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Q-Zeta and Elliptic Hall Polynomials (MPI, August 2025)

The fundamental property of zeta functions and L-functions is that their meromorphic continuations provide a lot of information about the corresponding objects. Complex values of s occur as a technical tool, with little direct arithmetic-geometric meaning. In the refined theory, $1/n^s$ are replaced by certain $q, t, \mathfrak{a}$-series, which are invariants of Lens Spaces $L(n,1)$ directly related to Elliptic Hall Polynomials. Superduality is one of their key features: under $q \leftrightarrow 1/t, \mathfrak{a} \mapsto \mathfrak{a}$. As $t \to 0$ they become Rogers-Ramanujan series (certain string functions), the limit $q \to 0$ is to Hall Polynomials, the limit $t \to 0$ is to Kac-Moody theory. We will begin with the q-deformation of Riemann's zeta: the case of A_1 when $t = q^{s-1/2}, a = t^2$. Basically, we lose $s \mapsto 1-s$ for Riemann's zeta when adding q, but gain the Hasse-Weil's $t \mapsto 1/(qt)$ (curves over $\mathbb{F}_q$) upon further adding $\mathfrak{a}$.

Three Major Challenges

(1) x^s is far from simple for complex s; Bernstein-Sato shift operators are needed for $(f(x))^s$. (2) Geometric meaning of $\zeta(s)$ for complex s is mysterious, including "Grand Conjectures". (3) $\zeta_X(s)$ is a weak invariant of $X/\mathbb{C}$; Hodge numbers of X can be recovered from it ($Nick\ Katz$). Hasse-Weil's $\zeta_{X;\mathbb{F}_q}(s)$ for $X/\mathbb{F}_q$ are much more geometric.

For plane curve singularities, (a) KhR polynomials, (b) DAHA superpolynomials, (c) motivic ones, (d) Galkin's L-functions, (e) p-adic ones, (f) ζ of the Dirac operator under the Schottky uniformization are known-conjectured to be full topological invariants. Thus, they "coincide".

(A) Selberg's zeta (closed geodesics) "almost" satisfies RH for compact Riemann surfaces X. It is related to "spectral" $\zeta_\Delta(s) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i^s}$ for $Spec(\Delta_X)$. They are more geometric. Say, $det(\Delta) = \lim_{s\to 0} exp(-\zeta'_\Delta(s))$ (Ray-Singer's regularization). Note that if $X=\Gamma(N)\backslash\mathbb{H}$, "$\lambda_1 \geq \frac{1}{4}$" is Selberg's conjecture (it follows from RH).

(B) In physics, the partition functions $Z_N(s)$ of Ising models with an external magnetic field satisfy RH, where $s=$ complex fugacity, and $|s|=1$ replaces $\Re s = \frac{1}{2}$ (Lee-Yang). Thus, $\lim_{N\to\infty} \log(Z_N)/N$ has only 1 phase transition at $s=1$ (only real $s>0$ matter).

A. STIRLING-MOAK FORMULA

Ueno, Nishizawa: $n^s \mapsto ([n]_q)^s$ **(**$x \mapsto \sinh(x)$**)**, **but the problem with** x^s **remains. Inspired by DAHA, we replace** x^{2k} **by the Macdonald measure:**

$$\boldsymbol{\delta_k(x;q) \stackrel{\text{def}}{=\!=} \prod_{j=0}^{\infty} \frac{(1-q^{j+2x})(1-q^{j-2x})}{(1-q^{j+k+2x})(1-q^{j+k-2x})}}.$$

Let $q = \exp(-1/a)$ **for** $\mathbb{R} \ni a > 0$, **and**

$$u^k = \exp(k \log u),\ u \notin -\mathbb{R}_+, k \in \mathbb{C}.$$

The Stirling-Moak limiting formula (A) vs. the straightforward limit to Gamma-functions (B):

$$\mathbf{A}: \lim_{a\to\infty} (a/4)^k \delta_k(\sqrt{az};q) \ = \ (-z)^k$$

if either $k \in C, z \notin \mathbb{R}_+$ **or** $z \in \mathbb{C}, k \in \mathbb{Z}$;

$$\mathbf{B}: \lim_{a\to\infty} a^{2k} \delta_k(\sqrt{z};q) =$$

$$= \frac{\Gamma(k+2\sqrt{z})\Gamma(k-2\sqrt{z})}{\Gamma(2\sqrt{z})\Gamma(-2\sqrt{z})} \text{ for } k, z \in \mathbb{C}.$$

