1. Introduction

In previous lectures, we discussed the spectral deformation theory of $p$-divisible groups. The main result we proved was (see [Lur16, Theorem 3.0.11]):

**Theorem 1.1.** Let $G_0$ be a nonstationary $p$-divisible group over a Noetherian $F$-finite $F_p$-algebra $R_0$. Then there is a universal deformation of $G_0$: in other words, there is a Noetherian connective $E_\infty$-ring $R_{G_0}$ equipped with a universal deformation $G$ of $G_0$.

In analogy with the classical story, one might hope that the universal deformation of a $p$-divisible formal group $G_0$ over a field $k$ of characteristic $p$ would give Morava $E$-theory $E(k, G_0)$ — but this is not true! Morava $E$-theory is 2-periodic, but $R_{G_0}$ is a connective $E_\infty$-ring.

The reason for this apparent failure can be boiled down to a very simple problem: we did not ask that these deformations of $G_0$ have anything to do with topology. At the moment, this a rather vague statement, but later in this lecture we will make it more precise. For now, let us illustrate with the concrete example of $G_0 = \mu_p^\infty$ (over an algebraically closed field $k$ of characteristic $p$). The Cartier dual of $G_0$ is just the constant group scheme $\mathbb{Q}_p/\mathbb{Z}_p$ (if $k$ was not algebraically closed, this would just be an étale group scheme), and the deformation theory of the constant group scheme is trivial. It follows that $\text{Def}_{\mu_p^\infty}$ is representable by $\text{Spf} S_p$, so that $R_{\mu_p^\infty} = S_p$, the $p$-complete sphere.

We already know that $E(k, \mu_p^\infty)$ is supposed to be $p$-adic $K$-theory, so we would like a way of constructing (via an algebro-geometric procedure) $K_p$ from $S_p$. To do this, we take a hint from a classical result of Snaith’s (see [Sna81]):

**Theorem 1.2 (Snaith).** There is an equivalence $\Sigma_+^{\infty} CP^\infty \lhd [\beta^{\pm 1}] \simeq K$.

There is therefore a canonical map of $E_\infty$-rings $\Sigma_+^{\infty} CP^\infty \to K$, given by localization at the Bott element.

**Remark 1.3.** This map of $E_\infty$-rings can be constructed without ever having to refer to Snaith’s theorem: the inclusion $CP^\infty \hookrightarrow GL \cdot K$ is adjoint to the $E_\infty$-ring map $\Sigma_+^{\infty} CP^\infty \to K$.

We are left with accomplishing the following two tasks:

1. Construct (again, via an algebro-geometric procedure) $\Sigma_+^{\infty} CP^\infty$ from $S_p$.
2. Define the Bott element in $\pi_2 \Sigma_+^{\infty} CP^\infty$.

---

1This just means that $G_0$ is classified by an unramified map $\text{Spec} R_0 \to M_{BT}$ over a ring $R_0$ with a finite Frobenius map $\phi : R_0 \to R_0$. 

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ORIENTATIONS OF DERIVED FORMAL GROUPS

SANATH DEVALAPURKAR
We will accomplish both of these tasks (and more) in this lecture, where \( S_p \) is replaced by a general \( \mathbb{E}_\infty \)-ring, and \( \mu_p \) is replaced by a general formal group. For the purpose of concreteness, we will illustrate (almost) everything with the example of the formal multiplicative group throughout these notes.

**Remark 1.4.** We used Snaith’s theorem as a motivating construction, but one can actually easily recover his result from the content of this and the following lectures.

### 2. Dualizing sheaves on formal groups

In the previous lecture, Robert defined the dualizing line of a formal group \( G_\mathbb{A} : \text{CAlg}_{\mathbb{R}} \to \text{Mod}_{\mathbb{R}}^\omega \) (with underlying formal hyperplane \( X = \Omega^\omega G_\mathbb{A} \)) over an \( \mathbb{E}_\infty \)-ring \( R \), with a fixed basepoint \( \eta \in X(R) \). This required us to be fairly careful: the naïve definition as the pullback \( \eta^! L_{X/R} \) of the cotangent complex is not sufficient. The primary issue with this construction is that if \( R \) is an ordinary ring, then \( \eta^! L_{X/R} \) is not concentrated in degree \( 0 \), so it does not agree with the cotangent space \( R \otimes_{O_X} \Omega_{O_X}/R \). These problems are remedied by the dualizing line, whose definition and key properties we will now recall.

We will fix an \( \mathbb{E}_\infty \)-ring \( R \) and a formal hyperplane (which will always be one-dimensional) \( X \) over \( R \), with a basepoint \( \eta \in X(\tau_{\geq 0} R) \). In all cases of interest, \( X \) will arise as \( \Omega^\omega G_\mathbb{A} \).

**Definition 2.1.** Define \( O_X(-\eta) \) by the cofiber sequence
\[
O_X(-\eta) \to O_X \xrightarrow{\eta} R;
\]
then the dualizing line \( \omega_{X, \eta} \) is defined to be \( O_X(-\eta) \otimes_{O_X} R \).

