Gieseker varieties, affine Springer fibers, and higher rank $q, t$-Catalan numbers

José Simental  
MPIM Bonn  
MIT Infinite Dimensional Algebra Seminar  

March 12, 2021
Plan

1. Quantized Gieseker varieties (joint work with P. Etingof, V. Krylov, I. Losev)
Plan

1. Quantized Gieseker varieties (joint work with P. Etingof, V. Krylov, I. Losev)
   - Gieseker varieties.
   - Quantizations.
   - Finite-dimensional representations.
Plan

1. Quantized Gieseker varieties (joint work with P. Etingof, V. Krylov, I. Losev)
   - Gieseker varieties.
   - Quantizations.
   - Finite-dimensional representations.

2. Geometric construction of representations (joint work with E. Gorsky, M. Vazirani)
Plan

1. Quantized Gieseker varieties (joint work with P. Etingof, V. Krylov, I. Losev)
   - Gieseker varieties.
   - Quantizations.
   - Finite-dimensional representations.

2. Geometric construction of representations (joint work with E. Gorsky, M. Vazirani)
   - Coulomb branches.
   - BFN Springer theory.
   - Hilbert schemes of points on singular curves.
Plan

1. Quantized Gieseker varieties (joint work with P. Etingof, V. Krylov, I. Losev)
   - Gieseker varieties.
   - Quantizations.
   - Finite-dimensional representations.

2. Geometric construction of representations (joint work with E. Gorsky, M. Vazirani)
   - Coulomb branches.
   - BFN Springer theory.
   - Hilbert schemes of points on singular curves.

3. Higher rank \((q, t)\)-Catalan numbers (joint work with V. Krylov)
Plan

1. Quantized Gieseker varieties (joint work with P. Etingof, V. Krylov, I. Losev)
   - Gieseker varieties.
   - Quantizations.
   - Finite-dimensional representations.

2. Geometric construction of representations (joint work with E. Gorsky, M. Vazirani)
   - Coulomb branches.
   - BFN Springer theory.
   - Hilbert schemes of points on singular curves.

3. Higher rank \((q, t)\)-Catalan numbers (joint work with V. Krylov)
   - Affine Springer fibers.
   - Affine pavings via the affine symmetric group.
   - Higher rank \((q, t)\)-Catalan numbers.
Gieseker varieties

Fix positive integers $n, r > 0$. Define

- $R := \mathfrak{gl}_n(\mathbb{C}) \oplus \text{Hom}(\mathbb{C}^r, \mathbb{C}^n)$
- $\overline{R} := \mathfrak{sl}_n(\mathbb{C}) \oplus \text{Hom}(\mathbb{C}^r, \mathbb{C}^n)$.

The group $G := \text{GL}_n(\mathbb{C})$ acts naturally both on $R$ and $\overline{R}$, and the action lifts to the cotangent bundles $T^*R, T^*\overline{R}$. This action is Hamiltonian, with the same formula for the moment map

$$
\mu(A, B, i, j) = [A, B] - ij \in \mathfrak{g}l_n^* = \mathfrak{g}l_n.
$$

$$
\mu^* : \mathfrak{g}l_n^* \rightarrow \mathbb{C}[T^*R]
$$

$$
\mathfrak{g}l_n^* \rightarrow \mathcal{E}_R
$$
Gieseker varieties

Definition

We define the Gieseker variety $M(n, r)$ (resp. $\overline{M}(n, r)$) to be the Hamiltonian reduction (at level 0) of $G$ acting on $T^*R$ (resp. $T^*\overline{R}$).

\[ M(n, r) = \mu^{-1}(0)/G = \text{Spec} \left( \frac{C[T^*R]}{(x_1, \ldots, x_n)} \right)^G \]

- Affine
- Poisson
- Conical $C^*G M(n, r)$ induced by $C^*G T^*R$ by dilations
- $\dim M(n, r) = 2nr$  $\dim \overline{M}(n, r) = 2nr - 2$
Gieseker varieties

**Definition**

We define the *Gieseker variety* $\mathcal{M}(n, r)$ (resp. $\overline{\mathcal{M}}(n, r)$) to be the Hamiltonian reduction (at level 0) of $G$ acting on $T^*R$ (resp. $T^*\overline{R}$).