The $\zeta_q(s) \stackrel{\text{def}}{=\!=} \sum_{n=1}^{\infty} \delta_{-s}(\sqrt{ain};q)$ **is singular anywhere. Nevertheless, this worked perfectly for generic** s**: for sufficiently general sequences** $a = a_m \to \infty$, $(-i/a)^s \zeta_q(s) \to \sum_{n=1}^{\infty} n^{-s},\ \Re s > 1.$

B. IMAGINARY INTEGRATION

Let $s = k + 1/2$. Following the classical

$$\mathfrak{Z}(k) \overset{\text{def}}{=\!=} 2\int_0^\infty (e^{x^2}+1)^{-1} x^{2k}\, dx$$

$$= (1 - 2^{\frac{1}{2}-k})\Gamma(k+\frac{1}{2})\zeta(k+\frac{1}{2}), \text{ let:}$$

$$\mathfrak{Z}_q(k) \overset{\text{def}}{=\!=} (-i)\int_{-\infty\imath}^{\infty i} (q^{x^2}+1)^{-1}\delta_k\, dx.$$

THM. Lim$_{a\to\infty}(\frac{a}{4})^{k-\frac{1}{2}}\mathfrak{Z}_q = \mathfrak{Z},\ q = e^{-\frac{1}{a}}$.
Here $\Re k > \frac{1}{2}$ or analytic continuations otherwise.
Warning: for the "standard" $e^{x^2} - 1$, let

$$\mathfrak{Z}^*_q(k) \overset{\text{def}}{=\!=} \frac{1}{i}\int_{-\infty\imath}^{\infty\imath} (q^{x^2}-1)^{-1}\delta_k\, dx; \text{ then}$$

$$(\frac{a}{4})^{k-\frac{1}{2}}\mathfrak{Z}^*_q(k) \to \Gamma(k+\frac{1}{2})\zeta(k+\frac{1}{2}),\ \Re k > \frac{1}{2},$$

$$a^{2k-1}\mathfrak{Z}^*_q(k) \to \tan(\pi k)\Gamma(k)^2,\ 0 < \Re k < \frac{1}{2}!!$$

Numerically, the zeros of $\mathfrak{Z}_q$ that are deformations of the classical zeros of $\zeta(s)$ for $s = k + 1/2$ go to the LEFT half-plane $\Re s < 1/2$ ($\Re k < 0$).

For a zero $k = z$ of $\zeta(k+1/2)$ at (near) $i\mathbb{R}$, and the corresponding zero $z(a)$ of $\mathfrak{Z}_q$, $z(a)/z \approx$

$$1 + \frac{4(z+\frac{1}{2})\zeta_+(z+\frac{3}{2}) - (z-1)\zeta_+(z-\frac{1}{2})}{12a\zeta'(z+\frac{1}{2})(1-2^{\frac{1}{2}-z})},$$

as $\zeta'(s) = \partial\zeta(s)/\partial s$, $\zeta_+(s) = (1-2^{1-s})\zeta(s)$.

The first zero z that may/might go to the right is $1977.27i$; it is exactly the first one "unusually" close to its neighbor, so the linear approximation cannot be trusted too much; see ["On q-analogues of Riemann's zeta function", 2001].

The periodicity (with some multiplier) of $\mathfrak{Z}_q$ in the imaginary direction gives the classical $T\log(T)$-formula. The limit $t = q^k \to 0$ is related to Rogers-Ramanujan identities.

THM. As $\Re k \to +\infty$, $\mathfrak{Z}_q$ tends to

$$\frac{1}{i}\int_0^{\infty i} \frac{\prod_{j=0}^{\infty}(1-q^{j+2x})(1-q^{j-2x})}{q^{x^2}+1}dx.$$

C. ANALYTIC CONTINUATION

First, δ is replaced by truncated θ-function: $\mu(x) =$

$$= \prod_{i=0}^{\infty} \frac{(1-q^{i+2x})(1-q^{i+1-2x})}{(1-q^{i+k+2x})(1-q^{i+k+1-2x})},$$

$$\Phi_{\epsilon}^{k}(F) \overset{\textbf{def}}{=\!=} \frac{1}{i}\int_{\epsilon+\imath\mathbb{R}} F(x)\mu(x)dx.$$

$$F(x):\ \mathcal{E}_0 = q^{-x^2}, \mathcal{E}_- = (q^{x^2}-1)^{-1}, \mathcal{E}_+ = (q^{x^2}+1)^{-1}.$$

THM. For sufficiently good F, $\Phi_{\epsilon}^{k}(F)$ is analytic as $\Re k > \max\{-2\epsilon, 2\epsilon-1\}$; $\epsilon = \frac{1}{4} \Rightarrow \Re k > -\frac{1}{2}$ (optimal). Example: $\Phi_{\frac{1}{4}}(\mathcal{E}_0) = \sqrt{\pi a}\prod_{j=1}^{\infty} \frac{1-q^{k+j}}{1-q^{2k+j}}$, $\Re k > -1/2$; **Lim**$_{a\to\infty} a^{k-\frac{1}{2}}\Phi_{\frac{1}{4}}(\mathcal{E}_0) = \sqrt{\pi}\frac{\Gamma(2k)}{\Gamma(k)}, q = e^{-\frac{1}{a}}$ **for the same** $k \in \mathbb{C}$.

