$\boxed{P=W}$ joint with S. Chiarello and T. Hausel

Higgs bundles

C genus g, $\xi \xrightarrow{\text{l.b. deg}=1} C$, K - can. bdle.

$$\mathcal{N} = \left\{ \begin{matrix} V \\ \downarrow \\ C \end{matrix} \quad \begin{matrix} rk=2 \\ \Lambda^2 V \cong \xi \end{matrix} \;\middle|\; L \subset V \Rightarrow \deg L \leq 0 \right\} \Big/ \sim$$

smooth projective $\dim = 3g-3$

ϕ : λ_i e.v. ; $Tr\,\phi = 0$

$$\mathcal{M} = \left\{ \left(\begin{matrix} V \\ \downarrow \\ C \end{matrix} \quad \begin{matrix} rk=2 \\ \Lambda^2 V \cong \xi \end{matrix}, \phi : V \to V \otimes K \right) \;\middle|\; \begin{matrix} L \subset V, \phi(L) \subset L \otimes K \\ \Rightarrow \deg L \leq 0 \end{matrix} \right\} \Big/ \sim$$

smooth qproj, $\dim = 6g-6$ $\mid \phi = 0 \Rightarrow$ Higgs stab $\Downarrow$ stab

- $T^*\mathcal{N} \subset \mathcal{M}$ open
- $G = \mathbb{C}^* \curvearrowright \mathcal{M}$: $(\lambda, (V, \phi)) \mapsto (V, \lambda\phi)$

$\mathcal{M}^G$ compact ; components $\mathcal{N}$ $(\phi = 0)$,

$V = L_1 \oplus L_2$ $F_i = S^{2g-2i-1} C$ $i = 1, \dots, g-1$

$\phi : L_1 \to L_2 \otimes K$

Cohomology of $\mathcal{M}$

$\mathcal{U} \to \mathcal{U} \otimes K$

$\downarrow \circlearrowleft G$ $\quad c_2(\mathrm{End}(\mathcal{U})) = 2\alpha \otimes \omega + 4 \sum \psi_i \otimes e^i - \beta \otimes 1$ (deg: 2, 3, 4)

- Generators: $\mathcal{M} \times C$ $\quad \gamma^6 = \sum \psi_i \psi^i$

Markman : $I_{\mathcal{M}} \subset \mathbb{Q}[\alpha, \beta, \psi_\cdot] \longrightarrow H^*_G(\mathcal{M})$ surjective

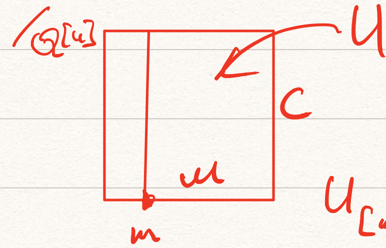
$\downarrow$

$H^*(\mathcal{N})$

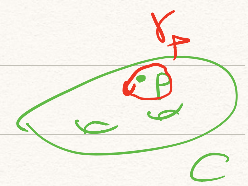
$I_{\mathcal{N}} > I_{\mathcal{M}}$

bigraded

$/\mathbb{Q}[u]$



$\mathcal{U}_{[m]} \cong m$

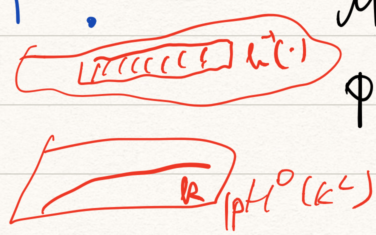


- Equivariant case ✓
- W: $\mathcal{M}_B \cong \{\pi_1 \Sigma\setminus\{p\} \longrightarrow SL_2(\mathbb{C}) \mid \gamma_p \to -I\} / \mathrm{Ad}\, SL_2\mathbb{C}$

Affine $\quad \mathcal{M} \underset{diff}{\cong} \mathcal{M}_B$!!!

MHS ⇝ new grading: $\deg_w \alpha = \deg_w \beta = \deg_w \psi_i = 2$

- P:



$$\mathcal{M} \xrightarrow{h} H^0(K^2) \quad \text{dim } 3g-3$$
$$\varphi \longmapsto \det\varphi$$

P: filtration of $H^*(\mathcal{M})$ w.r.t. $\sigma \in P_k H^j$ if $\operatorname{supp}\sigma \subset h^{-1}(\mathrm{Aff}\ \dim_k)$

The enumerative P=W

P=W was proved in dC-H-M.
dC-M-Sh.