**Proposition 2.2.** The dualizing line satisfies the following properties:

1. \( \omega_{X, \eta} \) is an equivalence if and only if the map \( \omega_{X', \eta'} \) is an equivalence.
2. A map \( f : X \to X' \) of hyperplanes is an equivalence if and only if the map \( \omega_{X, \eta} \to \omega_{X', \eta'} \) is a fiber sequence of \( R \)-modules
\[
\Sigma \omega_{X, \eta} \to R \otimes_{O_X} R \xrightarrow{\text{aug}} R.
\]

**Remark 2.3.** When \( R \) is a classical ring, and \( X \) is a formal hyperplane over \( R \), we may identify \( \omega_{X, \eta} \) with \( \ker(\epsilon)/\ker(\epsilon)^2 \), where \( \epsilon : O_X \to R \) is the augmentation. This is exactly the cotangent space.

**Construction 2.4 (Linearization).** Using Proposition 2.2, we obtain a map, natural in the connective \( \mathbb{E}_\infty \)-ring-algebra \( A \):
\[
\Omega X(A) \xrightarrow{\text{linearization}} \text{Map}_{\text{CAlg}}(R \otimes_{O_X} R, A) \xrightarrow{\text{Map}_{\text{Mod}}(R \otimes_{O_X} R, A)} \text{Map}_{\text{Mod}}(\Sigma \omega_{X, \eta}, A).
\]

The linearization map is particularly important when \( A = \tau_{\geq 0} R \).

**Example 2.5.** The strict multiplicative group \( G_\mathbb{A} : \text{CAlg} \to \text{Mod}_{\mathbb{R}}^\omega \) is defined via
\[
G_\mathbb{A}(R) = \text{Map}_{\text{Sp}}(\mathbb{H}, \text{GL}_1(R)) \simeq \text{Map}_{\text{CAlg}}(\Sigma^\omega \mathbb{Z}, R).
\]
The last identification above shows that \( R \to \Omega^\omega G_\mathbb{A}(R) \) is represented by \( \text{Spec} \Sigma^\omega \mathbb{Z} \simeq \text{Spec} S[\mathbb{Z}^{\omega}] \). Of course, one can now define \( G_\mathbb{A} \) over any \( \mathbb{E}_\infty \)-ring by base change. Let
The third term is discrete. It follows that in the fiber sequence
\[ \tilde{G}_m \to G_m(R) \to G_m(R^{\text{red}}). \]
By construction, this is representable by \( S[t^\pm 1] / (t-1) \). Therefore,
\[ S \otimes_{\mathcal{O}_{\tilde{G}_m}} S \cong S \otimes \mathbb{Z}[t^\pm] \mathbb{Z} \cong \Sigma_+^\infty \mathbb{Z} \cong \Sigma^{\infty} S^1 \cong \Sigma^{\infty} S^1 \vee S. \]

By Proposition 2.2, we learn that \( \omega_{\tilde{G}_m} \cong S \). It follows that the diagram defining the linearization map becomes (our base scheme here is \( S \), so \( A \) is any connective \( E_\infty \)-ring)
\[
\begin{align*}
\Omega^{\infty+1} \tilde{G}_m(A) & \xrightarrow{\text{linearization}} \text{Map}_{\text{CAlg}}(\Sigma_+^\infty S^1, A) \xrightarrow{\Omega(\omega \otimes - \otimes -)} \text{Map}_{\text{sp}}(\Sigma_+^\infty S^1, A) \\
\Omega^{\infty+1} & \xrightarrow{\text{linearization}} \text{Map}_{\text{sp}}(\Sigma^{\infty} S^1, \Sigma^{-1} A) \xrightarrow{\Omega(\omega \otimes - \otimes -)} \text{Map}_{\text{sp}}(\Sigma^{\infty} S^1, A)
\end{align*}
\]
The linearization map is therefore aptly named.

3. Classifying Orientations

In order to proceed, we will need to recall a classical bit of algebraic topology; namely, the following statements are equivalent for a spectrum \( E \):

1. the Atiyah-Hirzebruch spectral sequence computing \( E_*(\mathbb{C}P^\infty) \) degenerates.
2. the canonical unit element of \( \tilde{E}_2^2(S^2) \cong E_2^2(*) \cong \pi_2 E \) lies in the image of \( \tilde{E}_2^2(\mathbb{C}P^\infty) \to \tilde{E}_2^2(S^2) \).

The unit element can be thought of as a pointed map \( S^2 \to \Omega^\infty E \) (however, this is dependent on the choice of a basepoint of \( S^2 \subseteq \mathbb{C}P^\infty \)). This motivates:

**Definition 3.1.** A preorientation of a formal hyperplane \( X \to \text{Spec} R \) is a pointed map \( S^2 \to X(\tau_{\geq 0}) \).

