From diagram algebras via Ringel duality to highest weight categories

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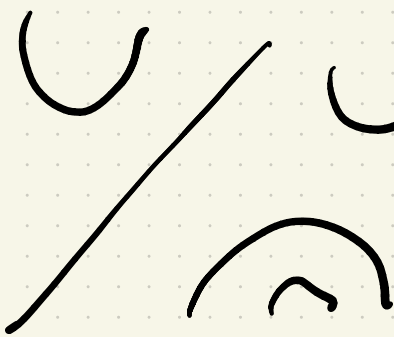


(jt with J. Brundan)

Ringel duality (classical form)

A f.d. quasi-hereditary algebra

T tilting generator

$A' = \mathrm{End}_A(T)$

$$\left\{ \begin{array}{c} \text{quasi-hereditary} \\ \text{algebras with} \\ \text{poset } \Lambda \end{array} \right\} \overset{RD}{\longleftrightarrow} \left\{ \begin{array}{c} \text{quasi-hereditary} \\ \text{algebras with} \\ \text{poset } \Lambda^{op} \end{array} \right\}$$

$$A \longleftrightarrow A'$$

Today:

- Show: only a shadow of a more general picture (including infinite dimensional (co) algebras)

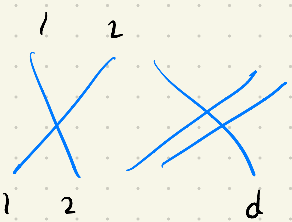
  Motivated by some Tannakian construction

- Incorporating Cline – Parshall – Scott
- algebraic versus categorical
- common framework for different types of stratified categories / algebras

<u>Motivation / Guiding principle</u>. Diagram algebras

e.g. $\mathbb{C}[\overset{\substack{\omega \\ \cap}}{S_d}]$ basis: permutation diagrams

eg



$\omega(1) = 2$

diagrammatical comp.

A zoo

0) $\mathbb{C}[S_d]$ symmetric group — permutation diagrams

centraliser of GL_n, $\mathfrak{gl}_n$

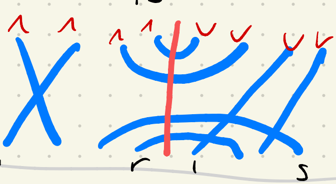
1) $TL_d(\delta)$ Temperley-Lieb algebra — crossingless matchings

$\mathfrak{sl}_2$, $\mathfrak{sl}_2$, $U_q(\mathfrak{sl}_2)$

2) $Br_d(\delta)$ Brauer algebra — pairings

$\mathfrak{so}$, $\mathfrak{sp}$ $\quad OSp(m|2n)$

3) $WBr_{r,s}(\delta)$ walled Brauer algebra — oriented pairings

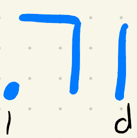


$\mathfrak{gl}_n \curvearrowright V^{\otimes r} \otimes V^{*\otimes s}$

$GL(m|n)$

4) $Par_d(\delta)$ partition algebra — partitionings

GL_n
$\cup\!|$
$S_n \curvearrowright V^{\otimes d}$



⋮ ⋮ ⋮

loops

- Diagrammatical multiplication ~ replace by $\delta \in k$
- They all naturally appear as centraliser algebras

Better: Consider the corresponding categories

$\mathcal{A} = \mathbb{C}[S_d]_{d\geq 0}$, $Br(\delta)$, $TL(\delta)$, $oBr(\delta)$, ...

objects: $\mathbb{N}_0$ (resp. finite sequences from $\{\wedge, \vee\}$ for oBr)

morphisms: $m \to n$

$\mathcal{A}(\delta)$ - linear combinations of the same types of diagrams but from m to n points

e.g 

$11 \uparrow 5$

Composition: concatenation and replacing $\circ$ by δ

Rk: Can view $\mathcal{A}$ as an algebra A

but only locally unital, locally finite dim

i.e. $A = \bigoplus_{i,j \in Ob(\mathcal{A})} \underbrace{e_i A e_j}_{\text{finite dimensional!}}$ $\quad \mathrm{Hom}_{\mathcal{A}}(i,j)$

"A-modules" means locally fd A-modules

$A\text{-mod}_{\ell fd}$, resp. functors from $\mathcal{A}$ to Vect.

A locally unital locally fd algebra

might have more finiteness condition

$\boxed{\text{finite dimensional}}$

$\boxed{\text{essentially fd.}}$ $\quad \dim Ae_i < \infty$, $\dim e_i A <$

- Semisimple fd algebras A have decompositions $A = \bigoplus_i Ae_i \otimes e_iA$

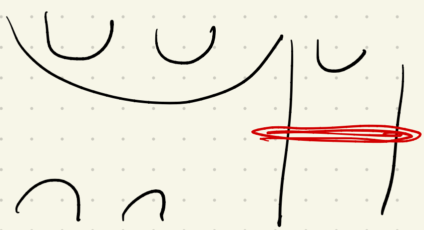
$\mathrm{End}(V) = V \otimes V^*$

$k[S_d]$ decomposition in RS-correspondence.

- Diagram algebras come with a nice basis

Observation: can combine this:

e.g. TL has basis $y \circ x$



cup diagram y

number of trough strands $\circ$ e_2

cap diagram $\circ$ x

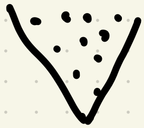
Natural ordering on idempotents e_i $i \in I := \mathbb{N}_0$

$\rightsquigarrow$ poset $\Lambda = \mathbb{N}_0$ with actually opposite natural ordering (upper finite)

Λ poset

- finite $|\Lambda| < \infty$

$\forall \lambda, \mu \in \Lambda$

- interval finite, e.g. $\mathbb{Z}$: $|\{\nu \mid \mu \le \nu \le \lambda\}| < \infty$
- upper finite  $|\{\nu \mid \nu \ge \lambda\}| < \infty$
- lower finite  $|\{\nu \mid \nu \le \lambda\}| < \infty$

Def: (Based quasi-hereditary algebra)

BqhA

A finite, essentially finite, upper finite resp. BqhA is a

finite dim., essentially fd., locally fd

locally unital algebra $A = \bigoplus_{i,j \in I} e_i A e_j$

with

(QH1) A subset $\Lambda \subseteq I$ indexing special idempotents

(QH2) Partial ordering on Λ which makes Λ finite, interval finite, upper finite, resp.

