The Segal–Sugawara construction

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1 Introduction

1.1 Reminders from previous talks

1.1.1 Definition. The (complex) Witt algebra is the complex Lie algebra $\mathfrak{Witt}$ of polynomial vector fields on $S^1$. Explicitly, $\mathfrak{Witt}$ has generators $L_m := i e^{i m \theta} \frac{d}{d \theta}$ for $m \in \mathbb{Z}$ with Lie bracket

$$[L_m, L_n] = (m - n)L_{m+n}$$

for all $m, n \in \mathbb{Z}$.

1.1.2. Ignoring regularity issues, the Witt algebra is the complexification of the Lie algebra of the group $\text{Diff}^+(S^1)$ of orientation-preserving diffeomorphisms of the circle.

1.1.3. We have that $H^2_{\text{Lie}}(\mathfrak{Witt}; \mathbb{C}) \cong \mathbb{C}$, so there is a 1-dimensional space of central extensions of the Witt algebra.
1.1.4 Definition. The (complex) Virasoro algebra $\text{Vir}$ is the central extension

$$1 \rightarrow C\text{chg} \rightarrow \text{Vir} \rightarrow \text{Witt} \rightarrow 1$$

of Witt with generators $L_m$ for $m \in \mathbb{Z}$ and a central element $\text{chg}$, and nontrivial Lie bracket given by

$$[L_m, L_n] := (m - n)L_{m+n} + \delta_{m,-n}m^3 - m \frac{m-1}{12} \text{chg}$$

for all $m, n \in \mathbb{Z}$.

We call the central element $\text{chg} \in \text{Vir}$ the central charge.

1.1.5. Again, ignoring regularity issues, the Virasoro algebra is the complexification of the Lie algebra of the Virasoro group $\tilde{\text{Diff}}^+(S^1)$.

1.2 Talk overview

1.2.1. Let $G$ be a simply-connected, simple, compact Lie group with Lie algebra $\mathfrak{g}$. Last time we looked at central extensions

$$1 \rightarrow S^1 \rightarrow \tilde{\mathcal{G}} \rightarrow \mathcal{G} \rightarrow 1$$

of the loop group $\mathcal{L}G := C^\infty(S^1, G)$.

The group $\tilde{\text{Diff}}^+(S^1)$ of orientation-preserving diffeomorphisms of the circle acts on $\mathcal{L}G$ by precomposition. So we might expect an action of the Virasoro group $\tilde{\text{Diff}}^+(S^1)$ on $\mathcal{L}G$. Last time we saw that even though there is not an action of $\tilde{\text{Diff}}^+(S^1)$ on $\mathcal{L}G$, roughly, the Virasoro group acts on any positive energy representation of $\mathcal{L}G$. However, the Virasoro action on positive energy representation of $\mathcal{L}G$ is very inexplicit, and we can only guarantee the existence of the Virasoro action up to ‘essential equivalence’, which is not actually an equivalence relation. In particular, the Pressley–Segal Theorem [8, Theorem 13.4.3] does not explicitly explain how the central circle $S^1 \subset \mathcal{L}G$ acts.

1.2.2 Goal. The goal of this talk is to explain the Lie algebra version of the Pressley–Segal Theorem, which gives an explicit representation of the Virasoro algebra on any ‘positive energy’ representation of the Kac–Moody algebra $\tilde{\mathcal{L}\mathfrak{g}}$ associated to a simple Lie algebra $\mathfrak{g}$ (over the complex numbers). We’ll be able to do this by writing down explicit universal formulas for ‘elements’ of the universal enveloping algebra $U(\tilde{\mathcal{L}\mathfrak{g}})$ that satisfy the Virasoro relations. The catch is that these universal formulas involve infinite sums, so they do not actually make sense as elements of $U(\tilde{\mathcal{L}\mathfrak{g}})$, but they do make sense whenever we act on a representation where only finitely many of the terms don’t act by zero (this is the ‘positive energy’ condition).