Comments. **(1) For even F, one can use δ for $\Re k > 0$:** $\Phi_0^k(F) = \frac{1+q^k}{2i}\int_{\imath\mathbb{R}} F(x)\delta(x)dx$, **but μ gives $\Re k > -\frac{1}{2}$. (2) Analytic continuation is needed for $\Re k < -\frac{1}{2}$; Bad** $k = \{2C-1-Z_+, -2C-Z_+\}$, $C = \{\epsilon+\imath\mathbb{R}\}$. **Generally, "picking up residues" due to H.Weyl-Arthur-Heckman-Opdam is used. No analytic continuation is needed for** $\int_{\imath\epsilon+\mathbb{R}} F\mu$; **connecting imaginary and real q-zetas (unknown) is a counterpart of Riemann's 2nd proof of FA.**

D. GAUSS INTEGRALS AND ZETAS

The starting point is the q-analog of Mehta-Macdonald formula: for $\int_{\imath\mathbb{R}^n} \gamma(x)\, \mu(q^x) dx$ for the "plus- Gaussian" $\gamma(x) = q^{-x^2/2}$ and the mu-function $\mu(q^x;t) = \prod_{\alpha, j\geq 0} \frac{(1-q^{(x,\alpha)+\nu_\alpha j})\,(1-q^{-(x,\alpha)+\nu_\alpha (j+1)})}{(1-t_\alpha q^{(x,\alpha)+\nu_\alpha j})(1-t_\alpha q^{-(x,\alpha)+\nu_\alpha (j+1)})}$. Here α are positive roots of $R\subset\mathbb{R}^n$, a reduced irreducible root system, normalized by $(\alpha_{sht},\alpha_{sht}) = 2$; we set $\nu_\alpha = 1$ for short α, and $\nu = 2,3$ for long. Let $t_{sht} = q^{k_{sht}}$, $t_{lng} = q^{k_{lng}}$; t_α depends only on $|\alpha|$. Also, $\rho_k = \frac{1}{2}\sum_{\alpha>0} k_\alpha \alpha$, $t^\rho = q^{\rho_k}$, and $\mu_1 = \mu/CT(\mu)$ for the constant term functional CT for Laurent series in terms of $X_b = q^{(x,b)}$ for $b\in P$ (the weight lattice). Finally, P_b for $b\in P_+$ will be (symmetric) Macdonald polynomials. We assume that $0<q<1$ and $t_\alpha > 1$.

The Jackson integration is $\mathfrak{J}\big(f(x)\big) = \sum_{\widehat{w}\in\widehat{W}} \widehat{w}\big(f(x)\big)$ for the extended affine Weyl group $\widehat{W} = W\ltimes P$, where $(wb)(X_a) = X_{w(a)} q^{-(a,b)}$. Elliptic Hall Polynomials of level $\ell > 0$ are $\mathcal{H}_b^{(\ell)} = \mathfrak{J}\big(\frac{P_b(q^x)}{P_b(t^\rho)} q^{\ell x^2/2}\, \mu_1(q^x;t^{-1})\big)$, or $\mu_1^{-1}\mathcal{H}_b^{(\ell)}$ [*Kapranov, 2000*]. For $\ell\in\mathbb{N}$, their space is isomorphic to that of KM-characters of level ℓ. For instance, $\mathcal{H}_b^{(1)} \sim \theta(x) \stackrel{\text{def}}{=\!=} \sum_{a\in P} q^{(a,a)/2} X_a$ (they are series).