(QH3) Sets $Y(i,\lambda) \subseteq e_i A e_\lambda$

$X(\lambda, j) \subseteq e_\lambda A e_j \quad \forall \lambda \in \Lambda$, $i,j \in I$

such that

write $Y(\lambda) := \bigcup_{i \in I} Y(i,\lambda)$, $X(\lambda) = \bigcup_{j \in I} X(\lambda, j)$

(QH4) The yx with $(y,x) \in \bigcup_\lambda Y(\lambda) \times X(\lambda)$ is a basis of A

(QH5) $Y(\mu,\lambda) \neq \emptyset$ or $X(\lambda,\mu) \neq \emptyset \Rightarrow \mu \leq \lambda$

(QH6) $Y(\lambda,\lambda) = \{e_\lambda\} = X(\lambda,\lambda)$

triangular basis

Passage to categories

Theorem A finite / upper finite / essentially finite Bgh A

$\Rightarrow$ $A\text{-mod}_{fd}$ / $A\text{-mod}_{efd}$ / $A\text{-mod}_{fd}$

is a finite / essentially finite / upper finite

highest weight category

image of e_λ in $A_{\leq\lambda}$

with

$$\Delta(\lambda) := A_{\leq\lambda}\,\overline{e_\lambda}$$

$$\nabla(\lambda) := \left(\overline{e_\lambda}\, A_{\leq\lambda}\right)^{\circledast}$$

$$A_{\leq\lambda} := A/_{(e_\mu \mid \mu \not\leq \lambda)}$$

Warning: $\Delta(\lambda)$, $\nabla(\lambda)$ might have infinite length!

Cor 1: Examples 0) – 4) give rise to upper finite hw. categories

$\mathcal{R}$ abelian category

Stratification of $\mathcal{R}$

- labelling of irreducibles $L : \mathcal{B} \to \mathrm{Ob}(\mathcal{R})$ (set)

  s.t. $L(b)$ $b \in \mathcal{B}$ full system of representables for isoclasses of irreducibles

- stratification function $\rho : \mathcal{B} \to \Lambda$ (poset)

  such that $\rho^{-1}(\lambda) =: \mathcal{B}_\lambda$ is finite $\forall \lambda$

(→ Losev–Webster)

$\lambda \in \Lambda \rightsquigarrow \mathcal{B}_\lambda \qquad \mathcal{B}_{\leq \lambda} \qquad \mathcal{B}_{< \lambda}$

Corresponding Serre subcategories of $\mathcal{R}$: $\qquad \mathcal{R}_{\leq \lambda} \qquad \mathcal{R}_{< \lambda}$

Here recollement:

Strata

Hom$_A$(Ae, $-$)

$Ae \otimes_{eAe} -$

$$\begin{tikzcd}
R_{<\lambda} \arrow[r, hook, "i_\lambda"] & R_{\leq\lambda} \arrow[r, two heads, "j^\lambda"] \arrow[l, bend right, "i^!_{<\lambda}"'] \arrow[l, bend left, "i^*_{<\lambda}"] & \underbrace{R_{\leq\lambda}/R_{<\lambda}}_{\text{Serre quotient}} =: R_\lambda \arrow[l, bend right, "j^\lambda_*"'] \arrow[l, bend left, "j^\lambda_!"]
\end{tikzcd}$$

Def: $\boxed{\begin{array}{ll} \Delta(b) := j^\lambda_! P_\lambda(b) & \nabla(b) := j^\lambda_* I_\lambda(b) \\ \bar{\Delta}(b) := j^\lambda_! L_\lambda(b) & \bar{\nabla}(b) := j^\lambda_* L_\lambda(b) \end{array}}$

(proper) (co)standard objects

Special case: • All strata are (semi) simple

then $\Delta = \bar{\Delta}$

• All strata are basic, local

maybe even weakly Frobenius

$$\Delta = \begin{matrix} & \bar\Delta & \\ \bar\Delta & & \bar\Delta \\ & \bar\Delta & \end{matrix}$$

$(\hat{P}_\Delta)$ $\forall b \in \mathcal{B}$ $\exists$ P_b projective with Δ-flag such that

$$\underbrace{K}_{\text{has } \Delta\text{-flag with } \Delta(c)\text{'s } \rho(c) \geq \rho(b)} \hookrightarrow P_b \twoheadrightarrow \Delta(b)$$

(P_Δ) $\forall b \in \mathcal{B}$ $P(b)$ has Δ-flag and

$$\underbrace{K}_{\text{has } \Delta\text{-flag with } \Delta(c)\text{'s } \rho(c) \geq \rho(b)} \hookrightarrow P(b) \twoheadrightarrow \Delta(b)$$

$\mathrm{Ext}^i(\Delta, \bar{\nabla}) = 0$, $i > 0$

$\mathrm{Ext}(\bar{\Delta}, \nabla) = 0$

<u>Lemma</u>: $(\hat{P}_\Delta) \Leftrightarrow (P_\Delta) \Leftrightarrow (\hat{I}_{\bar{\nabla}}) \Leftrightarrow (I_{\bar{\nabla}})$ $+$

$(\hat{P}_{\bar{\Delta}}) \Leftrightarrow (P_{\bar{\Delta}}) \Leftrightarrow (\hat{I}_\nabla) \Leftrightarrow (I_\nabla)$ $-$

(Might think of Δ's as thick Vermas $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \underbrace{S(\mathfrak{h})/I}_{\text{fd}}$ and $\bar{\Delta}$'s as Vermas) finite

$\mathcal{R} = A\text{-mod}_{fd}$

$\uparrow$ fd

Λ finite

upper finite

$\mathcal{R} = A\text{-mod}_{lfd}$

$A = \bigoplus_{i,j \in I} e_i A e_j$

Λ upper finite

essentially finite

$\mathcal{R} = A\text{-mod}$

$A = \bigoplus_i \underbrace{Ae_i}_{fd}$

also $e_i A$

Λ essentially finite

blocks of classical $\mathcal{O}$

rational Cherednik $\mathcal{O}$

Schur algebras

diagram cat.

$\mathcal{O}$ (affine Kac-Moody) positive level

Rep $(GL(1|1))$

$\mathcal{O}(\mathfrak{gl}(m|n))$

G_rT-modules

$\mathcal{O}'$ (affine Kac-Moody) restricted ~~level~~ critical level

⋮

Lemma: $(\hat{P}\Delta) \Leftrightarrow (P\Delta) \Leftrightarrow (\hat{I}\bar{\nabla}) \Leftrightarrow (I\bar{\nabla})$ $+$

$(\hat{P}\bar{\Delta}) \Leftrightarrow (P\bar{\Delta}) \Leftrightarrow (\hat{I}\nabla) \Leftrightarrow (I\nabla)$ $-$

$L : \mathbb{B} \to \mathrm{Ob}(\mathcal{R})$ $\rho : \mathbb{B} \to \Lambda$

$\mathcal{R}$ finite, upper finite, essentially finite

+-stratified if $\oplus$ holds

−-stratified if $\ominus$ holds

fully-stratified if both hold (stratified algebras)

fibred highest weight if both and ρ bijection (properly stratified Dlab, Ringel, König, Kleshchev)

highest weight $\leftarrow$ if ρ is bijection and strata are simple

$\oplus$ & $\ominus$ holds

($\Leftrightarrow$ ρ bij. and $\Delta = \bar{\Delta}$, $\nabla = \bar{\nabla}$)

E-stratified ....