2 Loop algebras & Kac–Moody algebras

The first thing we need to explain in order to state the Segal–Sugawara construction is what the Kac–Moody algebra $\tilde{\mathcal{L}\mathfrak{g}}$ is. As the notation suggests, $\tilde{\mathcal{L}\mathfrak{g}}$ is the Lie algebra analog of the central extension $\mathcal{L}G$ of the loop group $\mathcal{L}G$ (with suitable finiteness hypotheses). Before talking about Kac–Moody algebras, we need to talk about loop algebras.
2.1 Loop algebras

2.1.1 Recollection. Let $\mathfrak{g}$ be a Lie algebra over a ring $R$, and let $S$ be an $R$-algebra. The base change $\mathfrak{g} \otimes_R S$ of $\mathfrak{g}$ to $S$ is the Lie algebra over $S$ with underlying $S$-module the base change $\mathfrak{g} \otimes_R S$ of the underlying $R$-module of $\mathfrak{g}$ to $S$ with Lie bracket extended from pure tensors from the formula

$$[X_1 \otimes s_1, X_2 \otimes s_2]_{\mathfrak{g} \otimes_R S} := [X_1, X_2]_{\mathfrak{g}} \otimes s_1 s_2.$$

2.1.2 Definition. Let $\mathfrak{g}$ be a complex Lie algebra. The loop algebra $L\mathfrak{g}$ of $\mathfrak{g}$ is the Lie algebra $L\mathfrak{g} := \mathfrak{g} \otimes C[[t^{\pm 1}]]$, regarded as a Lie algebra over $C$ (rather than $C[t^{\pm 1}]$).

2.1.3 Notation. Let $\mathfrak{g}$ be a complex Lie algebra, $X \in \mathfrak{g}$, and $m$ an integer. We write $X(m) := X \otimes t^m \in L\mathfrak{g}$.

2.1.4. If $\{u_i\}_{i \in I}$ is a Lie algebra basis for $\mathfrak{g}$, then

$$\{u_i(m)\}_{(i, m) \in I \times \mathbb{Z}}$$

is a basis for $L\mathfrak{g}$.

2.1.5 Remark. The loop algebra functor

$$L : \text{Lie}_C \to \text{Lie}_C$$

preserves finite products.

2.1.6 Recollection. A finite dimensional Lie algebra $\mathfrak{g}$ is simple if $\mathfrak{g}$ is not abelian and the only ideals of $\mathfrak{g}$ are $\mathfrak{g}$ and $0$.

2.1.7 Theorem (Garland [3, §§1 & 2]). If $\mathfrak{g}$ is a simple Lie algebra over $C$, then

$$H^2_{\text{Lie}}(L\mathfrak{g}; C) \cong C.$$

In particular, if $\mathfrak{g}$ is simple there is a 1-dimensional space of central extensions of $L\mathfrak{g}$.

2.2 Recollection on bilinear forms & semisimplicity

2.2.1 Notation. Let $\mathfrak{g}$ be a complex Lie algebra. We write $\text{ad} : \mathfrak{g} \to \text{End}_C(\mathfrak{g})$ for the adjoint representation, defined by

$$\text{ad}(X) := [X, -].$$

2.2.2 Example. A Lie algebra $\mathfrak{g}$ is abelian if and only if the adjoint representation of $\mathfrak{g}$ is trivial.
2.2.3 Recollection (Killing form). Let \( \mathfrak{g} \) be a finite dimensional Lie algebra. The Killing form on \( \mathfrak{g} \) is the bilinear form

\[
K_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \\
(X, Y) \mapsto \text{tr}(\text{ad}(X) \circ \text{ad}(Y)).
\]

The Killing form is symmetric and invariant in the sense that

\[
K_{\mathfrak{g}}([X, Y], Z) = K_{\mathfrak{g}}(X, [Y, Z])
\]

for all \( X, Y, Z \in \mathfrak{g} \).

2.2.4 Example. If \( \mathfrak{g} \) is a simple Lie algebra, then every invariant symmetric bilinear form on \( \mathfrak{g} \) is a \( \mathbb{C} \)-multiple of the Killing form \( K_{\mathfrak{g}} \). See [2] for a nice exposition of this fact.