Note that

$$\mathcal{M}(n, r) = \overline{\mathcal{M}}(n, r) \times \mathbb{C}^2.$$
Gieseker varieties

Definition

We define the Gieseker variety $\mathcal{M}(n, r)$ (resp. $\overline{\mathcal{M}}(n, r)$) to be the Hamiltonian reduction (at level 0) of $G$ acting on $T^*R$ (resp. $T^*\overline{R}$).

Note that

$$\mathcal{M}(n, r) = \overline{\mathcal{M}}(n, r) \times \mathbb{C}^2.$$ 

Example

- $\mathcal{M}(n, 1) = (\mathbb{C}^{2n})/S_n$, $\overline{\mathcal{M}}(n, 1) = (\mathfrak{h} \oplus \mathfrak{h}^*)/S_n$, where $\mathfrak{h}$ is the reflection representation of $S_n$.
- $\overline{\mathcal{M}}(1, r) = \overline{O}_{\text{min}}$, the closure of the minimal nilpotent orbit in $\mathfrak{sl}_r$.

$$\{ x \in \mathfrak{sl}_r \mid x^2 = 0, \ \text{rank} \ x \leq 1 \}$$
Gieseker varieties

Note that we have an action of the 1-dimensional torus $\mathbb{C}^\times$ on $T^*R$ and $T^*\overline{R}$ by dilations. This action commutes with that of $G$, and descends to $\mathcal{M}(n, r)$, $\overline{\mathcal{M}}(n, r)$. These varieties are conical, singular, of dimension $2nr$ and $2nr - 2$, respectively. Moreover, they carry a Poisson bracket of degree $-2$. We can construct symplectic resolutions $\mathcal{M}^\theta(n, r)$, $\overline{\mathcal{M}}^\theta(n, r)$ using GIT Hamiltonian reduction.

$$\Theta: \mathbb{G}_m \to \mathbb{C}^\times$$

$$\mathcal{M}^\theta(n, r) = \mu^{-1}(0) / G$$

$\mu$ is flat $\Rightarrow$ $\mathcal{M}^\theta(n, r)$ ($\theta \neq 1$) is a resolution of $\overline{\mathcal{M}}(n, r)$.
Gieseker varieties

Note that we have an action of the 1-dimensional torus $\mathbb{C}^\times$ on $T^*R$ and $T^*\overline{R}$ by dilations. This action commutes with that of $G$, and descends to $\mathcal{M}(n, r)$, $\overline{\mathcal{M}}(n, r)$. These varieties are conical, singular, of dimension $2nr$ and $2nr - 2$, respectively. Moreover, they carry a Poisson bracket of degree $-2$. We can construct symplectic resolutions $\mathcal{M}^\theta(n, r)$, $\overline{\mathcal{M}}^\theta(n, r)$ using GIT Hamiltonian reduction.

**Example**

- $\mathcal{M}^\theta(n, 1) = \text{Hilb}^n(C^2)$. 
- $\overline{\mathcal{M}}^\theta(1, r) = T^*\mathbb{P}^{r-1}$. 

$\Theta_{m,n}$
Gieseker varieties

lifted from action on $\mathcal{Z}$ (resp. $\overline{\mathcal{Z}}$)

We have an action of the group $\mathbb{C}^\times \times \text{GL}_r$ on $T^* R$ (resp. $T^* \overline{R}$):

$$(t, g). (A, B, i, j) = (tA, t^{-1}B, ig^{-1}, g j)$$

Note that this descends to an action of $\mathbb{C}^\times \times \text{PGL}_r$ on $\mathcal{M}(n, r)$ (resp. $\overline{\mathcal{M}}(n, r)$).

Hamiltonian

$$A, B \in \mathfrak{sl}_n, \quad g \in \text{GL}_r$$
Quantized Gieseker varieties

We will be interested in quantizations of $\mathcal{M}(n, r)$, $\overline{\mathcal{M}}(n, r)$ which can be produced using quantum Hamiltonian reduction, as follows. Fix a parameter $c \in \mathbb{C}$. For each element $\xi \in \mathfrak{gl}_n$, let $\xi_R$ denote the vector field given by the infinitesimal action on $R$. Note that we can see $\xi_R$ as a differential operator, $\xi_R \in D(R)$. Then,

$$\mathcal{A}_c(n, r) := \left[ \frac{D(R)}{D(R)\{\xi_R - c \text{ tr}(\xi) \mid \xi \in \mathfrak{gl}_n\}} \right]^G$$

and similarly, define $\overline{\mathcal{A}}_c(n, r)$ with $\overline{R}$ instead of $R$. Note that

$$\mathcal{A}_c(n, r) = \overline{\mathcal{A}}_c(n, r) \otimes D(\mathbb{C}).$$

$\mathcal{A}_c(n, r)$ has a fil filtration by Bernstein filtration on $D(R)$

$$\text{gr } \mathcal{A}_c(n, r) \cong \mathcal{M}(n, r)$$ as graded Poisson algebra.
Quantized Gieseker varieties

The action of $\mathbb{C}^* \times \text{PGL}_r$ on $\mathcal{M}(n, r)$ is Hamiltonian, and gives a quantum comoment map $\Upsilon : \mathbb{C} \oplus \mathfrak{sl}_r \to \mathcal{A}_c(n, r)$.