<u>Back to examples</u> $\mathcal{A} = TL$ is monoidal □ □

Have monoidal functor:

$$F : \mathcal{A} \longrightarrow (\mathrm{Vect}_{fd})^{op}$$

$$\bullet \longmapsto V$$

$\cap$ $\quad (v_i \otimes v_j \mapsto (v_i, v_j))$

$\cup$ $\quad X$ $\quad (1 \longmapsto v_2 \otimes v_1 - q\, v_1 \otimes v_2)$

$V = \langle v_1, v_2 \rangle$ 2-dim k-vsp

$(-,-) : V \otimes V \to k$

$v_1 \otimes v_2 \longmapsto 1$

$v_2 \otimes v_1 \longmapsto \cancel{1}\ -q^{-1}$

$v_1 \otimes v_1 \longmapsto 0$

$v_2 \otimes v_2 \longmapsto 0$

$\bigcirc = -q - q^{-1} = \delta$, $\quad X = \asymp + q \,||$ $\quad X = \asymp + q^{-1} ||$

$\Updownarrow$

Tensor algebra $T = T(V) \in \mathrm{mod}_{ef}\text{-}A$

$T^{\otimes} = T(V^*) \in A\text{-}\mathrm{mod}_{ef}$

Tannakian reconstruction

$$C = \mathrm{Coend}_A(T^{\otimes}) := T \otimes_A T^{\otimes} = T(V) \otimes_A T(V^*)$$

commutative Hopf algebra

⌜ • tensor product of algebras

• $v_i^{(d)}$ basis of $T(V)^d$

$u_i^{(d)}$ (dual) basis of $T(V^*)^d$

generate C as algebra

$$c_{i,j}^{(d)} := v_i^{(d)} \otimes u_j^{(d)}$$

$$\Delta(c_{ij}^{(d)}) = \sum_r c_{ir}^{(d)} \otimes c_{rj}^{(d)}$$

$$\eta(c_{ij}^{(d)}) = \delta_{ij}$$ ⌟

Rk: To get a full set of relations between generators it's enough to take

$v\, x \otimes u = v \otimes x u \quad \forall\, x$ monoidal generator

corresp. $v \in T,\ u \in T^{\otimes}$ homog

In our example: enough $c_{r,s} := v_r \otimes u_s$

$$\boxed{\begin{aligned} (v_j \otimes v_i \cup) \otimes 1 &= v_i \otimes v_j \otimes \cup 1 \\ (1 \cap) \otimes u_i \otimes u_j &= 1 \otimes \cap (u_i \otimes u_j) \end{aligned}}$$

Calculation

$\rightsquigarrow$ generators $c_{11}, c_{12}, c_{21}, c_{22}$

relations:

$$\boxed{\begin{aligned} &c_{i2}c_{i1} = q\, c_{i1} c_{i2}, \quad c_{2j} c_{1j} = q\, c_{1j} c_{2j} \\ &c_{12} c_{21} = c_{21} c_{12}, \quad c_{22} c_{11} = c_{11} c_{22} - (q - q^{-1}) c_{12} c_{21} \\ &c_{11} c_{22} - q^{-1} c_{12} c_{21} = 1 \end{aligned}}$$

$\Rightarrow$ $C \cong k_q[SL_2]$ quantum coordinate ring

and $\text{comod}_{fd} - C \cong$ rational representations of quantum SL_2

In case $q=1$ or q generic irred.

rational reps are labelled by $\mathcal{B} = \Lambda^+$

(dominant integral weight) which is a lower

finite poset with usual ordering 

lower finite

stratification

$\wedge$

lower finite

$\mathcal{R}$ locally finite

locally finite

(EGNO)

$\mathcal{R} \cong \mathrm{comod}_{fd}\text{-}C$

Ind-completion

$\mathrm{comod}\text{-}C$

$\cup$

$\mathrm{comod}_{fd}\text{-}C$

(Ass): $L(b)$ has proj. cover and injective hull in $\mathcal{R}_{\leq \rho(b)}$ Δ $\bar{\Delta}$

$\Rightarrow$ earlier defs make sense if we truncate

Def: whatever stratified or highest weight as before if that properties hold for $\mathcal{R}^{\downarrow} \subseteq \mathcal{R}$, the Serre subcat. to an arbitrary lower set $\Lambda^{\downarrow} \subseteq \Lambda$

→ Cline – Parshall – Scott
(Marko – Zubkov)

Prop: (Homological characterisation)

Assume ∇ with (ASS)

If $\mathrm{Ext}^1(\bar{\Delta}(b), \nabla(c)) = 0 = \mathrm{Ext}^2(\bar{\Delta}(b), \nabla(c)) = 0$

and $\mathrm{Ext}^1(\Delta(b), \nabla(c)) = 0 = \mathrm{Ext}^2(\Delta(b), \bar{\nabla}(c)) = 0$

then $\mathcal{R}$ is lower finite fully stratified

If additionally $\mathrm{Hom}(\Delta(\lambda), \nabla(\lambda))$ is 1-dim.

then $\mathcal{R}$ is highest weight

(→ Riche – Williamson, Coulembier ..)

Ex: • Rep(G) G reductive algebraic group

• blocks of $\mathcal{O}$ (Affine Kac-Moody)

for negative level

are lower finite hw. categories

• Q quiver

- if kQ locally fd then

$\mathrm{Rep}(Q)_{fd} = \mathrm{comod}_{fd}\text{-}kQ^{op}$ is

$\boxed{\text{lower finite hw. for } \underline{\text{any}} \text{ lower}}$

finite ordering on vertices

- General case of arbitrary quiver

$\mathrm{Rep}(Q)^{\text{nil-finite length}} \simeq \mathrm{comod}\text{-}kQ^{op}$

not covered, but it is if we fix a nilpotence degree.