2.2.5 Example. Let \( \mathfrak{a} \) be a finite dimensional abelian Lie algebra over \( \mathbb{C} \). Then since the adjoint representation of \( \mathfrak{a} \) is trivial, the Killing form of \( \mathfrak{a} \) is identically zero. Also note that every bilinear form on the underlying vector space of \( \mathfrak{a} \) is an invariant bilinear form on \( \mathfrak{a} \).

2.2.6 Proposition ([9, Chapter II, Theorems 2 & 4]). Let \( \mathfrak{g} \) be a finite dimensional complex Lie algebra. The following conditions are equivalent:

(2.2.6.1) The center of \( \mathfrak{g} \) is trivial.

(2.2.6.2) The only abelian ideal in \( \mathfrak{g} \) is 0.

(2.2.6.3) The Lie algebra \( \mathfrak{g} \) is isomorphic to a product of simple Lie algebras.

(2.2.6.4) Cartan–Killing criterion: the Killing form of \( \mathfrak{g} \) is nondegenerate.

If the equivalent conditions (2.2.6.2)–(2.2.6.4) are satisfied, we say that \( \mathfrak{g} \) is semisimple.

2.3 Kac–Moody algebras

Now we define the Lie algebra analogue of the central extension \( \tilde{L}G \) of the loop group \( L\mathbb{G} \). In addition to the Lie algebra \( \mathfrak{g} \), we need to input an invariant symmetric bilinear form on \( \mathfrak{g} \); the Killing form provides a canonical choice.

2.3.1 Definition. Let \( \mathfrak{g} \) be a Lie algebra over \( \mathbb{C} \) with invariant symmetric bilinear form \( B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \). The Kac–Moody algebra of \( \mathfrak{g} \) with respect to the form \( B \) is the central extension

\[
1 \longrightarrow \mathbb{C}c \longrightarrow \tilde{L}_B \mathfrak{g} \longrightarrow L\mathfrak{g} \longrightarrow 1
\]

with central element \( c \) and with Lie bracket extended from the relation

\[
[X^{\langle m \rangle}, Y^{\langle n \rangle}]_{\tilde{L}_B \mathfrak{g}} = [X^{\langle m \rangle}, Y^{\langle n \rangle}]_{\mathfrak{g}} + \delta_{m,-n}mB(X, Y)c
\]

\[
= [X, Y]^{\langle m + n \rangle} + \delta_{m,-n}mB(X, Y)c
\]

for all \( X, Y \in \mathfrak{g} \).
2.3.2. If \( \{ u_i \}_{i \in I} \) is a Lie algebra basis for \( \mathfrak{g} \), then
\[
\{ u_i(m) \}_{(i,m) \in I \times \mathbb{Z}} \cup \{ c \}
\]
is a basis for \( \tilde{\mathfrak{g}} \).

2.3.3 Remark. The Kac–Moody algebra \( \tilde{\mathfrak{g}} \) is usually denoted by \( \hat{\mathfrak{g}} \) and is also known as the affine Lie algebra of \( \mathfrak{g} \).

2.3.4 Remark. Let \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) be complex Lie algebras equipped with invariant symmetric bilinear forms
\[
B_1 : \mathfrak{g}_1 \times \mathfrak{g}_1 \rightarrow \mathbb{C} \quad \text{and} \quad B_2 : \mathfrak{g}_2 \times \mathfrak{g}_2 \rightarrow \mathbb{C}.
\]
Write \( B \) for the bilinear form on the product Lie algebra \( \mathfrak{g}_1 \times \mathfrak{g}_2 \) defined by
\[
B((x_1, x_2), (y_1, y_2)) := B_1(x_1, y_1) + B(x_2, y_2).
\]
Then we have a canonical isomorphism
\[
\tilde{\mathfrak{g}}(\mathfrak{g}_1 \times \mathfrak{g}_2) \cong \tilde{\mathfrak{g}}_1 \mathfrak{g}_1 \times \tilde{\mathfrak{g}}_2 \mathfrak{g}_2.
\]

3 The Segal–Sugawara construction

We now have enough of the background on Lie algebras to give a vague statement of the Segal–Sugawara construction.