In particular, the space \( \text{Pre}(X) \) of preorientations is exactly \( \Omega^2 X(\tau_{\geq 0}) \). Note that this space is functorial in \( R \). The linearization map above gives a map:
\[
\text{Pre}(X) \cong \Omega(\Omega X(\tau_{\geq 0})) \to \Omega \text{Map}_{\text{Mod}}(\omega_{X, r}, \Sigma^{-1} R) \cong \text{Map}_{\text{Mod}}(\omega_{X, r}, \Sigma^{-2} R).
\]
The choice of a preorientation of \( X \) therefore determines a map \( \omega_{X, r} \to \Sigma^{-2} R \) of \( R \)-modules; this is called the Bott map.

If \( X \) arises as \( \Omega^\infty \circ G_\emptyset \) for some formal group \( G_\emptyset \), then
\[
\text{Pre}(G_\emptyset) \cong \Omega^{\infty+2} G_\emptyset(\tau_{\geq 0}) \cong \text{Map}_{\text{Mod}}(\Sigma^2 \mathbb{Z}, G_\emptyset(\tau_{\geq 0})).
\]

**Example 3.2.** By the above discussion, we know that \( \text{Pre}(\tilde{G}_m) \cong \text{Map}_{\text{Mod}}(\Sigma^2 \mathbb{Z}, \tilde{G}_m(\tau_{\geq 0})) \). In the fiber sequence
\[ \tilde{G}_m(\tau_{\geq 0}) \to G_m(\tau_{\geq 0}) \to G_m(\tau_{\emptyset}(R)^{\text{red}}), \]
the third term is discrete. It follows that
\[
\text{Pre}(\tilde{G}_m) \cong \text{Map}_{\text{Mod}}(\Sigma^2 \mathbb{Z}, G_m(\tau_{\geq 0})) \approx \text{Map}_{\text{CAlg}}(\Sigma^\infty \Omega^\infty \Sigma^2 \mathbb{Z}, R) = \text{Map}_{\text{CAlg}}(\Sigma^\infty \mathbb{C}P^\infty, R).
\]
Proposition 3.4. Let $R$ be an $\text{CAlg}$ functor of $R$.

Remark 3.3. Note that a preorientation of $X = \Omega^\infty \circ \hat G_m$ gives a map $\omega_{\hat G_m} \simeq R \to \Sigma^{-2} R$ of $R$-modules, i.e., an element of $\pi_2 R$.

This representability result holds in general:

**Proposition 3.4.** Let $R$ be an $E_{\infty}$-ring. Suppose $X$ is a formal hyperplane over $R$. The functor $\text{CAlg}_R \to \text{Top}$ given by $R \mapsto \text{Pre}(X_R)$ is representable by an affine scheme $\text{Spec}A$.

**Proof.** The functor $\Omega X : \text{CAlg}^\text{co}_{R} \to \text{Top}$ is corepresentable by the connective $E_{\infty}$-ring $B = R \otimes_{O_X} R$. We noted above that $\text{Pre}(X) \simeq \Omega^2 X(\tau_{\geq 2} R)$, so the functor in the proposition is corepresentable by the connective $E_{\infty}$-ring $A = R \otimes_{B} R$, as desired. \qed

**Remark 3.5.** In particular, there is an $E_{\infty}$-ring $A$ with a ring map $R \to A$ such that there is a universal preorientation of $X_A$. This gives a universal Bott map $\omega_{X_A, \eta} \to \Sigma^{-2} A$ of $A$-modules.

Let $E$ be an even periodic complex oriented $E_{\infty}$-ring; then $\hat G_0 = \text{Spf}E^0(\text{CP}^\infty)$ is a formal group over $\pi_0 E$. Picking a coordinate $t$ for $\hat G_0$, we learn that the cotangent space to $\hat G_0$ is exactly $(t)/(t)^2$, which is isomorphic to $\pi_2 E$. One should therefore think of an identification of the cotangent space with $\pi_2 \Sigma^{-2} E$ as providing a complex orientation (and not just a “preorientation”) of $E$. In fact, this comes from a spectral identification, as we will now discuss.

**Example 3.6.** Let $R$ be a complex oriented weakly even periodic $E_{\infty}$-ring, i.e., what Jacob calls a complex periodic $E_{\infty}$-ring. We will denote by $\hat G_R^Q$ the Quillen formal group; this is the functor $\text{Lat}_Z^\text{co} \to \text{coCAlg}^\text{sm}_R$ defined by sending $M$ to $R \otimes \Sigma^\infty_+ \text{CP}^\infty$. Last time, we proved that this is a smooth formal group over $R$ of dimension 1. Then

$$\Theta_{\hat G_R^Q} \simeq \text{Map}_{\text{Sp}}(\Sigma^\infty_+ \text{CP}^\infty, R) =: C^*(\text{CP}^\infty; R).$$

There is a canonical base point $\eta \in \hat G_R^Q(\tau_{\geq 0} R)$, given by the map $C^*(\text{CP}^\infty; R) \to R$ defined by evaluation on the basepoint of $\text{CP}^\infty$. It follows from Proposition 2.2 that there is a fiber sequence

$$\begin{array}{c}
\Sigma \omega_{\hat G_R^Q, \eta} \\
\downarrow \simeq \Sigma^{-1} R \\
\downarrow \\
\Sigma^{-1} R \simeq C_{{\text{red}}}(S^1, R) \\
\downarrow \text{evaluate} \\
R
\end{array}$$

It follows that there is a canonical equivalence $\omega_{\hat G_R^Q, \eta} \simeq \Sigma^{-2} R$.