Tilting objects

Assume (ε) for $\varepsilon = +$ or $\varepsilon = -$

$$\Delta_\varepsilon := \begin{cases} \Delta & \text{if } \varepsilon = + \\ \bar{\Delta} & \text{if } \varepsilon = - \end{cases} \qquad \nabla_\varepsilon := \begin{cases} \bar{\nabla} & \text{if } \varepsilon = + \\ \nabla & \text{if } \varepsilon = - \end{cases}$$

$$\mathrm{Tilt}_\varepsilon(\mathcal{R}) := \Delta_\varepsilon(\mathcal{R}) \cap \nabla_\varepsilon(\mathcal{R})$$

objects with both Δ_ε and ∇_ε-flag

Lemma $\mathrm{Tilt}_\varepsilon(\mathcal{R})$ is Karoubian

Prop

1) In finite, essentially finite case have

$$\left\{ \begin{array}{l} \text{indec.} \\ \varepsilon\text{-tiltings} \end{array} \right\} \overset{1:1}{\longleftrightarrow} \Lambda$$

$\bar{\nabla}(\lambda)$

$\bar{\nabla}(\mu)$'s

$\Delta(\mu)$'s

$\Delta(\lambda)$

2) In upper finite same statement if we define $\mathrm{Tilt}_\varepsilon(\mathcal{R}) := \Delta_\varepsilon^{asc}(\mathcal{R}) \cap \nabla_\varepsilon^{desc}(\mathcal{R})$

Standard example: $V^{\otimes d}$ for $U_q(\mathfrak{sl}_2)$ or GL_2

is tilting

T is tilting generator if

$T = \bigoplus_{i} T_i$ such that each $T_\varepsilon(b)$ is isomorphic to a summand of T appearing with finite multiplicity

$\in$ $\mathrm{Ind}(\mathcal{R})$ for lower finite

$T_i \in \mathrm{Tilt}_\varepsilon(\mathcal{R})$

In essentially finite case assume also that

matrix $\left(\mathrm{Hom}(T_\varepsilon(a), T_\varepsilon(b))\right)_{a,b\in B}$ finitely many entries $\neq 0$ in each row, column

tilting bounded

$(\Leftrightarrow$ $\forall a$ $\exists$ finitely many b $(T(b):\nabla(a)) \neq 0$

" " " " $(T(b):\Delta(a)) \neq 0$ $)$

$\Rightarrow$ dim $\mathrm{Hom}(T_\varepsilon(a), T_\varepsilon(b)) < \infty$ in all cases

Fix tilting generator T

Theorem (Semiinfinite Ringel duality I)

$$\left\{\begin{matrix}\text{lower finite}\\ \varepsilon\text{-stratified cat}\end{matrix}\right\} \rightleftarrows \left\{\begin{matrix}\text{upper finite}\\ (-\varepsilon)\text{-stratified cat}\end{matrix}\right\}$$

$$\mathcal{R} \longmapsto \mathcal{R}' = A\text{-}\mathrm{mod}^{d}_{lf}$$

with $F = \bigoplus_i \mathrm{Hom}_{\mathcal{R}}(T_i, -)$ for $T = \bigoplus_i T_i$

tilting generator $A = \left(\bigoplus_{i,j} \mathrm{Hom}_{\mathcal{R}}(T_i, T_j)\right)^{op}$

$$\mathcal{R}' \longleftarrow\!\!| \ \mathcal{R}$$

$\| $

$\mathrm{Comod}_{fd}\text{-}C$

$C = \mathrm{Coend}_{\mathcal{R}}(T) = T^{\circledast} \otimes_A T$

Restricts to:

$$\left\{\begin{array}{c}\text{lower finite}\\ \text{hw - categories}\end{array}\right\} \rightleftarrows \left\{\begin{array}{c}\text{upper finite}\\ \text{hw. categories}\end{array}\right\}$$

Rk: Restricts further to classical Ringel duality. In the finite dim case can pass between modules and comodules

$$A\text{-mod}_{fd} \cong \text{mod}_{fd}\text{-}C \qquad C = A^*$$

Tilting-boundedness also appears naturally:

$$\left\{\begin{array}{c}\text{Tilting bounded}\\ \text{lower finite hw.}\\ \text{categories} \quad \Lambda\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Costen bounded}\\ \text{hw. categories}\\ \text{upper finite}\\ \Lambda^{op}\end{array}\right\}$$

Theorem (Semi-infinite Ringel-duality II)

$$\left\{\begin{array}{c}\text{essentially finite}\\ \text{tilting bounded}\\ \varepsilon\text{-stratified cat}\end{array}\right\} \overset{RD}{\longleftrightarrow} \left\{\begin{array}{c}\text{essentially finite}\\ \text{tilting bounded}\\ (-\varepsilon)\text{-stratified cat}\end{array}\right\}$$

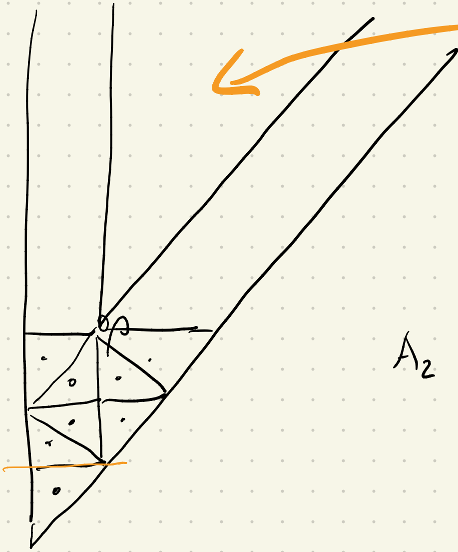
Warning: In the lower finite case tilting-boundedness is <u>not</u> automatic

e.g. • Rep(G) is <u>not</u> tilting-bounded except if it is semisimple.

• principal block of $U_q(\mathfrak{g})$-mod

Lusztig's divided power quantum grp q primitive odd root of unity (order > Coxeter number)

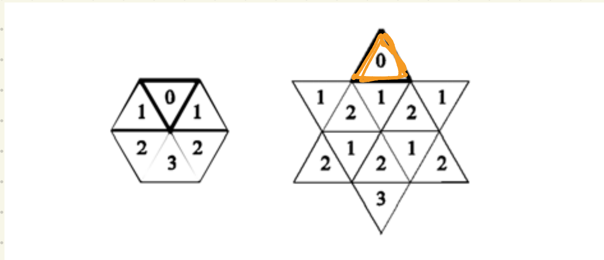
Soergel's Conj/Thm $\Longrightarrow$ tilting boundedness is a question about affine KL-polys



