Z}) \).

2.7 Construction. Define a map \( i : \Omega^{k-1}(M) \to \hat{H}^k(M; \mathbb{Z}) \) by setting

\[
i(\omega)(z) := \exp \left( 2\pi i \int_z \omega \right)
\]

for every smooth \((k - 1)\)-cycle \( z \). By Stokes' Theorem, we see that \( \text{curv}(i(\omega)) = d\omega \).

We have an \( \mathbb{R} \)-valued lift of \( i(\omega) \) given by setting

\[
i(\omega)(z) := \int_z \omega
\]

for every smooth \((k - 1)\)-cycle \( z \). So by Stokes' Theorem we have

\[
I(i(\omega))(c) = -i(\omega)(\partial c) + \int_c \text{curv}(i(\omega)) = -\int_{\partial c} \omega + \int_c d\omega = 0
\]

for every smooth \( k \)-chain \( c \). Hence \( n(i) = 0 \).

We see that \( i : \Omega^{k-1}(M) \to \hat{H}^k(M; \mathbb{Z}) \) has kernel those closed forms \( \omega \) such that \( \int_z \omega \) is an integer for all \( z \in Z^m_{k-1}(M; \mathbb{Z}) \). That is,

\[
\ker(i) = \Omega^{k-1}_c(M) \mathbb{Z}
\]

is the group of closed \((k - 1)\)-forms with integral periods. Hence \( i \) descends to an injection

\[
i : \Omega^{k-1}(M)/\Omega^{k-1}_c(M) \mathbb{Z} \to \hat{H}^k(\cdot; \mathbb{Z}).
\]

2.8 Notation. Write \( \text{Man} \) for the category of smooth manifolds and \( \text{GrAb} \) for the category of graded abelian groups.

2.9 Theorem (Simons–Sullivan \([9, \text{Theorem 1.1}]\)). There is an essentially unique functor \( \hat{H}^*(-; \mathbb{Z}) : \text{Man}^{op} \to \text{GrAb} \) equipped with natural transformations

\[
\langle -, - \rangle : H^{*-1}(-; \mathbb{R}/\mathbb{Z}) \to \hat{H}^*(-; \mathbb{Z}),
\]

\[
i : \Omega^{*-1}(M)/\Omega^{*-1}_c(M) \mathbb{Z} \to \hat{H}^*(-; \mathbb{Z}),
\]

\[
\text{ch} : \hat{H}^*(-; \mathbb{Z}) \to H^*(-; \mathbb{Z}),
\]

\[
\text{curv} : \hat{H}^*(-; \mathbb{Z}) \to \Omega^*_c(-; \mathbb{Z})
\]

and

\[
\hat{H}^*(-; \mathbb{Z}) \to \Omega^*_c(-; \mathbb{Z})
\]
filling in the ‘differential cohomology diagram’

\[
\begin{array}{c}
0 \\ H^{*-1}(M; \mathbb{R}/\mathbb{Z}) \\ H^{*}(M; \mathbb{Z}) \\ H_{dR}^{*}(M) \\ \Omega^{*-1}(M) \\ \Omega^{*}(M) \\
\end{array}
\begin{array}{c}
\beta \\ \beta \\ d \\ d \\ d \\
\end{array}
\begin{array}{c}
0 \\ H^{*}(M; \mathbb{Z}) \\ H_{dR}^{*}(M) \\ \Omega^{*}(M) \\ \Omega^{*}(M) \\
\end{array}
\begin{array}{c}
0 \\ 0 \\
\end{array}
\]

so that the diagonal sequences are exact.

Any functor \( \hat{H}^{*}(-; \mathbb{Z}) : \text{Man}^{op} \to \text{GrAb} \) satisfying these properties is called ordinary differential cohomology.

2.10 Remark (Deligne’s model). Motivated by Deligne cohomology in Hodge theory \([12, \S 12.3]\), we can consider the smooth version of the Deligne complex on a manifold \( M \). Write \( \mathbb{Z}_{D}(k) \) for the complex of sheaves on \( M \)

\[
0 \longrightarrow \mathbb{Z} \longrightarrow \Omega^{0} \longrightarrow \Omega^{1} \longrightarrow \cdots \longrightarrow \Omega^{k-1} \longrightarrow 0,
\]

where \( \Omega^{i} \) is in degree \( i + 1 \). The \( k \)th smooth Deligne cohomology group of \( M \) is the hypercohomology group \( H^{k}(M; \mathbb{Z}_{D}(k)) \). We will see later in the seminar that smooth Deligne cohomology agrees with ordinary differential cohomology \([2, \S 4.3; 6, \S 3]\).

2.11 Questions.

(2.11.1) Is there differential K-theory?

Yes! Simons–Sullivan tell a similar story, and define differential K-theory in terms of vector bundles with connection \([10; 11]\).

(2.11.2) What about differential [favorite cohomology theory]?

Also yes, but the theory is more complicated. The fundamental observation is that everything we’ve considered comes from a sheaf of abelian groups or chain complexes (which we regard as spectra) on the category of all smooth manifolds.

The category \( \text{Sh}((\text{Man}; \text{Sp}) \) has rich structure that gives rise to a ‘differential cohomology diagram’ associated to every object (see \([2, \S 3; 4]\)).

2.12 Remark. The category \( \text{Sh}((\text{Man}; \text{Set}) \) is really the right place for moduli spaces of manifolds to live, and both Banach \([5]\) and Fréchet manifolds \([7; 8, \text{Theorem 3.1.1; 13, Theorem A.1.5}] \) embed as full subcategories of \( \text{Sh}((\text{Man}; \text{Set}) \).
2.13 Applications. The following are some applications we will study throughout the semester.

(2.13.1) Lifting characteristic classes to differential cohomology. In particular, there are applications to the existence of conformal immersions [3, §6].


References