3.0.1 Definition. Let \( \mathfrak{g} \) be a Lie algebra over \( \mathbb{C} \) and \( B \) an invariant symmetric bilinear form on \( \mathfrak{g} \). A representation \( \rho : \tilde{\mathfrak{g}} \rightarrow \text{End}_{\mathbb{C}}(V) \) has positive energy if for all \( v \in V \) and \( X \in \mathfrak{g} \) there exists an integer \( m > 0 \) such that
\[
\rho(X(m))v = 0.
\]

3.0.2 Remark. In the theory of Kac–Moody algebras, positive energy representations are more often called admissible. We have chosen the term 'positive energy' to align with the loop group terminology.

3.0.3 Theorem (Segal–Sugawara construction, vague formulation). Let \( \mathfrak{g} \) be an abelian or simple Lie algebra over \( \mathbb{C} \) and let \( B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C} \) be a nondegenerate invariant symmetric bilinear form on \( \mathfrak{g} \). Write \( \text{Cas}_B(\mathfrak{g}) \in \mathbb{U}(\mathfrak{g}) \) for the Casimir element of \( \mathfrak{g} \) with respect to the bilinear form \( B \). Let
\[
\rho : \tilde{\mathfrak{g}} \rightarrow \text{End}_{\mathbb{C}}(V)
\]
be a positive energy representation of \( \tilde{\mathfrak{g}} \) such that
\[
(3.3.4.1) \quad \text{the central element } c \in \tilde{\mathfrak{g}} \text{ acts by multiplication by a complex number } \ell,
\]
\[
(3.3.4.2) \quad \text{and the complex number } -\ell \text{ is not equal to } \lambda_B(G) := \frac{\text{tr}(\text{ad}(\text{Cas}_B(\mathfrak{g})))}{2 \dim(\mathfrak{g})}.
\]
Then there is an explicit action of the Virasoro algebra on \( V \) where the central charge \( \text{chg} \in \text{Vir} \) acts by multiplication by
\[
\ell \dim(\mathfrak{g}) \left/ \frac{\ell}{\ell + \lambda_B(\mathfrak{g})} \right.
\]
As special cases:

(3.3.4.3) If \( \mathfrak{g} \) is abelian, then \( \lambda_B(\mathfrak{g}) = 0 \) for any nondegenerate invariant symmetric bilinear form \( B \), and central charge \( \text{chg} \in \text{Vir} \) acts by multiplication by \( \dim(\mathfrak{g}) \).

(3.3.4.4) If \( \mathfrak{g} \) is simple and \( B \) is the normalization of the Killing form such that the long roots of \( \mathfrak{g} \) have square length 2, then \( \lambda_B(\mathfrak{g}) \) is a positive integer known as the dual Coxeter number of \( \mathfrak{g} \).

3.0.4. The complex number \( \ell \) in Theorem 3.3.4 is known as the level of the positive energy representation \( \rho \).

3.0.5 Goal. The goal for the rest of the talk is to explain this construction, the Casimir element \( \text{Cas}_B(\mathfrak{g}) \), and give a better description of the normalized trace \( \lambda_B(\mathfrak{g}) \) as an eigenvalue of \( \text{ad}(\text{Cas}_B(\mathfrak{g})) \).

3.1 Motivating case: the Heisenberg algebra

As motivation for the Segal–Sugawara construction, we start with the most simple case, where \( \mathfrak{g} \) is the 1-dimensional abelian Lie algebra. Since the constant \( \lambda_B(\mathfrak{g}) \) will be zero in this case, we can do this without yet introducing the Casimir element.

3.1.1 Definition. The Heisenberg algebra is the Kac–Moody algebra
\[
\text{Heis} := \hat{\mathfrak{c}}
\]
of the 1-dimensional abelian Lie algebra \( \mathfrak{c} \) with respect to the bilinear form \( \mathfrak{c} \times \mathfrak{c} \to \mathbb{C} \) given by multiplication.