**Remark 3.7.** If $\hat G_0$ is a formal group over a complex periodic $E_{\infty}$-ring $R$, then an identification of $\omega_{\hat G_0}^Q$ with $\Sigma^{-2} R$ (via a preorientation) is canonically the same as an identification of $\omega_{\hat G_R^Q, \eta}^Q$ with $\omega_{\hat G_R^Q, \eta}$. By Proposition 2.2 this is the same as an identification of $\hat G_0$ with $\hat G_R^Q$. 
Remark 3.8. The astute reader might argue that we were initially talking about an identification of the cotangent space with $\pi_0^{\text{SO}} R = \pi_2 R$, which is a priori not the same as an identification of the spectral $R$-modules $\omega^\infty_{\mathcal{G}_R}$ with $\Sigma^{-2} R$. This will be made clear in Theorem 3.12.

Our discussion above motivates the following definition.

Definition 3.9. An orientation of a formal hyperplane $X \to \text{Spec } R$ is a preorientation for which the associated Bott map $\omega_{X, \eta} \to \Sigma^{-2} R$ is an equivalence.

As we proved above, this is the same as an identification of $X$ with $\Omega\infty \circ \mathcal{G}_R^Q$.

Remark 3.10. As $\omega_{X, \eta}$ is locally free of rank 1 as an $R$-module, $R$ must be weakly even periodic in order for $X$ to admit an orientation. In particular, although preorientations of $X \to \text{Spec } R$ are equivalent to preorientations of $X_{\tau \geq 0} \to \text{Spec } \tau \geq 0 R$, it is not true that orientations of $X \to \text{Spec } R$ are the same as orientations of $X_{\tau \geq 0} \to \text{Spec } \tau \geq 0 R$.

Lemma 3.11. Let $G_0$ be a formal group over an $E_{\infty}$-ring $R$. Then there is an equivalence $\text{Pre}(G_0) \simeq \text{Map}(\hat{G}_R^Q, G_0)$.

Proof. We argued above that $\text{Pre}(G_0) \simeq \text{Map}^{\text{Mod}_{\mathbb{Z}}} (\Sigma^2 \mathbb{Z}, G_0(\tau \geq 0 R)) \simeq \text{Map}^{\text{Ab}(\text{Top})} (\mathbb{C}P^\infty, \text{Map}^{\text{coCAlg}_{\text{R}}}(R, \mathcal{O}_{G_0}))$.

This reflects the slogan “$\mathbb{C}P^\infty$ is generated by $\mathbb{C}P^1$ as a topological abelian group”. Therefore

$$\text{Pre}(G_0) \simeq \text{Map}^{\text{Ab(coCAlg}_{\text{R}})(R \otimes \Sigma^2 \mathbb{C}P^\infty, \mathcal{O}_{G_0}^\vee) \simeq \text{Map}(\hat{G}_R^Q, G_0).$$

The following result makes everything run.

Theorem 3.12. Fix an $E_{\infty}$-ring $R$.

1. Let $X$ be a formal hyperplane over $R$. Then there is an $E_{\infty}$-ring $R[X]$ with a ring map $R \to R[X]$ such that there is a universal orientation of $X_{R[X]}$.

2. Suppose $\hat{G}$ is a formal group over $R$ with a preorientation $e \in \text{Pre}(\hat{G})$. Then $e$ is an orientation if and only if
   a. $R$ is complex periodic.
   b. The associated map $\hat{G}_R^Q \to \hat{G}$ is an equivalence.

Proof. We begin by proving (1); this is equivalent to proving that the functor $\text{CAlg}_{R} \to \text{Top}$ given by $R' \to \{\text{orientations of } X_{R'}\}$ is corepresentable. In Proposition 3.4, we showed that the functor $R' \to \text{Pre}(X_{R'})$ is corepresented by an $E_{\infty}$-$R$-algebra $A$. By construction, this is equipped with a universal Bott map $\omega_{X, \eta} \to \Sigma^{-2} A$. In order to prove (1), it therefore suffices to prove the following result: let $R$ be an $E_{\infty}$-ring, and suppose $u : L \to L'$ is a map of invertible $R$-modules. Then there is an object $\hat{R}[u^{-1}]$ such that for every $A \in \text{CAlg}_{R}$, we have:

$$\text{Map}^{\text{CAlg}}_{R}(\hat{R}[u^{-1}], A) \simeq \left\{ \begin{array}{ll} * & \text{if } u : A \otimes R L \to \sim A \otimes R L' \\ \emptyset & \text{else.} \end{array} \right.$$
The proof of this result is just algebra, so we will omit it. There is an equivalence of $R$-modules
\[ \text{colim}(R \xrightarrow{u} L^{-1} \otimes_R L' \xrightarrow{u} (L^{-1})^2 \otimes_R L'^2 \xrightarrow{u} \cdots ) \simeq R[u^{-1}]. \]
Applying this to the Bott map $\beta : L = \omega_{X_{\mathbf{A}^1}} \to \Sigma^{-2}A = L'$, we get the $E_{\infty}$-$R$-algebra $R^{\text{or}} = A[\beta^{-1}]$.