Let $Z_n^+(q,t)=\frac{\int_{\varepsilon+\imath R^n}\gamma(x)/(1+\gamma(x))\,\mu(q^x)dx}{\int_{\varepsilon+\imath R^n}\gamma(x)\,\mu(q^x)dx}$, and we set: $s=k_{sht}|R_+^{sht}|+k_{lng}|R_+^{lng}|+\frac{n}{2}$. Then one has: $\lim_{q\to 1_-}Z_n^+(q,t)=\eta(s)\stackrel{\mathbf{def}}{=\!=}(1-2^{1-s})\zeta(s)$ for the Riemann's $\zeta(s)$. The translation by ε combined with the usage of μ (instead of δ) improves the range of k_ν where Z_n^+ is analytic. The basic range is $\Re k_\nu>0$, which is for $\varepsilon=0$. If $k_{lng}=k=k_{sht}$ and $\varepsilon=\rho/h$ for the Coxeter number, then $Z_n^+(q,t)$ is analytic for $\Re k>-1/h$, which corresponds to $\Re s>0$, i.e. we cover the critical strip $0<s<1$, which can be potentially useful for "*q-Lindelöf*".

The convergence to $\eta(s)$ holds for any $s\in\mathbb{C}$ upon the analytic continuation. The latter is not needed for $\int_{\varepsilon\imath+\mathbb{R}^n}$, but then Appell's functions occur in "*q-Macdonald-Mehta*" instead of q-Γ ones.

Type A_n. Setting, $\upsilon=n+1$ and $\upsilon^\circ=-k\upsilon$: $s=k\frac{\upsilon(\upsilon-1)}{2}+\frac{\upsilon-1}{2}=-\frac{1}{2}(\upsilon-1)(\upsilon^\circ-1)$. It occurred in [*Etingof-Gorsky-Losev, 2013*]: $T(l,m)=T(m,l)$ via rational DAHA. Thus, $\mathcal{I}_n^+=\int_{\varepsilon+\imath R^n}\frac{\gamma(x)}{1+\gamma(x)}\,\mu(q^x)dx$ for $\varepsilon=\rho/\upsilon$ is an analytic function for $\Re k>-1/\upsilon$ and, accordingly, for $s>-\frac{1}{\upsilon}\frac{\upsilon(\upsilon-1)}{2}+\frac{\upsilon-1}{2}=0$.

Almost a theorem. **In type A, there exists meromorphic $\mathcal{Z}(q,t,\mathfrak{a})$ satisfying the superduality, which is $\boldsymbol{\mathcal{Z}(t^{-1},q^{-1},\mathfrak{a})=\mathcal{Z}(q,t,\mathfrak{a})}$, and such that $\eta(s)$ is the limit $q\to 1_-$ of $\mathcal{Z}(q,t=q^k,\mathfrak{a}=t^{\upsilon})$. Similarly, $\mathcal{H}_b^{(\ell)}/\big(\mathcal{H}_0^{(1)}\big)^{\ell}$ are stable and superdual, where $\lambda\mapsto\lambda'$ for the corresponding Young diagrams λ.**

In terms of k,υ, the superduality $q\leftrightarrow 1/t,\mathfrak{a}\mapsto\mathfrak{a}$ becomes $\boldsymbol{k\mapsto 1/k,\upsilon\to -k\upsilon=\upsilon^{\circ}}$. Indeed, $t=q^k$ becomes $(1/q)^{1/k}=1/t$ and this is compatible with $|q|<1<|t|$. The corresponding s remains fixed (it must!), but we have a non-trivial connection between the values of $\mathcal{Z}$ at k and $1/k$ in the q-theory. For instance, the $\mathfrak{a}$-coefficients of $\mathcal{Z}$ are bounded for "large" $|k|$ and $\Re s>0$, or for $\mathcal{Z}(q,t,\mathfrak{a}=\alpha)$ with super-invariant α (say, $\alpha=\frac{t}{q}$), a $q,t,\mathfrak{a}$-variant of the Lindelöf hypothesis.

Algebraically, $\mathcal{Z}(q,t,\mathfrak{a})$ is a generating function of the simplest elliptic Hall polynomials, that of invariants of $L(n,1)$. Geometrically:

Problem. **Find M (conifold?) with $L(n,1)$ as $\mathbb{C}^*$-invariant fibers such that $\mathcal{Z}$ is the Poincare polynomial in proper homology (upon *"localization"*).**

E. DAHA-SATAKE MAP

A nontrivial fact is that $\mathscr{S}\big(X_b\big)$ is regular for generic q,t, where $\mathscr{S}(f) = \sum_{hw\in\widehat{W}} \widehat{w}\big(f(x)\mu(q^x;t^{-1}\big)$. This requires an alternative approach based on Affine Symmetrizer in terms of the DAHA operators $T_{\widehat{w}}$ acting in $\mathscr{X} = \mathbb{C}[X_b, b\in P]$. The DAHA-Satake map is

$$\mathscr{P}_+(f) \stackrel{\text{def}}{=\!=} \sum_{\widehat{w}\in\widehat{W}} t^{-l(\widehat{w})/2} T_{\widehat{w}}^{-1}(f),\ \ f\in q^{\ell x^2/2}\mathscr{X}.$$