3.1.2. Write \( u \in \mathfrak{c} \) for the element 1, which we regard as a basis for \( \mathfrak{c} \) as a 1-dimensional abelian Lie algebra. Then the Heisenberg algebra has generators \( \{c\} \cup \{u(m)\}_{m \in \mathbb{Z}} \), where \( c \) is central and the nontrivial bracket relation is given by
\[
[u(m), u(n)] = \delta_{m,-n}mc.
\]

3.1.3 Definition. Let \( \mu, h \in \mathbb{C} \). Write \( u \in \mathfrak{c} \) for the element 1, which we regard as a basis for \( \mathfrak{c} \) as a 1-dimensional abelian Lie algebra. The Fock representation \( \text{Fock}(\mu, h) \) is the representation of the Heisenberg algebra on the polynomial ring
\[
\text{Fock}(\mu, h) := \mathbb{C}[x_1, x_2, \ldots]
\]
in infinitely many variables, where
\[
c \mapsto h \text{id}
\]
\[
u(n) \mapsto \begin{cases} 
\frac{\partial}{\partial x_n}, & n > 0 \\
-hx_{-n}, & n < 0 \\
\mu \text{id}, & n = 0
\end{cases}
\]
The following fact about the irreducibility of Fock representations is easy:

3.1.4 Lemma ([6, Lemma 2.1]). Let \( \mu, h \in \mathbb{C} \). If \( h \neq 0 \), then the Heis-representation \( \text{Fock}(\mu, h) \) is irreducible.

3.1.5. If \( h = 0 \), then the constants \( \mathbb{C} \subset \text{Fock}(\mu, 0) \) are invariant.

3.1.6 Properties. The following are some important properties of the Fock representations of the Hiesenberg algebra.

(3.1.6.1) The elements \( u(0) \) and \( c \) of Heis act by multiplication.

(3.1.6.2) For every polynomial \( p \in \text{Fock}(\mu, h) \), there exists an integer \( n \gg 0 \) such that \( u(n)p = 0 \); let \( n \) be any positive such that the variable \( x_n \) does not appear in \( p \). That is, the Fock representation \( \text{Fock}(\mu, h) \) is ‘positive energy’ in the sense of Definition 3.0.1.

(3.1.6.3) For each integer \( n > 0 \), the element \( u(n) \) of Heis acts locally nilpotently on \( \text{Fock}(\mu, h) \).

Now we can give the Segal–Sugawara construction for the Fock representations of the Heisenberg algebra.

3.1.7 Construction (Virasoro action of Fock representations). For each integer \( m \in \mathbb{Z} \), define an infinite sum of elements of \( U(\text{Heis}) \) by

\[
L^S_m = \frac{1}{2} \sum_{j \in \mathbb{Z}} :u(-j)u(j + m):.
\]

Here, \( :u(-j)u(j + m): \) denotes the normal ordering on \( u(-j)u(j + m) \), defined by

\[
:u(-j)u(j + m): = \begin{cases} u(-j)u(j + m), & -j \leq j + m \\ u(j + m)u(-j), & -j \geq j + m. \end{cases}
\]

Explicitly,

\[
L^S_m = \begin{cases} \frac{1}{2}u(n)^2 + \sum_{j \geq 0} u(n - j)u(n + j), & m = 2n \\ \sum_{j \geq 0} u(n + 1 - j)u(n + j), & m = 2n + 1. \end{cases}
\]

The operators \( L^S_m \) are not well-defined elements of \( U(\text{Heis}) \), but since the Fock representations of Heis are positive energy (3.1.6.2), the operators \( L^S_m \) make sense as operators on \( \text{Fock}(\mu, h) \).

3.1.8 Theorem (Segal–Sugawara for \( \text{Fock}(\mu, 1) \) [6, Proposition 2.3]). Under the representation of Heis on the Fock space \( \text{Fock}(\mu, 1) \), the operators \( L^S_m \) on \( \text{Fock}(\mu, 1) \) satisfy the commutation relation

\[
[L^S_m, L^S_n] = (m - n)L^S_{m+n} + \delta_{m,n} \frac{m^3 - m}{12}.
\]
Hence the assignment

\[
\begin{align*}
\text{Vir} & \to \text{End}_\mathbb{C}(\text{Fock}(\mu,1)) \\
L_m & \mapsto L^m_m \\
\text{chg} & \mapsto \text{id}
\end{align*}
\]

is a Vir-representation with central charge 1.

3.1.9 Remark. To derive the Segal–Sugawara action on Fock(\mu, \hbar) for \hbar \neq 0, let \(L_m\) act by \(\frac{1}{\hbar} L^S_m\).