Let us now turn to the proof of (2). Our discussion above establishes that if $R$ is complex periodic and the associated map $\widehat{G}^Q_R \to G_\circ \omega$ (from Lemma 3.11) is an equivalence, then $e$ is an orientation. It suffices to prove the other direction.

Suppose $e$ is an orientation. As (b) is equivalent to the map $\widehat{G}^Q_R \to G_\circ \omega$ being an equivalence (by Proposition 2.2), it suffices to show that $R$ is complex periodic. As $R$ is weakly even periodic by Remark 3.10, it suffices to show that $R$ is complex oriented. In other words, we need to show that the map $\pi_{-2}C^*_{\text{red}}(\mathbb{C}P^{\infty}; R) \to \pi_{-2}C^*_{\text{red}}(\mathbb{C}P^1; R)$ is surjective. To prove this, we will use the following diagram:

\[ \begin{array}{ccc}
\mathbb{O}_{G_\circ}(-1) & \to & \mathbb{O}_{G_\circ} \to R \\
\downarrow & & \downarrow \\
C^*_{\text{red}}(\mathbb{C}P^{\infty}; R) & \to & C^*(\mathbb{C}P^{\infty}; R) \to R \\
\downarrow & & \downarrow \\
C^*_{\text{red}}(\mathbb{C}P^1; R) & \to & C^*(\mathbb{C}P^1; R) \to R
\end{array} \]

where $\mathbb{O}_{G_\circ}(-1)$ is defined as the fiber of the augmentation $\mathbb{O}_{G_\circ} \to R$. The map $C^*_{\text{red}}(\mathbb{C}P^{\infty}; R) \to C^*_{\text{red}}(\mathbb{C}P^1; R)$ therefore factors the map $\mathbb{O}_{G_\circ}(-1) \to C^*_{\text{red}}(\mathbb{C}P^1; R)$, so it suffices to prove that the latter map is surjective on $\pi_{-2}$. This map can be identified with the composite
\[ \mathbb{O}_{G_\circ}(-1) \to R \otimes_{\mathbb{O}_{G_\circ}} \mathbb{O}_{G_\circ}(-1) = \omega_{G_\circ} \xrightarrow{\beta} \Sigma^{-2}R \simeq C^*_{\text{red}}(\mathbb{C}P^1; R). \]

The Bott map $\beta$ is an equivalence since $e$ is an orientation. The proof is now completed by observing that the map $\mathbb{O}_{G_\circ}(-1) \to \omega_{G_\circ}$ is surjective on homotopy.

**Remark 3.13.** Let $R$ be an $E_{\infty}$-ring and $\widehat{G}$ a preoriented formal group over $R$. Denote by $\widehat{G}_\circ$ the underlying classical formal group of $\widehat{G}$, living over $\pi_2 R$. It follows from Theorem 3.12 that a preorientation $e \in \Omega^2 \widehat{G}(R)$ is an orientation if and only if:

1. $\widehat{G}_\circ \to \text{Spec } \pi_2 R$ is smooth of relative dimension 1.
2. The map $\omega_{\widehat{G}_\circ} \to \pi_2 R$ induces isomorphisms
\[ \omega_{\widehat{G}_\circ} \otimes_{\pi_2 R} \pi_n R \xrightarrow{\beta} \pi_n R \otimes_{\pi_2 R} \pi_n R \to \pi_{n+2} R \]
for every integer $n$.

See [Lur09, Definition 3.3] for this definition of an orientation.

**Example 3.14.** It follows from Example 3.2 and Remark 3.3 that the universal Bott element $\beta$ for $\widehat{G}_\circ \to \text{Spec } R$ lies in $\pi_2 \Sigma^{\infty} \mathbb{C}P^{\infty}$. By Example 2.5, we learn that $\beta$ is exactly given by the inclusion of $S^2 = \mathbb{C}P^1$ into $\mathbb{C}P^{\infty}$; in other words, $\beta$ is the usual Bott map. We’ve now accomplished task (2) as well. It follows from Theorem 3.12 that $S^{\text{or}} = \Sigma^{\infty} \mathbb{C}P^{\infty}[\beta^{-1}]$. 
Example 3.15. Let us return to the discussion in the introduction. Fix a nonstationary $p$-divisible group $G_0$ over a Noetherian $F$-finite $\mathbb{F}_p$-algebra $R_0$. Denote by $G$ the universal deformation of $G_0$ over the $E_\infty$-ring $R_{G_0}^{un}$, and let $G^\circ$ be the connected component of the identity. Then $G^\circ$ is a formal group over $R_{G_0}^{un}$. By Theorem 3.12 there is an $E_\infty$-$R_{G_0}^{un}$-algebra $R_{G_0}^{or}$ such that there is a universal orientation of $G^\circ \otimes_{R_{G_0}^{un}} R_{G_0}^{or}$. This $E_\infty$-ring is the desired analogue of Morava $E$-theory for $p$-divisible groups (compare with Example 3.14 and Snaith’s theorem).