Let $l>0$. The operators $\mathscr{S}$ and converges absolutely at any given $f\in \mathscr{X} q^{\,lx^2/2}$ for any generic $k=\{k_{\text{sht}},k_{\text{lng}}\}$. For $\mathscr{P}$, this requires $\Re(h_k^{\text{lng}}), \Re(h_k^{\text{sht}})<1$, where $h_k^{\text{sht}} = (\rho_k,\vartheta)+k_{\text{sht}}$, $h_k^{\text{lng}} = (\rho_k,\theta^\vee)+k_{\text{lng}}$ and θ is the maximal root in R_+. These inequalities become $\Re k<\frac{1}{h}$ for $h_k = kh$ in the simply-laced case, where h is the Coxeter number. For such k: $\boldsymbol{\mathscr{P} = CT\big(\mu(q;t^{-1})\big)\mathscr{S}}$.

Application: $\mathscr{S}(f)$ is well-defined for any k but may have poles; $\mathscr{P}(f)$ is regular for $\Re k<1/h$. Combining, the poles of $\mathscr{S}(X_b)$ are those due to $CT\big(\mu(q;t^{-1}\big)^{-1}$.

The adelic version of this argument is expected to provide an alternative approach to the fact that the Langlands formula for the inner product of pseudo-Eisenstein series is non-singular (Kazhdan,Okounkov,Opdam, ...).

F. TOPOLOGICAL VERTEX

Recall that $X_a(q^b)=q^{(a,b)}$. **Setting** $q_\alpha=q^{\nu_\alpha}$, $P_b(q^{\rho_k})=q^{-(\rho_k,b)}\prod_{\alpha>0}\prod_{j=0}^{(\alpha^\vee,b)-1}\Big(\frac{1-q_\alpha^j t_\alpha X_\alpha(q^{\rho_k})}{1-q_\alpha^j X_\alpha(q^{\rho_k})}\Big)$.
Setting $b^\iota=-w_0(b)$, $CT(f)=\langle f\rangle$, $\langle P_b P_c^\iota\mu\rangle=\langle\mu\rangle\,\delta_{bc}\times\prod_{\alpha>0}\prod_{j=0}^{(\alpha^\vee,b)-1}\frac{(1-q_\alpha^{j+1}t_\alpha^{-1}X_\alpha(q^{\rho_k}))(1-q_\alpha^j t_\alpha X_\alpha(q^{\rho_k}))}{(1-q_\alpha^j X_\alpha(q^{\rho_k}))(1-q_\alpha^{j+1}X_\alpha(q^{\rho_k}))}$.

Let $\theta_u(X)\stackrel{\mathbf{def}}{=\!=}\sum_{a\in P}u(a)q^{(a,a)/2}X_a$ **for characters** $u:P/Q\to\mathbb{C}^*$, **playing the role of the classical theta-characteristics. Finally, let** $\widetilde{P}_a\widetilde{P}_b=\sum_c\mathbb{C}_{ab}^{c\mathbf{u}}\widetilde{P}_c$ **for** $\widetilde{P}_a\stackrel{\mathbf{def}}{=\!=}P_a\theta_{\mathbf{u}}^{(\ell)}$, $a,b\in P_+$, **where** $\theta_{\mathbf{u}}^{(\ell)}=\theta_{u_1}\cdots\theta_{u_\ell}$ **for** $\mathbf{u}=\{u_1,\ldots,u_\ell\},\ell\ge 0$. **Setting** $\mathbb{C}_{0b}^{cu}\stackrel{\mathbf{def}}{=\!=}\frac{\langle P_b P_c^\iota\theta_u\,\mu\rangle}{\langle P_c P_c^\iota\mu\rangle}=\frac{q^{b^2/2+c^2/2+(b+c,\rho_k)}}{u(b-c)\langle P_c P_c^\iota\mu\rangle}P_b^\iota(q^{c+\rho_k})P_c(q^{\rho_k})\langle\theta\mu\rangle$, $\mathbb{C}_{\mathbf{0b}}^{\mathbf{cu}}=\sum_{\mathbf{c_1,c_2,\ldots,c_{\ell-1}}\in P_+}\mathbb{C}_{\mathbf{0b}}^{\mathbf{c_1u_1}}\mathbb{C}_{\mathbf{0c_1}}^{\mathbf{c_2u_2}}\mathbb{C}_{\mathbf{0c_2}}^{\mathbf{c_3u_3}}\cdots\mathbb{C}_{\mathbf{0c_{\ell-1}}}^{\mathbf{cu_\ell}}$.