3.2 The Casimir element

In the general case, the idea is to try to mimic the formulas that we wrote down defining the operators on the Fock representations that satisfy the Virasoro relations. First, we need to explain the Casimir element and normalized trace \(\lambda_B(\mathfrak{g})\) appearing in Theorem 3.3.4.

3.2.1 Recollection. Let \(\mathfrak{g}\) be a finite dimensional Lie algebra over \(\mathbb{C}\) and let \(B\) be a nondegenerate invariant symmetric bilinear form on \(\mathfrak{g}\). The Casimir element \(\text{Cas}_B(\mathfrak{g})\) of \(\mathfrak{g}\) with respect to the form \(B\) is the element of the universal enveloping algebra \(U(\mathfrak{g})\) given by the image of \(\text{id}_{\mathfrak{g}}\) under the composite

\[
\text{End}_\mathbb{C}(\mathfrak{g}) \cong \mathfrak{g} \otimes \mathfrak{g}^\vee \to \mathfrak{g} \otimes \mathfrak{g} \to T_C(\mathfrak{g}) \to U(\mathfrak{g}).
\]

Here the isomorphism \(\mathfrak{g} \otimes \mathfrak{g}^\vee \to \mathfrak{g} \otimes \mathfrak{g}\) is the identity on the first factor and the isomorphism \(\mathfrak{g}^\vee \to \mathfrak{g}\) induced by the form \(B\) on the second factor, and \(T_C(\mathfrak{g})\) is the tensor algebra of \(\mathfrak{g}\) over \(\mathbb{C}\).

The following are some key properties that we need to know about the Casimir element:

(3.2.1.1) The Casimir element \(\text{Cas}_B(\mathfrak{g})\) is a central element of \(U(\mathfrak{g})\).

(3.2.1.2) If \(\{u_1, \ldots, u_d\}\) and \(\{u^1, \ldots, u^d\}\) are bases of \(\mathfrak{g}\) that are dual with respect to the bilinear form \(B\) in the sense that

\[
B(u_i, u^j) = \delta_{i,j},
\]

then \(\text{Cas}_B(\mathfrak{g}) = \sum_{i=1}^d u_i u^i\).

(3.2.1.3) Assume that \(\mathfrak{g}\) is simple. Then the Casimir element of the Killing form of \(\mathfrak{g}\) acts by the identity in the adjoint representation. Hence for any nondegenerate invariant symmetric bilinear form \(B\) on \(\mathfrak{g}\), the Casimir element \(\text{Cas}_B(\mathfrak{g})\) acts by scalar multiplication in the adjoint representation of \(\mathfrak{g}\). If \(B\) is the normalization of the Killing form on \(\mathfrak{g}\) such that long roots have square length 2, then in the adjoint representation \(\text{Cas}_B(\mathfrak{g})\) acts by multiplication by an even positive integer.
If $\mathfrak{g}$ is abelian, then since the adjoint representation of $\mathfrak{g}$ is trivial, for any non-degenerate invariant symmetric bilinear form $B$ on $\mathfrak{g}$ we have $\text{ad}(\text{Cas}_B(\mathfrak{g})) = 0$. In particular, in the adjoint representation $\text{Cas}_B(\mathfrak{g})$ acts by scalar multiplication.

Even though there are no Lie algebras that are both abelian and simple, it is important for us that both types of Lie algebras have the property that the Casimir element associated to any nondegenerate invariant symmetric bilinear form acts by scalar multiplication in the adjoint representation. In particular, if $\mathfrak{g}$ is abelian or simple, then $\text{ad}(\text{Cas}_B(\mathfrak{g}))$ only has exactly one eigenvalue.

3.2.2 Definition. Let $\mathfrak{g}$ be a finite dimensional abelian or simple Lie algebra over $\mathbb{C}$ and let $B: \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. Define a complex number $\lambda_B(\mathfrak{g})$ by

$$\lambda_B(\mathfrak{g}) := \frac{1}{2} \left( \text{eigenvalue of } \text{ad}(\text{Cas}_B(\mathfrak{g})) \right).$$

3.2.3. If $\dim(\mathfrak{g}) > 0$, then

$$\lambda_B(\mathfrak{g}) = \frac{\text{tr}(\text{ad}(\text{Cas}_B(\mathfrak{g})))}{2 \dim(\mathfrak{g})},$$

which aligns with the vague formulation of the Segal–Sugawara construction (Theorem 3.3.4).