It is not clear that $R_{G_0}^{or}$ agrees with Morava $E$-theory when $R_0$ is an algebraically closed field of characteristic $p$ and $G_0$ is a $p$-divisible formal group over $R_0$; this will be the content of the following two lectures. The method of proof of this result is a generalization of the moduli-theoretic proof of Snaith’s theorem (see [Mat12]). In order to prove this result, it will be simpler to work in the $K(n)$-local category: it turns out that this does not lose any information since one can prove that $R_{G_0}^{or}$ is itself $K(n)$-local. We will now develop some methods allowing us to prove that an $E_\infty$-ring is $K(n)$-local, which will be useful in the sequel.

4. $K(n)$-locality of complex periodic $E_\infty$-rings

Let us begin with a classical observation.\(^3\)

Proposition 4.1. Let $R$ be a complex oriented ring spectrum (not necessarily an $E_\infty$-ring). Then there is an equivalence

\[
\begin{array}{ccc}
R & \overset{I_n}{\rightarrow} & L_{K(n)} R \\
\downarrow & & \downarrow \\
\text{holim}_{j \in \mathbb{N}^n} v^{-1} R/I_n^j =: R_{v_n} & \leftarrow & \\
\end{array}
\]

where $I_n = (p, v_1, \cdots, v_{n-1}) \subseteq BP_{\mathbb{Z}}$ and $I_n^j = (p^j, v_1^j, \cdots, v_{n-1}^j)$.

Proof. We must first show that the map $R \rightarrow R_{v_n}$ factors through $L_{K(n)} R$. It suffices to show that each $v^{-1}_n R/I_n^j$ is $K(n)$-local. The spectrum $v^{-1}_n R/I_n^1$ is built from $v^{-1}_n R/I_n$ by a finite number of cofiber sequences, so it suffices to prove that the spectrum $v^{-1}_n R/I_n^1$ is $K(n)$-local. This spectrum is a $v^{-1}_n BP/I_n$-module, hence $v^{-1}_n BP/I_n$-local. As $\langle v^{-1}_n BP/I_n \rangle = \langle K(n) \rangle$, it is also $K(n)$-local.

To prove that the map $L_{K(n)} R \rightarrow R_{v_n}$ is an equivalence, we must show that $K(n)_* R \rightarrow K(n)_* R_{v_n}$. It suffices to prove this after smashing the map $R \rightarrow R_{v_n}$ with a finite complex of type $n$. Consider the type $n$ complex $X = S/(p^b, v_1^j, \cdots, v_{n-1}^j)$ for some cofinal $(I_0, I_1, \cdots, I_{n-1})$ coming from the Devinatz-Hopkins-Smith nilpotence technology (see [DEHS88, HS98]); then

\[ R_{v_n} \wedge X \simeq \text{holim}_{j \in \mathbb{N}^n} (v^{-1}_n R/I_n^j \wedge X) \simeq v^{-1}_n R/I_n^j. \]

Therefore, as $K(n)_* (R \wedge X) \simeq K(n)_* (R/I_n^j)$, we learn that

\[ K(n)_* (R \wedge X) \simeq K(n)_* (v^{-1}_n R/I_n^j) \simeq K(n)_* (R_{v_n} \wedge X), \]

as desired.\(\Box\)

\(^3\)I don’t know of a reference for this statement.
Corollary 4.2. A complex oriented ring spectrum \( R \) is \( K(n) \)-local iff \( R \) is \( I_n \)-complete and \( v_n \) is a unit modulo \( I_n \) (in other words, the underlying formal group of the Quillen formal group over \( \pi_2 R \) has height at most \( n \)).

Our goal in this section is to give another proof of Corollary 4.2 for \( E_{\infty} \)-rings which does not rely on Devinatz-Hopkins-Smith.

Recall (a standard reference is Paul Goerss’ paper \([Goe08]\) on quasicoherent sheaves on \( \mathcal{M}_{fg} \)):

Definition 4.3. Let \( G_0 \) be a formal group over a (classical) \( F_p \)-scheme \( S \). Then \( G_0 \) has height \( \geq n \) if there is a factorization

\[
G_0 \xrightarrow{\phi} G_2(p) \xrightarrow{\phi(p)} \cdots \xrightarrow{\phi(p^{n-1})} G_0^{(p^n)} \xrightarrow{[p]} G_0
\]

Construction 4.4. The map \( T \) induces a map \( T^* : \omega_{G_0} \to \omega_{G_0^{(p^n)}} \simeq \omega_{G_0}^{\otimes p^n} \). As \( \omega_{G_0} \) is a line bundle, this is the same as a map \( \mathcal{O}_S \to \omega_{G_0}^{\otimes (p^n-1)} \). This defines a global section \( v_n \in \omega_{G_0}^{\otimes (p^n-1)} \), called the \( n \)th Hasse invariant. Let \( \mathcal{M}(n+1) \) be the closed substack of \( \mathcal{M}_{fg} \) defined by the line bundle \( \omega_{G_0}^{\otimes p^n-1} \) and the section \( v_n \).