The expansion for $\mathbb{C}_{0b}^{c\mathbf{u}}$ **for minuscule** $b,c\in P_+$ **are refined Rogers-Ramanujan sums; they become modular** 0**-weight functions as** $t\to 0$ [*Ch,B.Feigin, 2013*].

G. STABILIZATION AND SUPERDUALITY

Let $\mathfrak{b}=(b_i, 1\leq i\leq \ell)\subset P_+ \ni c$, $P_b\mapsto P_{\mathfrak{b}}=\prod_i P_{b_i}$. Then $\mathbb{C}^{cu}_{0\mathfrak{b}} = \frac{\dot\tau_-^{-1}(P_{\mathfrak{b}}P_{c^\iota})(q^{\rho_k})\langle\theta\mu\rangle}{u(\sum_i b_i - c)\,\langle P_c P_c^\iota \mu\rangle}$, where $\dot\tau_-$ is the action of τ_- in $\mathscr{X}$: $\dot\tau_-(P_b) = q^{-(b,b)/2-(b,\rho_k)}P_b$. This formula (one of the key in DAHA) is a link to superpolynomials:

THM. For A_n and Young diagrams λ,μ corresponding to $b,c\in P_+$, there exists a unique series $\mathfrak{C}^{\mu}_{\lambda}(q,t,\mathfrak{a})$ such that $\mathbb{C}^{cu}_{0b}/\langle\theta\mu\rangle$ for trivial u equals $\mathfrak{C}^{\mu}_{\lambda}(q,t,\mathfrak{a}=t^{n+1})$ (stabilization). Then $\mathfrak{C}^{\mu}_{\lambda}(q,t,\mathfrak{a}) = \mathfrak{C}^{\mu'}_{\lambda'}(t^{-1},q^{-1},\mathfrak{a})$ (superduality), where λ' is the transposition.

Comments. **(1) Here $\dot\tau_-^{-1}(P_{\mathfrak{b}}P_{c^\iota})(q^{\rho_k})/P_c(q^{\rho_k})$ is the DAHA-Jones polynomial from** [*Ch,Danilenko,2015*] **for Hopf $(\ell+1)$-link with the pairwise linking numbers -1 for the components colored by $\mathfrak{b}$ and 1 between $\mathfrak{b}$ and c.**

(2) There is a connection to Delign's GL_z at the level of "HOMFLY-PT" (Etingof-Losev), but the actual focus of refined theories is superduality (where adding $\mathfrak{a}$ is necessary); motivically, it is Hasse-Weil's $t\mapsto 1/(qt)$.

H. TQFT WITH LEVELS

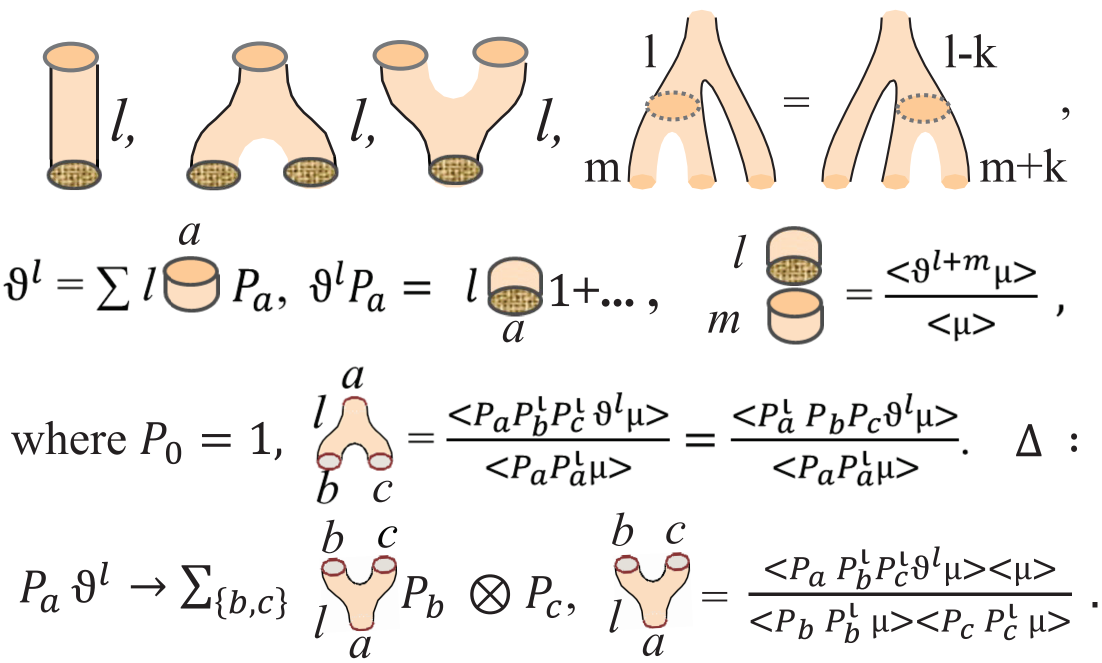
Following TQFT (the unoriented one due to Turaev-Tuner with ι), the relations between $\mathbb{C}^{c\mathbf{u}}_{0\mathfrak{b}}$ can be interpreted as follows. Let $\mathscr{A}$ be a commutative algebra with 1 and a symmetric non-degenerate form $\langle f,g\rangle=\langle fg^{\iota}\mu_1\rangle$ for $\epsilon:\mathscr{A}\ni f\mapsto\langle f\mu_1\rangle$, $\mu_1^{\iota}=\mu_1$, $1^{\iota}=1, \epsilon(1)=1$. Define $\Delta:\mathscr{A}\to\mathscr{A}\widehat{\otimes}\mathscr{A}$ via $\langle\Delta(f), x\otimes y\rangle=\langle f, xy\rangle$.