3.2.4 Example. If $\mathfrak{g}$ is simple and $B$ is the normalization of the Killing form on $\mathfrak{g}$ such that long roots have square length 2, then $\lambda_B(\mathfrak{g})$ is a positive integer (3.2.1.3) known as the dual Coxeter number of $\mathfrak{g}$.

3.2.5 Example. If $\mathfrak{a}$ is an abelian Lie algebra, then for any nondegenerate invariant symmetric bilinear form $B$ on $\mathfrak{a}$, we have $\lambda_B(\mathfrak{a}) = 0$.

3.3 The general case

Now let us try 'the same' formula to write down a Virasoro action on positive energy representations of $\tilde{\mathfrak{g}}$ as we did for the Heisenberg algebra. The first modification is that we need to sum over a basis of $\mathfrak{g}$.

3.3.1 Construction. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$ and let $B: \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. Given a positive energy representation $\rho: \tilde{\mathfrak{g}} \to \text{End}_\mathbb{C}(V)$, for each integer $m \in \mathbb{Z}$ define

$$T_m^\rho := \frac{1}{2} \sum_{i=1}^d \sum_{j \in \mathbb{Z}} :\rho(u_i(-j)) \rho(u^i(j+m)):\ \in \text{End}_\mathbb{C}(V).$$

Note that even though the formula defining $T_m^\rho$ involves an infinite sum, since $\rho$ is a positive energy representation, for each $v \in V$, all but finitely many terms in the sum defining $T_m^\rho$ annihilate $v$. Hence $T_m^\rho$ is well-defined as an element of $\text{End}_\mathbb{C}(V)$. 
We used the letter ‘$T$’ instead of ‘$L$’ because the commutation relation is not quite right:

**3.3.2 Lemma** ([6, Theorem 10.1]). Let $\mathfrak{g}$ be a finite dimensional abelian or simple Lie algebra over $\mathbb{C}$ and let $B$ be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. For every positive energy representation $\rho : \tilde{\mathfrak{L}} \mathfrak{g} \to \text{End}_{\mathbb{C}}(V)$, we have the following commutation relation in $\text{End}_{\mathbb{C}}(V)$:

$$\begin{align*}
[T^\rho_m, T^\rho_n] &= (\rho(c) + \lambda_B(g))(m - n)T^\rho_{m+n} \\
&\quad + \delta_{m,-n} \dim(g) \frac{m^3 - m}{12} \rho(c)(\rho(c) + \lambda_B(g)).
\end{align*}$$

**3.3.3 Idea.** The naïve guess that the operators $T^\rho_m$ satisfy the Virasoro relations is not correct. However, if we could invert $\rho(c) + \lambda_B(g)$, then the operators

$$\frac{1}{\rho(c) + \lambda_B(g)} T^\rho_m$$

would satisfy the Virasoro relations. We can do this provided that the central element $c \in \tilde{\mathfrak{L}} \mathfrak{g}$ acts by a scalar $\ell$ on $V$, and $\ell \neq -\lambda_B(g)$.

**3.3.4 Theorem** (Segal–Sugawara construction [6, Corollary 10.1]). Let $\mathfrak{g}$ be a finite dimensional abelian or simple Lie algebra over $\mathbb{C}$ and let $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be a nondegenerate invariant symmetric bilinear form. Let $\rho : \tilde{\mathfrak{L}} \mathfrak{g} \to \text{End}_{\mathbb{C}}(V)$ be a positive energy representation of $\tilde{\mathfrak{L}} \mathfrak{g}$ such that

1. the central element $c \in \tilde{\mathfrak{L}} \mathfrak{g}$ acts by multiplication by a complex number $\ell$,
2. $\ell \neq -\lambda_B(g)$.