Definition 4.5. Let \( J_n \) denote the ideal sheaf defining the closed substack \( \mathcal{M}(n) \), so that \( J_n \) is the image of the injection \( v_n : \omega^{\otimes (p^n-1)}_{\mathcal{M}(n)} \to \mathcal{O}_{\mathcal{M}_{fg}} \). If \( S = \text{Spec} \ R \) is a \( F_p \)-scheme and \( G_0 \) is given by a map \( f : S \to \mathcal{M}_{fg} \), the pullback \( f^* J_n = I_n^{G_0} \) defines an ideal of \( R \). This is called the \( n \)th Landweber ideal of \( G_0 \).

Notation 4.6. If \( R \) is an \( E_{\infty} \)-ring and \( G \) is a formal group over \( R \), we set \( I_n^G = I_n^{G_0} \subseteq \pi_2^R \).

Let \( R \) be an \( E_{\infty} \)-ring, and \( G \) be a formal group over \( R \). Say that \( G \) has height \( < n \) if \( I_n^G = \pi_2^R \).

Definition 4.7. If \( R \) is complex periodic, we set \( I_n = I_n^{\mathcal{G}^Q} \), with \( R \) left implicit; this is the \( n \)th Landweber ideal of \( R \).

Let \( \mathcal{G}_R^{Q} \) denote the base change \( \mathcal{G}_R^{G_0} \otimes_{\pi_2 R} \pi_2^R / I_n \). By construction, \( \mathcal{G}_R^{Q} \) has height \( \geq n \). Moreover, it follows from Proposition 2.2 that \( \omega_{\mathcal{G}_R^{Q}} = \pi_2^R / I_n \). The section \( v_n \) is now an element of \( \pi_{2(p^{n-1})}^R / I_n \). Let \( \overline{v}_n \) denote any lift of \( v_n \) to \( \pi_{2(p^{n-1})}^R \); then \( I_n^{G_0} \) is generated by \( I_n \) and \( \overline{v}_n \).

We can now state the generalization of Corollary 4.2: Assume that we have \( p \)-localized everywhere.

Theorem 4.8. Let \( R \) be a complex periodic \( E_{\infty} \)-ring and let \( n > 0 \). The \( R \)-module \( M \) is \( K(n) \)-local if and only if the following conditions are satisfied:

1. \( M \) is complete with respect to \( I_n \subseteq \pi_2^R \).
2. Multiplication by \( \overline{v}_n \) induces an equivalence \( \Sigma^{2(p^{n-1})} M \to M \).

\[ \text{Jacob denotes this by } J_n^R, \text{ but again, I do not know how to write a fraktur I.} \]
Proof. Assume that (1) and (2) are satisfied. It suffices to prove the following statement for all $0 \leq m \leq n$: if $N$ is a perfect $R$-module which is $I_m$-nilpotent, then $M \otimes_R N$ is $K(n)$-local. Indeed, when $n = 0$, choosing $N = R$ gives us that $M = M \otimes_R R$ is $K(1)$-local.

This statement is proved by descending induction along $m$. We first prove the statement in the case $m = n$. To prove that $M \otimes_R N$ is $K(n)$-local, we need to show that for any $K(n)$-acyclic $\Gamma$-spectrum $X$, the space $\text{Map}_{\text{Sp}}(X, M \otimes_R N) \simeq \text{Map}_{\text{Sp}}(X \otimes N^\vee, M)$ is contractible. As usual, $N^\vee$ denotes the $R$-linear dual of $N$. It therefore suffices to prove that $X \otimes N^\vee$ is zero.

The spectrum $M U P \otimes R$ is faithfully flat over $R$; this is a classical result (e.g., in Adams’ blue book) but we have chosen to rephrase it in fancy language. Therefore it suffices to prove that $X \otimes N^\vee \otimes R \simeq X \otimes N^\vee \otimes M U P$ is contractible.

Let $n \in \pi_2 M U P$ be an invertible element. As $\nu_m \in \pi_2\{p=1\} M U P/\pi_n^{M U P}$, we can choose elements $w_m \in \pi_2 M U P$ such that $w_m = \pi_n^{M U P}(-p_n^{n-1})$. By construction, $(w_0, \ldots, w_{n-1})$ generate $I_n^{M U P}$. Clearly $I_n^{M U P}$ and $I_n$ generate the same ideal inside $\pi_0(R \otimes M U P)$, so perfectness and $I_n$-nilpotence of $N$ implies that $N^\vee \otimes M U P$ is a perfect module over $R \otimes M U P$ which is $I_n^{M U P}$-nilpotent.

$N^\vee \otimes M U P$ is a retract of $(N^\vee \otimes M U P)/(w_0^k, \ldots, w_{n-1}^k)$ for $k \gg 0$ by construction, so it suffices to prove that each $X \otimes N^\vee \otimes M U P/(w_0^k, \ldots, w_{n-1}^k)$ vanishes. However, as we can build $M U P/(w_0^k, \ldots, w_{n-1}^k)$ from $M U P/(w_0, \ldots, w_{n-1})$ by a finite number of cofiber sequences, it suffices to show that $X \otimes N^\vee \otimes M U P/(w_0, \ldots, w_{n-1})$ vanishes.