In the basis of "orthogonal polynomials" $\{P_a\in\mathscr{A}\}$, $\Delta(P_aV)=\sum_{b,c}\frac{\langle P_aV,P_bP_c\rangle P_b\otimes P_c}{\langle P_b,P_b\rangle\langle P_c,P_c\rangle}$ for a ι-invariant potential V. Let $P_0=1, \langle 1,1\rangle=1$. The invariant of S^2 is then $\langle V\mu_1\rangle$. Taking $V=\theta^{(\ell)}_{\mathbf{u}}$, $P_a(a\in P_+)$ etc., as above, it is $\langle\theta^{(\ell)}_{\mathbf{u}}\mu\rangle/\langle\mu\rangle=\mathbb{C}^{0\mathbf{u}}_{00}$, where $\mathbf{u}=(u_1,\ldots,u_\ell)$.

The corresponding invariant for the torus T^2 (the real test!) is $\sum_{b\in P_+}\frac{\langle\theta^{(\ell)}_{\mathbf{u}},P_bP_b\rangle}{\langle P_b,P_b\rangle}$. For $A_1,\ell=1,\theta^{(\ell)}_{\mathbf{u}}=\theta$ as $t\to 0$, it is proportional to $1+\sum_{m\geq 1}\frac{1}{(1-q)\cdots(1-q^m)}$, which is interesting ... but diverges as $|q|<1$ (roots of unity can be used or so). There is no convergence problem though for $\theta^{(\ell)}_{\mathbf{u}}$ $(\ell\geq 0)$ if no "cycles" are allowed in the interpretation via "tubes", "pants" and "caps".

Thus, different ways to present $C^{c\mathbf{u}}_{0b}$ for general ℓ via **2-point amplitudes** C^{cu}_{0b} **of level** 1 **are 1-1 with the identities between generalized** **Rogers-Ramanujan series** **and 1-1 with adding levels to the standard pictures from TQFT (the total must be** ℓ**). In contrast to usual TQFT,** **2-p amplitudes are sufficient and explicit if all levels are** > 0**.**

Generators, relations and some amplitudes:



Making here the levels $\ell = l/m$ **presumably leads to invariants of lens spaces** $L(l, m)$ **and is related to the so-called** $\widehat{Z}$**-construction** [*Gukov–Pei–Putrov–Vafa, 2020*].

I. NIL-THEORY: THE LIMIT t → 0

The usual Rogers-Ramanujan sums occur as $t \to 0$ ($t_\nu \to 0$, **to be exact). The** μ**-function and** P**-polynomials are well-defined at** $t=0$**; we put then** $\overline{\mu}, \overline{P}_b, \overline{\mathbb{C}}^{\cdots}_{\cdots}$**. Using** $\lim_{t\to 0} q^{(b,\rho_k)} P_b^\iota(q^{c+\rho_k}) = q^{-(b,c)}$**, one obtains:** $\overline{\mathbb{C}}^{cu}_{0b} \stackrel{\textbf{def}}{=\!=} \frac{\langle \overline{P}_b \overline{P}^\iota_c \theta_u\, \overline{\mu}\rangle}{\langle \overline{P}_c \overline{P}^\iota_c \overline{\mu}\rangle} = \frac{q^{(b-c)^2/2}}{u(b{-}c)\prod_{i=1}^n \prod_{j=1}^{(c,\alpha_i^\vee)}(1-q_i^j)}$, $\overline{\mathbb{C}}^{\mathbf{cu}}_{0b} =$