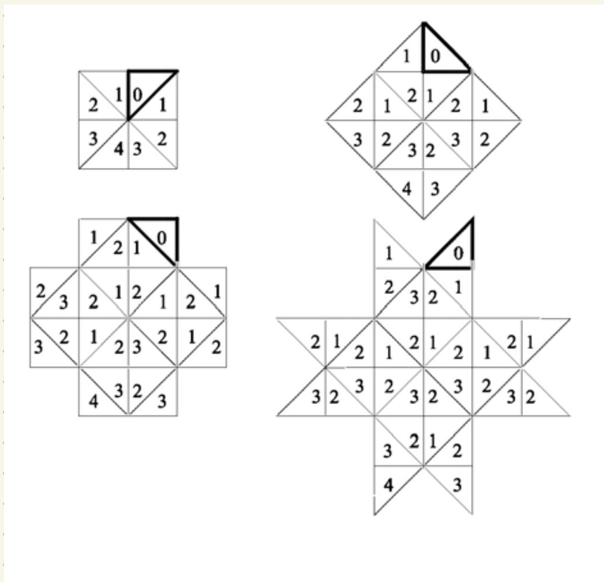
Lusztig's generic patterns

$P_{\lambda,\mu}(q)$

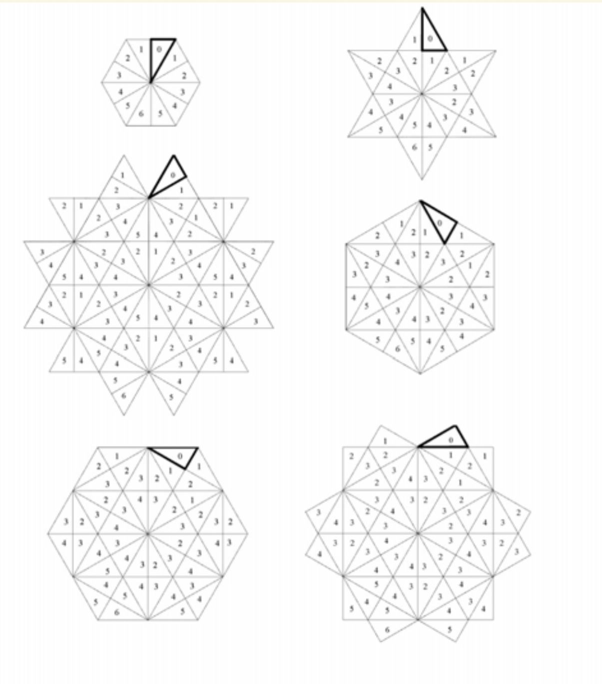
A_2

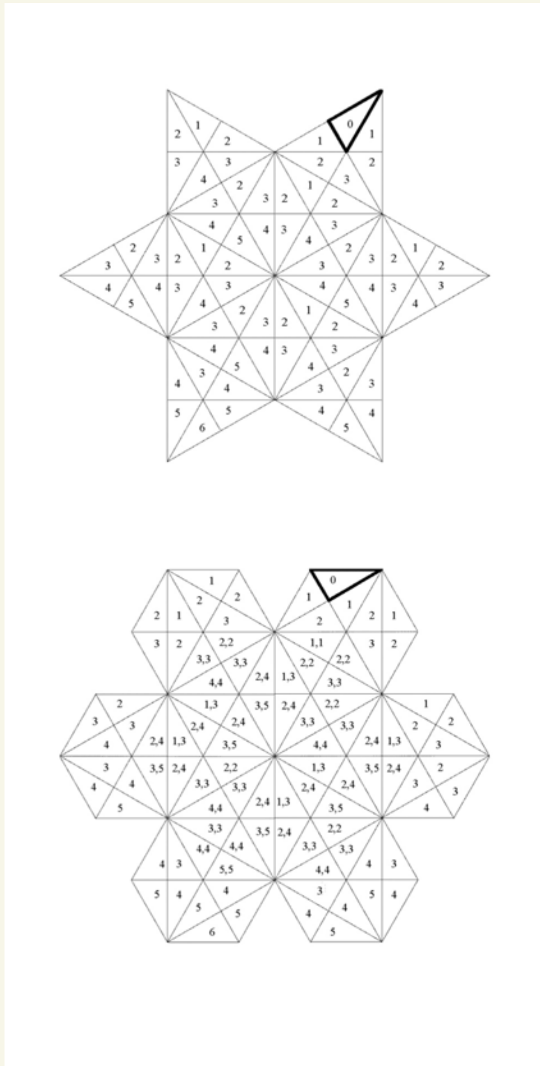


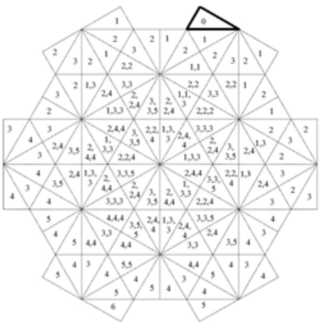
B_2:

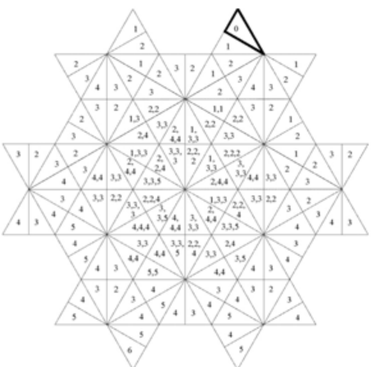


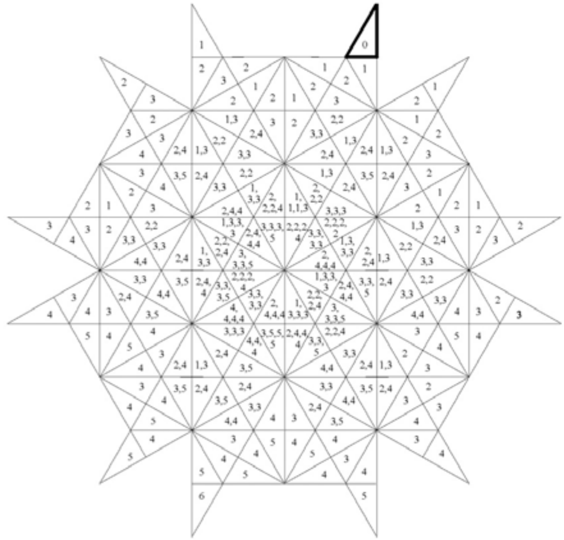
Beautiful pictures in type G_2

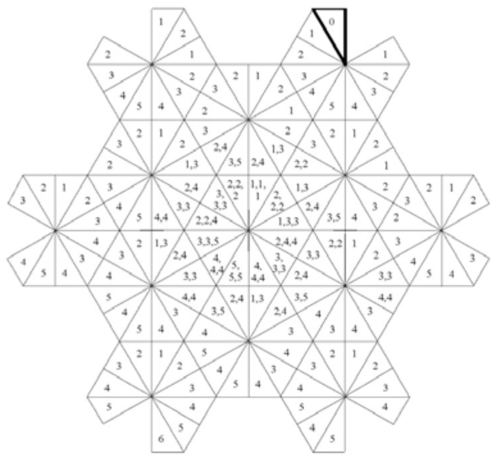






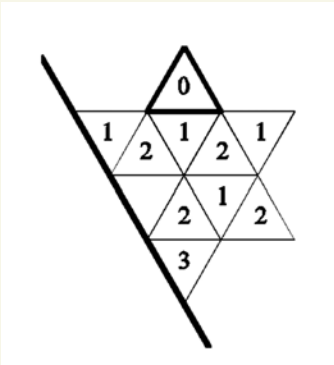


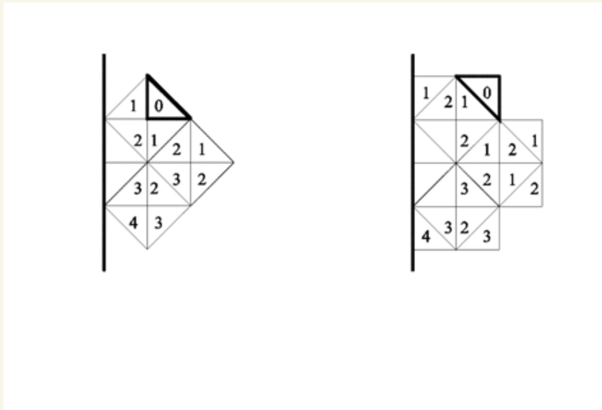


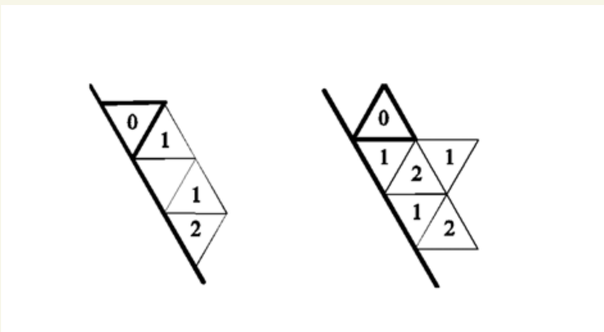


Along the walls

A_2

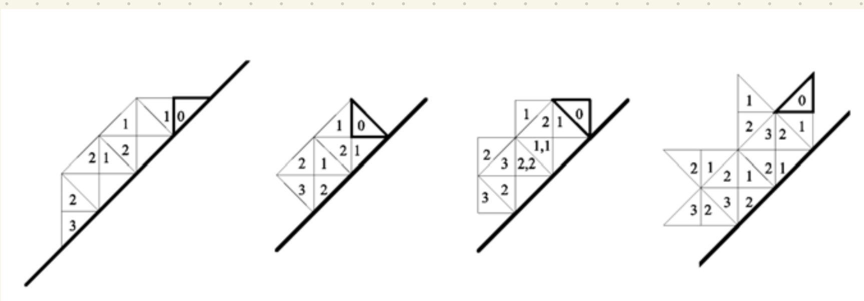




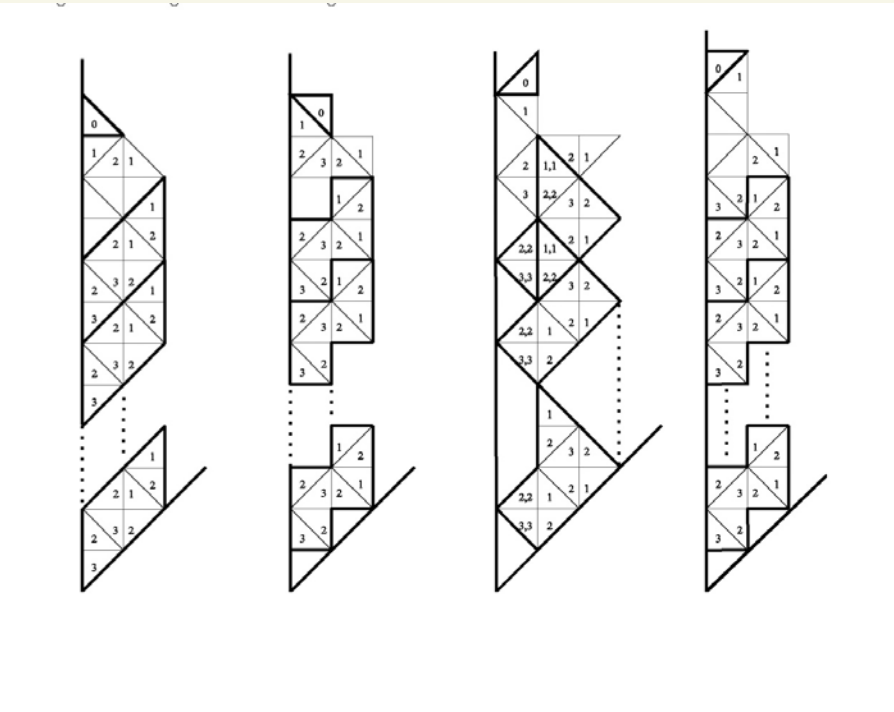


$\Rightarrow$ tiling bounded

B_2 :

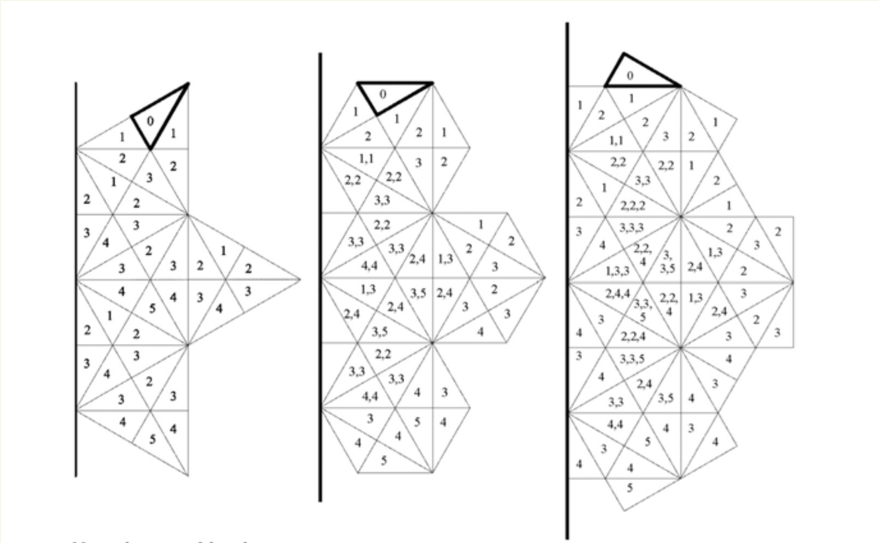


The growing snakes ...