As before, $w_n$ acts invertibly on $N^\vee \otimes M U P$, so it can be regarded as a $R \otimes M U P[w_n^{-1}]$-module. In particular, it suffices to show that $X \otimes N^\vee \otimes M U P/(w_0, \ldots, w_{n-1})[w_n^{-1}]$ vanishes. However, $M U P/(w_0, \ldots, w_{n-1})[w_n^{-1}]$ is $\nu_n^{-1}BP/I_n$-local, hence $K(n)$-local. As $X$ is $K(n)$-acyclic, we learn that $X \otimes M U P/(w_0, \ldots, w_{n-1})[w_n^{-1}]$ is contractible, as desired.

To prove that (1) and (2) imply that $M$ is $K(n)$-local, it remains to establish the inductive step. Concretely, we need to show that $N$ being a perfect $R$-module which is $I_m$-nilpotent implies that $M \otimes_R N$ is $K(n)$-local. Condition (1) says that $M$ is $I_n$-complete, so perfectness of $N$ implies that $M \otimes_R N$ is also $I_n$-complete. Therefore

$$M \otimes_R N = \text{holim}_n M \otimes_R (N/w_n^k).$$

Each $N/w_m^k$ is $I_m$-nilpotent, so $M \otimes_R (N/w_m^k)$ is $K(n)$-local by the inductive hypothesis.

It remains to establish that if $M$ is $K(n)$-local, then (1) and (2) are satisfied. To establish (1), we need to show that $M$ is $(x)$-complete for every $x \in I_n$. In other words, we must show that for every $R[1/x]$-module $N$, the space $\text{Map}_{\text{Mod}_x}(N, M)$ is contractible. As $M$ is $K(n)$-local, there is an equivalence

$$\text{Map}_{\text{Mod}_x}(N, M) \simeq \text{Map}_{\text{Mod}_x}(I_{K(n)} N, M).$$

It therefore suffices to show that $I_{K(n)} N = 0$, i.e., $K(n) \otimes N = 0$. This is a $K(n) \otimes R[1/x]$-module, so it suffices to show that $K(n) \otimes R[1/x] = 0$. This is easy: the ring $K(n) \otimes N$ carries two formal group laws, namely the height $n$ formal group law from $K(n)$, and the height $< n$ formal group law from $R[1/x]$. These cannot be isomorphic as they are of different heights, so $K(n) \otimes R[1/x] = 0$, as desired.

To establish (2), we need to show that the map $\Sigma^2(p^{n-1}) M \rightarrow M$ is an equivalence. As $M$ is $K(n)$-local, it suffices to show that $\Sigma^2(p^{n-1}) M \otimes K(n) \rightarrow M \otimes K(n)$ is an equivalence.

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5There is a typo in Jacob’s book.
As the formal group law over $\pi_\ast(R \otimes K(n))$ has height $n$, this map is an isomorphism on homotopy, as desired. □

This recovers a special case of Corollary 4.2

**Corollary 4.9.** Let $R$ be a complex periodic $E_\infty$-ring and let $n > 0$. Then $R$ is $K(n)$-local if and only if:

1. $R$ is $I_n$-complete.
2. $I_{n+1} = \pi_\ast R$, i.e., $G^Q_R$ has height $\leq n$.

**Proof.** Suppose (1) and (2) are satisfied. As $R$ is $I_n$-complete, Theorem 4.8 says that $R$ is $K(n)$-local if and only if multiplication by $\overline{\pi}_n$ induces an equivalence $\Sigma^{p^{n-1}}R \to R$ of $R$-modules. In other words, it suffices to establish that $\overline{\pi}_n$ is invertible in $\pi_\ast R$. We know that $I_{n+1} = \pi_\ast R$ is generated by $I_n$ and the image of $\overline{\pi}_n : \pi_{-2p^{n-1}}R \to \pi_\ast R$. Therefore, $\overline{\pi}_n$ is invertible modulo $I_n$. We are now done: the $I_n$-completeness of $\pi_\ast R$ implies that $\overline{\pi}_n$ is itself invertible.

The proof of the other direction is exactly the same, with the steps reversed. Assume $R$ is $K(n)$-local. Theorem 4.8 implies that $R$ is $I_n$-complete, so it suffices to establish that $I_{n+1} = \pi_\ast R$. Again, $I_{n+1}$ is generated by $I_n$ and the image of $\overline{\pi}_n : \pi_{-2p^{n-1}}R \to \pi_\ast R$ — but condition (2) implies that the latter map is an isomorphism (as $\overline{\pi}_n$ is invertible in $\pi_\ast R$ by Theorem 4.8). Therefore $I_{n+1} = \pi_\ast R$. □

**Remark 4.10.** Note that this result is strictly weaker than Corollary 4.2: it requires that $R$ be weakly even periodic and an $E_\infty$-ring.

**References**