$$\sum_{c_1,c_2,\ldots,c_{\ell-1}\in P_+} \overline{\mathbb{C}}^{c_1u_1}_{0b}\, \overline{\mathbb{C}}^{c_2u_2}_{0c_1}\, \overline{\mathbb{C}}^{c_3u_3}_{0c_2} \cdots \overline{\mathbb{C}}^{cu_\ell}_{0,c_{\ell-1}} =$$

$$\sum_{c_1,c_2,\ldots,c_{\ell-1}} \frac{q^{(c_0-c_1)^2/2+(c_1-c_2)^2/2+\ldots+(c_{\ell-1}-c_\ell)^2/2}}{\prod_{p=1}^{\ell} u_p(c_{p-1}{-}c_p)\prod_{i=1}^n \prod_{j=1}^{(c_p,\alpha_i^\vee)}(1-q_i^j)},$$

where $c_i \in P_+, q_i = q_{\alpha_i}, \alpha_i^\vee = 2\alpha_i/(\alpha_i,\alpha_i)$**, and we set** $c_0 = b, c_\ell = c \in P_+$**.**

Here **q-Hermite polynomials** $\overline{P}_b$ **coincide with dominant Demazure level-one characters (***Sanders, Ion***). Upon the division by their norms, they coincide with the characters of some natural quotients of the upper level-one Demazure modules and those of global Weyl modules; see [***Ch, S.Kato, 2018***] and references there.**

J. RELATION TO STRING FUNCTIONS

Let us discuss briefly the connections with string functions. Here $\widehat{\theta}_v(X) \stackrel{\mathbf{def}}{=} \sum_{a\in v+Q} q^{\frac{(a,a)}{2}} X_a$ **for** $v \in P/Q$ **are more convenient. Then** $\langle \overline{P}_b \overline{P}^{\iota}_c \widehat{\theta}_{\mathbf{v}} \overline{\mu}\rangle / \langle \overline{P}_c \overline{P}^{\iota}_c \overline{\mu}\rangle$ **for** $c_0 = b, c_\ell = c$ **become**

$$\widehat{\mathbb{C}}^{\mathbf{v}}_{b,c} = \sum_{c_1,c_2,\ldots,c_{\ell-1}\in P_+} \frac{q^{(c_0-c_1)^2/2+\ldots+(c_{\ell-1}-c_\ell)^2/2}}{\prod_{p=1}^{\ell}\prod_{i=1}^{n}\prod_{j=1}^{(c_p,\alpha_i^\vee)}(1-q_i^j)},$$

where $\mathbf{v} = \{v_1,\ldots,v_\ell\} \subset P/Q$ **and the summation is over** $c_i - c_{i+1} \in v_i + Q$**. They are zero unless** $b - c + v_1 + \ldots + v_\ell \in Q$**. When** $b = 0$**, they are modular weight-zero functions for minuscule** c**, w.r.t. some congruence subgroups of** $SL(2,\mathbb{Z})$ **and up to** $q^\bullet$**. Let** $\eta = q^{\frac{1}{24}} \prod_{i=1}^{\infty}(1-q^i)$**.**

First of all, we obtain the Rogers-Ramanujan "G". Namely, $q^{-\frac{1}{4}} \widehat{\mathbb{C}}^{111}_{0,1} = \prod_{j=1}^{\infty}(1+q^j)^2 \sum_{m=0}^{\infty} \frac{q^{2m^2}}{\prod_{j=1}^{m}(1-q^{2j})}$ **for** A_1 **and** $\ell = 3$**; it is** $\sum_{m=0}^{\infty}$ **after** $q^2 \mapsto q$**.**

Next, using parafermions and similar KM-tools, our $\widehat{\mathbb{C}}^{000}_{0,0}, \widehat{\mathbb{C}}^{110}_{0,0}, \widehat{\mathbb{C}}^{100}_{0,1}, \widehat{\mathbb{C}}^{111}_{0,1}$ **coincide with the basic string functions for** $\widehat{sl_3}$ **of level 2:** $C_0^{2\widehat{\omega}_0}, C_{\alpha_1+\alpha_2}^{2\widehat{\omega}_0}, C_{\omega_1}^{\widehat{\omega}_0+\widehat{\omega}_1}, C_{\omega_1+\alpha_2}^{\widehat{\omega}_0+\widehat{\omega}_1}$ [*Georgiev,1995*]**. This coincidence here is up to factors** $\frac{q^\bullet}{\eta^2}\times$**. Note that the level-rank duality becomes a relatively simple manipulation with the formulas for** $\widehat{\mathbb{C}}$**.**