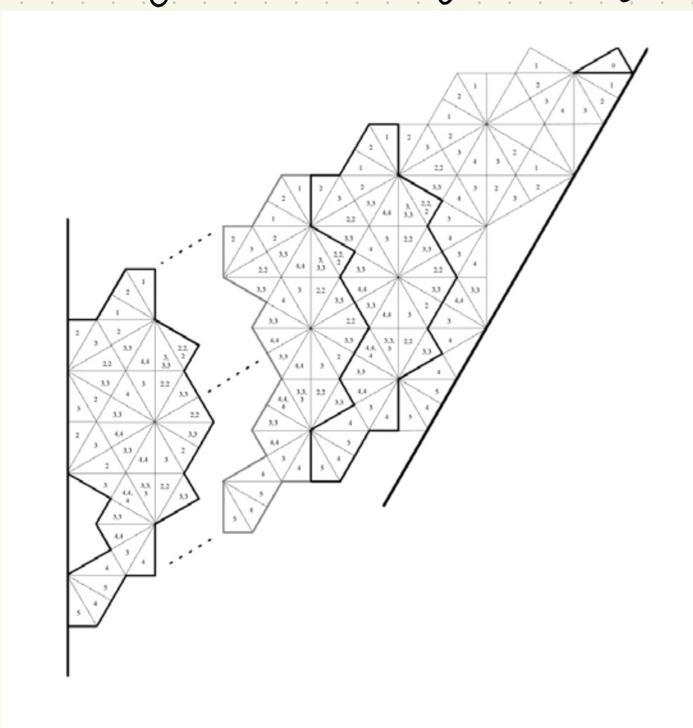
⇒ $\boxed{\text{not}}$ tiling bounded

G_2 First the usual picture along one wall

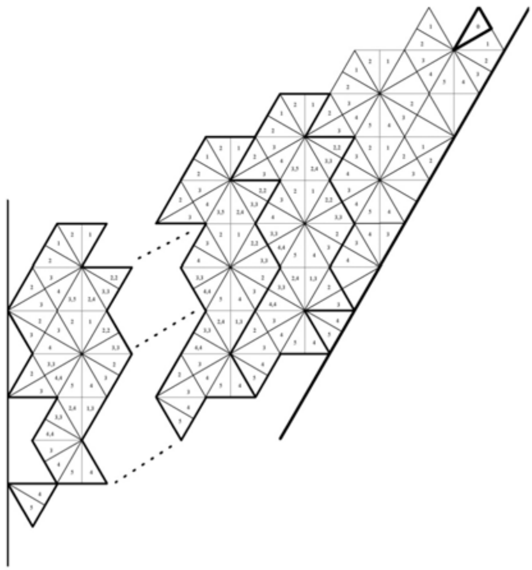


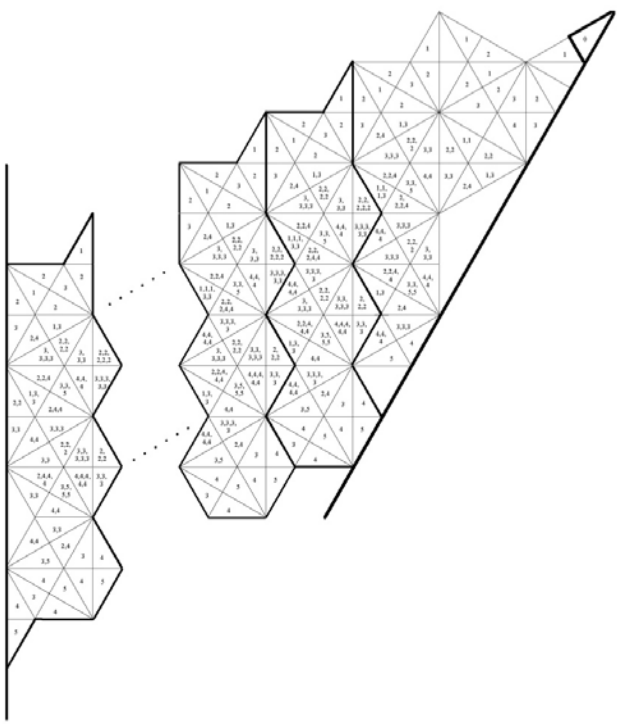
etc.

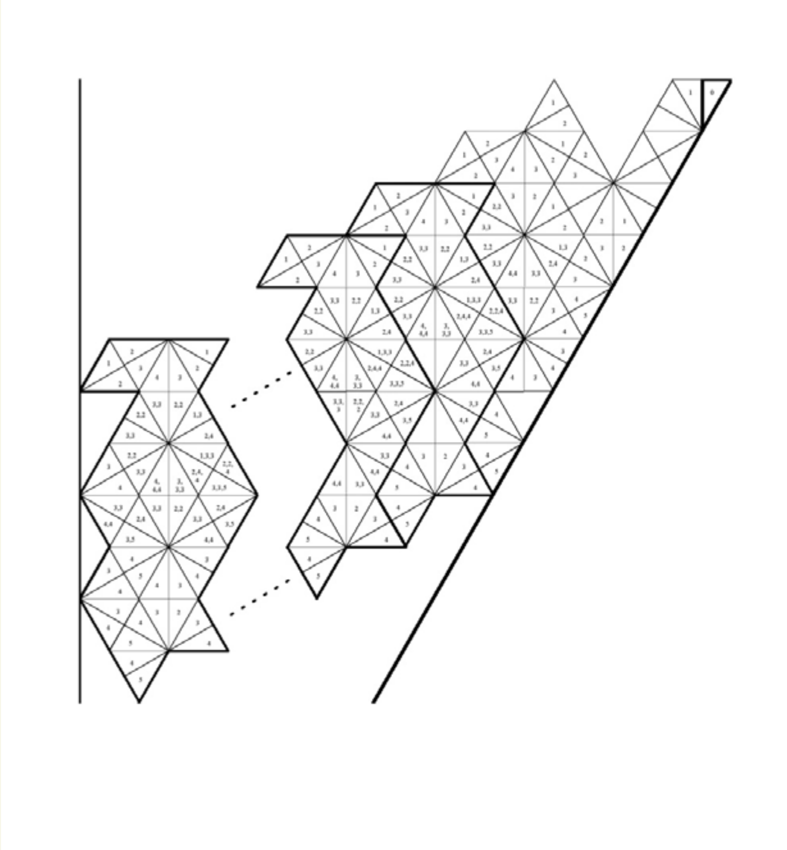
Growing snakes again along the other wall:

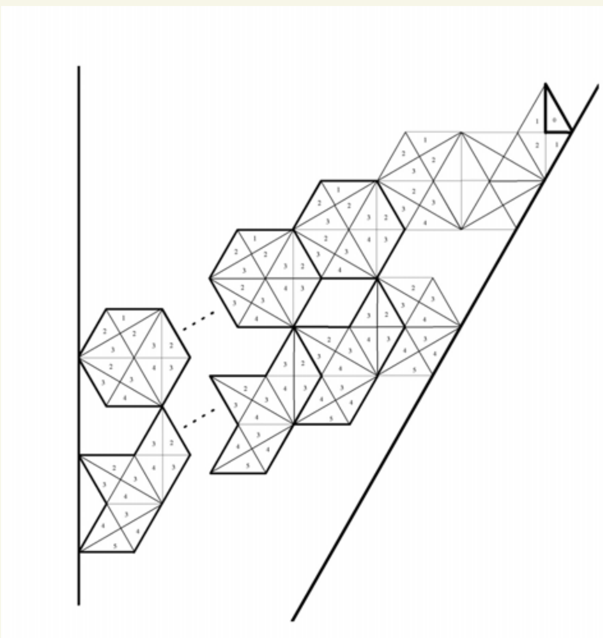


But this time the snake is tamed ☺









filling
bounded

Back to the beginning!