Choose bases $\{u_1, \ldots, u_d\}$ and $\{u^1, \ldots, u^d\}$ of $\mathfrak{g}$ that are dual with respect to the bilinear form $B$.

Then the assignment

$$L_m \mapsto L^\rho_m := \frac{1}{2(\ell + \lambda_B(g))} \sum_{i=1}^{d} \sum_{j \in \mathbb{Z}} :\rho(u_i(\langle -j \rangle))\rho(u^i(\langle j + m \rangle)):$$

extends to a Vir-representation on $V$ with central charge

$$\frac{\ell \dim(g)}{\ell + \lambda_B(g)}.$$  

That is, in $\text{End}_{\mathbb{C}}(V)$, the operators $L^\rho_m$ satisfy the commutation relation

$$[L^\rho_m, L^\rho_n] = (m - n)L^\rho_{m+n} + \delta_{m,-n} \frac{m^3 - m}{12} \frac{\ell \dim(g)}{\ell + \lambda_B(g)}.$$
3.3.5 Remark. For \( a, b \in \mathbb{Z} \), the sum \( \sum_{i=1}^{d} u_{i}(a)u_{i}^{\dagger}(b) \) is independent of the choice of basis \( \{ u_{1}, \ldots, u_{d} \} \) of \( \mathfrak{g} \). In particular, the operators \( L_{m}^{\rho} \) are independent of the choice of basis.

3.3.6 Remark. If \( \ell = -\lambda_{B}(\mathfrak{g}) \), then the formulas we wrote down for the Segal–Sugawara operators \( L_{m}^{\rho} \) do not make sense, and there is a fundamental difficulty in dealing with the ‘critical level’ \( \ell = -\lambda_{B}(\mathfrak{g}) \). At the critical level, the theory seems to resemble the positive characteristic situation rather than the classical one; see [5] for some discussion of this point.

3.3.7 Remark. In light of Remark 2.3.4, the Segal–Sugawara construction can be extended to the case where \( \mathfrak{g} \) is reductive, i.e., \( \mathfrak{g} \) decomposes as a product

\[
\mathfrak{g} \equiv \mathfrak{a} \times \mathfrak{g}_{1} \times \cdots \times \mathfrak{g}_{r},
\]

where \( \mathfrak{a} \) is an abelian Lie algebra and \( \mathfrak{g}_{1}, \ldots, \mathfrak{g}_{r} \) are simple Lie algebras. In this case, the central charge of the resulting Vir-representation is

\[
\dim(\mathfrak{a}) + \sum_{i=1}^{r} \frac{\ell_{i} \dim(\mathfrak{g}_{i})}{\ell_{i} + \lambda_{B}(\mathfrak{g}_{i})},
\]

where the central element of \( \mathfrak{a} \) acts by multiplication by a nonzero complex number and the central element of \( (\mathfrak{g}_{i}) \) acts by multiplication by \( \ell_{i} \in \mathbb{C} \), and \( \ell_{i} \neq -\lambda_{B}(\mathfrak{g}_{i}) \). This is rather useful as all of the ‘classical’ Lie algebras are reductive [7, Theorem 5.49]; see [6, Remark 10.3] for details.

3.3.8 Remark. The Segal–Sugawara construction is usually stated with the assumptions that \( \mathfrak{g} \) is simple and \( B \) is the normalization of the Killing form such that the long roots of \( \mathfrak{g} \) have square length 2 (so that \( \lambda_{B}(\mathfrak{g}) \) is the dual Coxeter number, often denoted by \( h^{\vee} \)). This is somewhat unfortunate; because the Killing form of an abelian Lie algebra is trivial, to include the abelian case (and the reductive extension) the ‘usual’ statement needs to be modified to include arbitrary nondegenerate invariant symmetric bilinear forms as in Theorem 3.3.4.

3.3.9 Remark. One of the motivations for the formula for the Segal–Sugawara operators \( L_{m}^{\rho} \) comes from the theory of vertex algebras. See [1, §3], in particular [1, Proposition 3.3.1], for more details on the relation to vertex algebras.

References


5. J. Humphreys, Math Overflow Question 25592: What role does the "dual Coxeter number" play in Lie theory (and should it be called the "Kac number")? MO:25592, May 2010.


