Symmetric Tensor Categories and New Constructions of Exceptional Simple Lie Superalgebras

Arun Kannan (MIT)

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Infinite-Dimensional Algebra Seminar

Goals of this talk

- Part 1: Explain some basic notions about symmetric tensor categories

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- Part 2: Given an application to constructing exceptional simple Lie superalgebras
- Part 3: Open problems

Part 1

Modern Representation Theory

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- Example: $\operatorname{Rep} G$, the finite-dimensional representations of a group.
- The properties of $\operatorname{Rep} G$ can be summarized by saying it is a **symmetric tensor category** or STC. STCs are a home to do commutative algebra, Lie theory, algebraic geometry, etc.

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5. Has a tensor product and a unit object $\mathbb{1}$ w.r.t tensor product, the trivial representation (monoidal structure)
6. Has a natural isomorphism $c_{V,W} : V \otimes W \to W \otimes V$ that squares to identity (symmetric structure), called the **braiding**.

What is a symmetric tensor category?

A symmetric tensor category is a $\mathbb{K}$-linear, abelian, locally finite rigid, symmetric monoidal category such that $\mathrm{End}(\mathbb{1}) = \mathbb{K}$ and $\otimes$ is bilinear on morphisms. We will denote the braiding as c.

- The category $\mathrm{Vec}_{\mathbb{K}}$ of finite-dimensional vector spaces over $\mathbb{K}$ is an STC with braiding $c_{V,W} : V \otimes W \to W \otimes V$ given by $v \otimes w \mapsto w \otimes v$. The unit object $\mathbb{1}$ is $\mathbb{K}$.

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- It is an STC. The braiding $c_{V,W} : V \otimes W \to W \otimes V$ is given by $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$ (on homogeneous elements).

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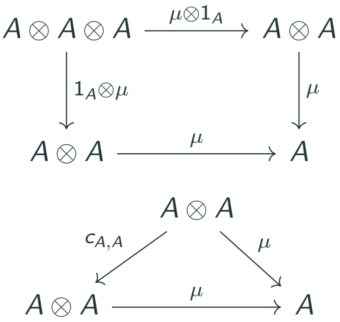
- Recall that a commutative algebra A is a vector space with multiplication and a unit object satisfying some axioms.
- Can phrase this categorically: A is an object in $\mathrm{Vec}_{\mathbb{K}}$ with two maps $\mu : A \otimes A \to A$ and $\eta : \mathbb{K} \to A$ satisfying some axioms. For instance, the following diagrams commute (associativity and commutativity):



Commutative algebras form a subcategory **CommAlg**.

An affine group scheme G is a representable functor $\mathrm{Hom}(\mathcal{O}(G), \cdot)$ from **CommAlg** to **Set** that factors through **Grp**.

Important Remark: Can extend this definition to any symmetric tensor category! Correspondence with Hopf algebras goes through in totality.

More Examples from Affine Group Schemes

- The affine group scheme GL_n assigns to any commutative algebra A the group of $n \times n$ invertible matrices w/ entries in A. It is represented by

$$\mathcal{O}(GL_n) = \mathbb{K}[\{x_{ij}\}_{i,j=1}^{n}, t]/(1 - t \cdot \det)$$

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- Representation categories of affine group schemes generalize representation categories of groups, Lie algebras, etc by process of taking matrix coefficients.
- Can do the same definitions but with commutative superalgebras to get *affine supergroup schemes*.

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- Otherwise, in characteristic zero, no: counterexamples include Deligne categories and STCs arising from oligomorphic groups.
- In characteristic $p > 0$, there are STCs with moderate growth that don't fiber over $\mathrm{sVec}_{\mathbb{K}}$, like the Verlinde category Ver_p.
- This gives us new kinds of algebra (and Lie theory, algebraic geometry, etc), one without vector spaces.

Part 2

- In [Eld06; CE07b; CE07a; Eld07], Elduque and Cunha constructed new exceptional simple Lie superalgebras (in characteristic 3)

The Elduque and Cunha Lie Superalgebras

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- Constructed using the *Elduque Supermagic Square*, a super analog of the *Freudenthal Magic Square*
- Associates a Lie superalgebra to two unital composition algebras.

Theorem (K). These Lie superalgebras (and many others) can be constructed using STCs. In particular, they are constructed by *semisimplifying* an exceptional Lie algebra equipped with a nilpotent derivation of degree at most 3.