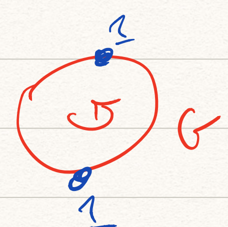
Thm [Enumerative P=W] $\quad Z = \mathcal{M}/\!/G^{\text{proj}}$ $\quad$ dim $6g-7$, $\;\downarrow h\;\; \mathbb{P}^{3g-4}$

surj $H^*(Z) \longleftarrow H^*(\alpha,\beta,\gamma,u)$

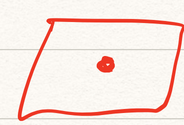
P=W is equivalent to $\exists$ extension $E = \beta^e\gamma^m + u\, q(\alpha,\beta,\gamma,u)$ s.t.

$$\int_Z \Big[\underbrace{u^{3g-3-2e-2m}\cdot E}_{=0 \text{ in } H^*(Z)}\Big]\cdot x = 0 \qquad \forall\, x \in H^*(Z).$$

$\mathbb{P}^1$



$$\oint_{\mathbb{P}^1} 1 = \frac{1}{2u} + \frac{1}{-2u} = 0$$



$$\oint_0 1 = \frac{1}{2u}\,; \quad \int_{\mathbb{C}/\!/\mathbb{C}^*} 1 = \frac{1}{2}$$

Equivariant Integration

$\mathbb{C}^\times \cong G \circlearrowright Y$, Y^G cpt $\rightsquigarrow \sigma \in H^\bullet_G(Y)$

$$\oint_Y \sigma \overset{\text{def}}{=} \sum_{F \subset Y^G \text{ comp}} \int_F \frac{\sigma|_F}{e_G(N_F)} \in \mathbb{Q}(u)/u^{\dim Y}$$

Kalkman: $\displaystyle\int_Z \sigma = -\operatorname*{Res}_{u=0} \oint_M \sigma \, du$

$$= -\operatorname*{Res}_{u=0} \left[\int_W \frac{\sigma|_W}{e_G(N_W)} + \sum_{i=1}^{g-1} \int \frac{\sigma|_{F_i}}{e_G(N_{F_i})} \right]$$

- Miracle 1 $\quad T \subset S^\bullet(\mathfrak{t}^*)^W = \mathbb{C}[y^2]$

$\rightsquigarrow T(u) \in H^*(M \times C)$

$\rightsquigarrow T_{(0)} = T(u) \wedge 1$, $T_{(2)} = T(u) \wedge \omega$

Thm

$$\oint T_{(0)} \exp Q_{(2)} = \frac{1}{2} \sum_{r \in \{0, \pm u\}} \operatorname*{Res}_{y=r} \frac{T \left(Q' - \frac{2u}{u^2-y^2}\right)^g dy}{u^{g-1} \left(e^{Q'} \frac{u-y}{u+y} - \cdot\right) y^{2g-2} (u^2-y^2)^{g-1}}$$

- Miracle 2.

$$\operatorname*{Res}_{u=0} \operatorname*{Res}_{y=0,\pm u} \cdot \, dy\, du = \operatorname*{Res}_{y=0} \left[\operatorname*{Res}_{u=0} + \operatorname*{Res}_{u=y \tanh(Q'/2)} \right] \cdot du\, dy$$

Thm

$$\int_{\overline{\mathcal{M}}} u^{3g-3-2k} T_{(0)} \exp Q_{(2)} =$$

$$= \operatorname*{Res}_{y=0} \frac{dy}{y^{4k-1}} T(u = y \tanh Q'/2) \cdot R^{Q}$$

$$R = \left(\frac{\sinh Q'/2}{y}\right)^{2g-2k-2} \cosh^{2k-2}(Q'/2) \left(Q'' + \frac{\sinh Q'}{y}\right)^{g}$$

$$\deg(\alpha) = 0, \quad \deg(\beta) = \deg(\gamma) = \deg(u) = 2$$

Thm for β^k There exists $F \in H^{*}(\overline{\mathcal{M}})$ s.t. $\forall P \in H^{*}(\overline{\mathcal{M}})$ $\deg(P) = 2k-2$ we have

F and P are polys in α, β, γ, u

$$\int u^{3g-3-2k} (\beta^k + uF) \cdot P = 0$$

- $k \leq g-1$, $\deg(F_k) = 2k-2$.

$F = F_{2k-2} + F_{2k-1} \ldots$

The Matrix of monomials

$$\left(M_k\right)^{\overset{P}{\downarrow}\, a_2 n_2}_{\underset{F}{\uparrow}\, a_1 n_1} = \binom{3g-3-a_1-n_1-a_2-n_2}{k-a_1-n_1} \binom{g-n_2}{n_1}$$

$0 < n_1 \leq a_1 \quad a_1 + n \leq k \longleftarrow F \quad \frac{k^2}{4}$

$0 \leq n_2 \leq a_2 \leq k-1 \longleftarrow P \quad k^2/2$

Solution:

unknown coeff

Let $\quad a_2 \quad n_2$

$$p_k(X,Z) = \sum L_{a_1,n_1} \binom{\cdots\; z-x}{\cdots} \binom{\cdots\; x}{\cdots}$$

Then for $\quad P_k = \operatorname*{Res}_{t=0} \frac{dt}{t^{k+1}} (1+t)^{Z-2X} (1+2t)^{X}$

$p_k(X,Z) = 0$ for $(X,Z) = (n_2, n_2 + a_2)$

in $\Gamma_k = \{ (X,Z) \in \mathbb{Z}^2 \mid 0 \le X,\ 2X \le Z \le X+k \}$