Theorem A finite / upper finite / essentially finite BGG A

$\Rightarrow$ $A\text{-mod}_{fd}$ / $A\text{-mod}_{lfd}$ / $A\text{-mod}_{fd}$

is a finite / upper finite / essentially finite highest weight category

Ringel duality gives converse! Here is the idea:

Given: upper finite hw. category $\mathcal{R} \cong A\text{-mod}_{lfd}$

$$A = \Big(\bigoplus_{i,j} \mathrm{Hom}(T_i, T_j)\Big)^{op}$$

Ringel duality

lower finite hw. category $\mathcal{R}'$

How to find basis? More general basis of $\mathrm{Hom}_{\mathcal{R}'}(M, N)$ $\quad M \in \Delta_\varepsilon(\mathcal{R}')$, $N \in \nabla_\varepsilon(\mathcal{R}')$

Answer is the following picture

(for fixed inclusions i_b, projections π_b $\forall b$)

$$\begin{tikzcd}
 & \Delta_\varepsilon(b) \arrow[d, hook, "i_b"] \arrow[dr, "\bar{x}"] & \\
M \arrow[r, dashed, "y"] \arrow[dr, "\bar{y}"'] & T_b \arrow[r, dashed, "x"] \arrow[d, two heads, "\pi_b"] & N \\
 & \nabla_\varepsilon(b) &
\end{tikzcd}$$

[Andersen – Tubbenhauer – S.
Cellular structures using tilting modules]

Theorem (Basis theorem)

Pick $\forall b \in B$

$Y_b \subseteq \mathrm{Hom}_{R'}(M, T_b)$ s.t. $\pi_b \circ Y_b$ basis of $\mathrm{Hom}_{R'}(M, \nabla_c(b))$

$X_b \subseteq \mathrm{Hom}_{R'}(T_b, N)$ s.t. $Y_b \circ i_b$ basis of $\mathrm{Hom}(\Delta_c(b), N)$

Then $\left\{ x \circ y \;\middle|\; (y, x) \in \bigcup_{b \in B} Y_b \times X_b \right\}$ is basis for $\mathrm{Hom}_{R'}(M, N)$

Generalises to stratified algebras, e.g.

Assume fully stratified, tilting rigid (i.e. $Tilt_+ = Tilt_-$) with weakly symmetric strata

$$
\begin{tikzcd}
 & \Delta(a) \arrow[r, "\bar{h}"] \arrow[d, hook, "i_a"] & \nabla(b) & \\
M \arrow[r, dashed, "y"] \arrow[dr, "\bar{y}"'] & T_a \arrow[r, dashed, "h"] \arrow[d, two heads, "\bar{\pi}_a"] & T_b \arrow[r, dashed, "x"] \arrow[u, two heads, "\bar{\pi}_b"'] & N \\
 & \bar{\nabla}(a) & \bar{\Delta}(b) \arrow[u, hook, "\bar{i}_b"'] \arrow[ur, "\hat{x}"'] &
\end{tikzcd}
$$

Theorem (Basis theorem II)

Pick $\forall b \in B$

$Y_a \subseteq \mathrm{Hom}_R(M, T_a)$ s.t. $\pi \circ Y_a$ basis of $\mathrm{Hom}_R(M, \bar{\nabla}(a))$

$X_b \subseteq \mathrm{Hom}_R(T_b, N)$ s.t. $Y_b \circ i_b$ basis of $\mathrm{Hom}(\bar{\Delta}(b), N)$

$H(a,b) \subseteq \mathrm{Hom}_R(T_a, T_b)$ s.t. $\pi_b \circ h \circ i_a$ basis of $\mathrm{Hom}(\Delta(a), \nabla(b))$

$$\boxed{\text{Then } \left\{ x \circ h \circ y \,\middle|\, (y,h,x) \in \bigcup_{b \in B} Y_a \times H(a,b) \times X_b \right\} \text{ is basis for } \mathrm{Hom}_R(M, N)}$$

→ generalisation of based quasi-hereditary algebras to algebras with triangular decomposition.

⇒ ZOO of algebras with based triangular decomposition

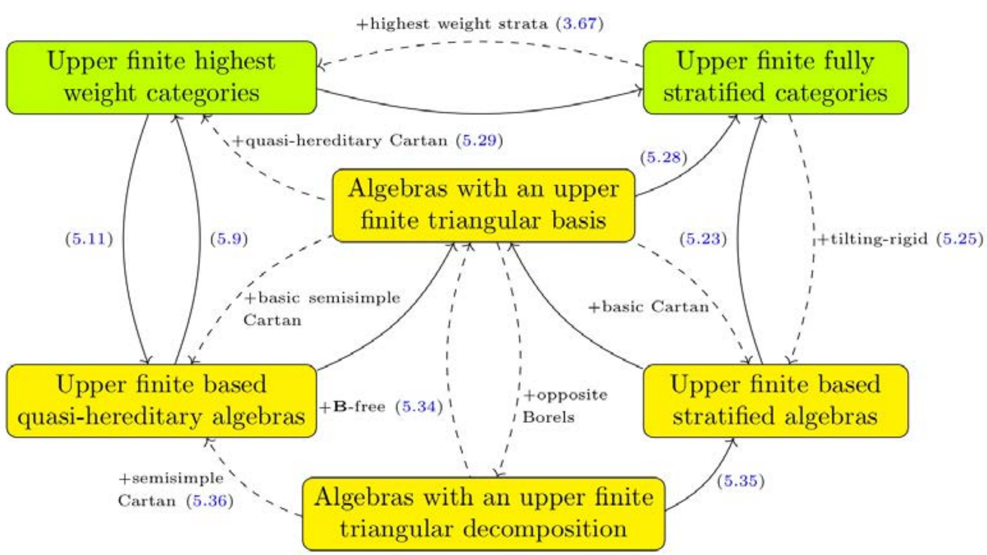


TABLE 2. Upper finite algebras and categories