Lie Algebras in STCs

- An operadic Lie algebra in an STC $\mathcal{C}$ is an object $\mathfrak{g} \in \mathcal{C}$ and a morphism $B : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that

$$B \circ (1_{\mathfrak{g}\otimes\mathfrak{g}} + c_{\mathfrak{g},\mathfrak{g}}) = 0;$$
$$B \circ (B \otimes 1_{\mathfrak{g}}) \circ (1_{\mathfrak{g}^{\otimes 3}} + (123)_{\mathfrak{g}^{\otimes 3}} + (132)_{\mathfrak{g}^{\otimes 3}}) = 0.$$

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- A Lie algebra as you know it is an operadic Lie algebra in $\mathrm{Vec}_{\mathbb{K}}$ $(\mathrm{char}\ \mathbb{K} \neq 2)$. A Lie superalgebra as you know it is an operadic Lie algebra in $\mathrm{sVec}_{\mathbb{K}}$ $(\mathrm{char}\ \mathbb{K} \neq 2, 3)$

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- In general might not satisfy PBW theorem but not a problem for us.

Another Example

- Let $\boldsymbol{\alpha}_p$ be the kernel of Frobenius endomorphism on $\mathbb{G}_a$ over algebraically closed field $\mathbb{K}$ of characteristic $p > 0$.
  $\operatorname{Rep} \boldsymbol{\alpha}_p \cong \operatorname{Rep} \mathbb{K}[t]/(t^p)$ is an STC.

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- $\operatorname{Rep}\alpha_p$ is not semisimple. Indecomposable objects are $J_n = \mathbb{K}^n$ for $1 \leq n \leq p$, where t acts as nilpotent Jordan block of size n:

$$t \mapsto \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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- A Lie algebra in $\operatorname{Rep} \alpha_p$ is an ordinary Lie algebra equipped with a nilpotent derivation d of degree at most p.

Semisimplification

- For $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, say f is *negligible* if $\mathrm{tr}(f \circ g) = 0$ for all $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$. Collection of negligible morphisms form a tensor ideal $\mathcal{N}$.

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- Informally, $\overline{\mathcal{C}}$ is obtained by declaring all indecomposables to be simple or if they have categorical dimension 0, zero.
- The Verlinde category Ver_p is by definition the semisimplification $\overline{\mathrm{Rep}\,\boldsymbol{\alpha}_p}$ of $\mathrm{Rep}\,\boldsymbol{\alpha}_p$.

Properties of Ver_p

- Simple objects: $L_1, \ldots, L_{p-1}$, images of $J_1, \ldots, J_{p-1}$ (resp.). The indecomposable J_p goes to zero as $\dim J_p = p = 0$.

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- Tensor product rule in general is given by the so called "truncated Clebsch-Gordan rule":

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- Theorem of Ostrik says that all semisimple STCs fiber over Ver_p.

Properties of Ver_p (cont.)

Proposition: Ver_p is not the representation category of an affine supergroup scheme in characteristics $p > 3$.

Proof.

For $p = 5$,

$$L_3 \otimes L_3 = L_1 \oplus L_3 \implies t^2 = 1 + t$$

So $t = \dim L_3$ is not integral. Similar idea for $p > 5$. □

Properties of Ver_p (cont.)

Proposition: $\mathrm{sVec}_{\mathbb{K}}$ is a full subcategory of Ver_p.

Proof.

Consider the full subcategory tensor generated by L_{p-1}. L_{p-1} has categorical dimension -1 with $L_{p-1} \otimes L_{p-1} = L_1$ and $S^2(L_{p-1}) = 0$. Hence we get the category of supervector spaces. □

Upshot: Going back to part 1, any commutative algebra, Lie theory, or algebraic geometry done in Ver_p is new but also generalizes known phenomena. Also, $\mathrm{Ver}_3 = \mathrm{sVec}_{\mathbb{K}}$.

Key idea

The semisimplification of an operadic Lie algebra $(\mathfrak{g}, B)$ in $\operatorname{Rep} \boldsymbol{\alpha}_p$ is an operadic Lie algebra $(\overline{\mathfrak{g}}, \overline{B})$ in Ver_p. In particular, when $p = 3$, we get a Lie superalgebra (that might not satisfy $[x, [x, x]] = 0$ for odd x, but not a concern for us).

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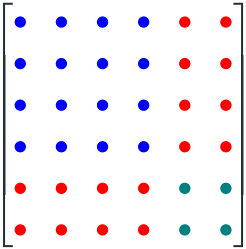
- It decomposes as $\mathfrak{gl}_6 = 16J_1 \oplus 8J_2 \oplus (J_1 \oplus J_3)$:



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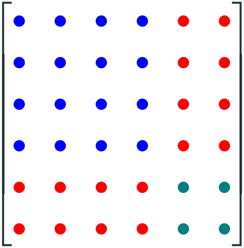

- Therefore, its semisimplification is $\mathfrak{gl}(4|1) = 16L_1 \oplus 8L_2 \oplus L_1$.

More semisimplifications:

- In any STC, define $\mathfrak{gl}(V) = V \otimes V^*$, with Lie bracket B

$$B = 1_V \otimes ev_{V^*,V} \otimes 1_{V^*} \circ \left(1_{\mathfrak{gl}(V)\otimes\mathfrak{gl}(V)} - c_{\mathfrak{gl}(V),\mathfrak{gl}(V)}\right)$$

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- Similar statement for Lie algebra that preserves a non-degenerate bilinear form (the semisimplification preserves the semisimplification of the form).

The Result, informally stated, again

Theorem (K). The Elduque and Cunha Lie superalgebras (and many others) can be constructed as the *semisimplification* of an exceptional Lie algebra equipped with a nilpotent derivation of degree at most 3.

- The setup: $A \in \mathrm{Mat}_n(\mathbb{Z})$ such that diagonal entries are either 2 or 0; if $a_{ii} = 2$, declare i to be an even index, if $a_{ii} = 0$, declare i to be an odd index. Define the Lie superalgebra $\tilde{\mathfrak{g}}(A)$ over $\mathbb{K}$ to be the free Lie superalgebra on generators $\{e_i, f_i, h_i\}_{1 \le i \le n}$ subject to the relations:

$$[e_i, f_j] = \delta_{ij} h_i; \quad [h, e_j] = a_{ij} e_j; \quad [h, f_j] = -a_{ij} f_j; \quad [h_i, h_j] = 0,$$

and let $\mathfrak{g}(A)$ be $\tilde{\mathfrak{g}}(A)/I$, where I is the maximal ideal trivially intersecting $\mathfrak{h} = \mathbb{K}h_1 \oplus \cdots \oplus \mathbb{K}h_n$.

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- The Elduque and Cunha Lie superalgebras are of this form (or "related").

The 133-dimensional simple exceptional Lie algebra $\mathfrak{e}_7$ can be written $\mathfrak{e}_7 = \mathfrak{g}(\hat{A})$, where

$$\hat{A} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

The generator e_1 is ad-nilpotent of degree 3, so can view $\mathfrak{e}_7$ as an object in $\operatorname{Rep} \boldsymbol{\alpha}_3$ w.r.t. $\operatorname{ad} e_1$.

Semisimplification in Action

Its semisimplification is a finite-dimensional simple exceptional Eldque and Cunha Lie superalgebra $\mathfrak{g}(A)$ of superdimension $(66|32)$, where

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix};$$

Idea: the copy of J_2 spanned by e_1 and $[e_1, e_3]$ in $\mathfrak{e}_7$ became an odd generator (resp. f) in the semisimplification.

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- A few of them, however, cannot be determined by looking at Cartan matrix alone; these must be manually determined. For instance, there is the Elduque Lie superalgebra in characteristic 5. This can be constructed by semisimplifying $\mathfrak{e}_8$ with respect to $e_2 + e_3 + e_4$.

Summary of Results

Lie algebra	Nilpotent element	Lie superalgebra
$\mathfrak{br}_3$	e_1, e_2	$\mathfrak{brj}_{2;3}$
$\mathfrak{f}_4$	e_1	see $(\star)$ below
	e_4	$\mathfrak{g}(1,6)$
	$e_1 + e_4$	see $(\star)$ below
$\mathfrak{e}_6^{(1)}$	e_1, e_2, e_6	$\mathfrak{g}(2,6)^{(1)}$
	$e_1 + e_2,\ e_2 + e_6,\ e_1 + e_6$	$\mathfrak{g}(3,3)^{(1)}$
	$e_1 + e_2 + e_6$	$\mathfrak{g}(2,3)^{(1)}$
$\mathfrak{e}_7$	e_1, e_2, e_7	$\mathfrak{g}(4,6)$
	$e_1 + e_2,\ e_2 + e_7,\ e_1 + e_7$	$\mathfrak{el}(5;3)$
	$e_1 + e_2 + e_7$	$\mathfrak{g}(4,3)$
	$e_2 + e_5 + e_7$	$\mathfrak{f}_4$; see $(\star\star)$ below
	$e_1 + e_2 + e_5 + e_7$	$\mathfrak{g}(1,6)$
$\mathfrak{e}_8$	e_1, e_2, e_8	$\mathfrak{g}(8,6)$
	$e_1 + e_2,\ e_2 + e_8,\ e_1 + e_8$	$\mathfrak{g}(6,6)$
	$e_1 + e_2 + e_8$	$\mathfrak{g}(8,3)$
	$e_1 + e_2 + e_6 + e_8$	$\mathfrak{g}(3,6)$

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- What other simple Lie superalgebras can be obtained this way? What about simple Lie algebras in Ver_p?
- Semisimplify other algebraic objects, like affine group schemes. What happens?

Part 3

Some Open Problems

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- More generally: what theorems that extend from vector spaces to supervector spaces extend to the Verlinde setting?

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