Example

- (Gan-Ginzburg, Losev) When $r = 1$, $\mathcal{A}_c(n, 1)$ (resp. $\overline{\mathcal{A}}_c(n, 1)$) is the spherical rational Cherednik algebra of type $\mathfrak{gl}_n$ (resp. $\mathfrak{sl}_n$).
- When $n = 1$, $\overline{\mathcal{A}}_c(1, r) = D_c(\mathbb{P}^{r-1})$ and $\mathcal{A}_c(1, r) = D_c(\mathbb{P}^{r-1} \times \mathbb{C})$.

$$c \in \mathbb{Z}, \quad \overline{\mathcal{A}}_c(1, r) = D_c(O(C))$$
Finite-dimensional representations

Theorem (Losev)

The algebra $\overline{A}_c(n, r)$ admits a finite-dimensional representation if and only if $c = \frac{m}{n}$ with $\gcd(m; n) = 1$ and $c \notin (-r, 0)$. In this case, $\overline{A}_c(n, r)$ admits a unique irreducible finite-dimensional representation that we call $\overline{L}_{m/n,r}$. This representation doesn’t admit self-extensions.

We remark that we have an isomorphism $\overline{A}_c(n, r) \cong \overline{A}_{-c-r}(n, r)$. So we will focus on the case $c > 0$.

\[
\dim_{\mathbb{C}} \overline{L}_{m/n,r} \\
\text{ch}_{\mathbb{C}^* \times S_n} \overline{L}_{m/n,r}
\]
Finite-dimensional representations

**Theorem (Losev)**

The algebra $\overline{A}_c(n, r)$ admits a finite-dimensional representation if and only if $c = \frac{m}{n}$ with $\gcd(m; n) = 1$ and $c \notin (-r, 0)$. In this case, $\overline{A}_c(n, r)$ admits a unique irreducible finite-dimensional representation that we call $\overline{L}_{m/n,r}$. This representation doesn’t admit self-extensions.

We remark that we have an isomorphism $\overline{A}_c(n, r) \cong \overline{A}_{-c-r}(n, r)$. So we will focus on the case $c > 0$.

**Example**

- When $r = 1$, this is a theorem of Berest-Etingof-Ginzburg.
- When $n = 1$, the theorem says that $D_c(\mathbb{P}^{r-1})$ admits a finite-dimensional representation if and only if we have an algebra of differential operators on a line bundle, $D(\mathcal{O}(m))$. In this case, $\overline{L}_{m/1,r} = \Gamma(\mathbb{P}^{r-1}, \mathcal{O}(m)) = \text{Sym}^m(\mathbb{C}^r^*)$. 

Cherednik for DAHA
Finite-dimensional representations

To describe the representation $\overline{L}_{m/n,r}$, we need finite-dimensional representations of the full rational Cherednik algebra.

**Definition (Etingof-Ginzburg)**

The rational Cherednik algebra of type $\mathfrak{sl}_n$, $\overline{H}_c(n)$, is the quotient of the semidirect product algebra $\mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle \rtimes S_n$ by the relations

- $x_1 + \cdots + x_n = y_1 + \cdots + y_n = 0$.
- $[x_i, x_j] = [y_i, y_j] = 0$.
- $[x_i, y_j] = \frac{1}{n} - c s_{ij}$ ($i \neq j$)

\[ e := \frac{1}{n!} \sum_{\omega \in S_n} \omega \epsilon \overline{H}_c(n) \]

Gen. Ginzburg: $\overline{A}_c(n,1) \cong e \overline{H}_c(n) e$
Finite-dimensional representations

To describe the representation $\overline{L}_{m/n,r}$, we need finite-dimensional representations of the full rational Cherednik algebra.

**Definition (Etingof-Ginzburg)**

The rational Cherednik algebra of type $\mathfrak{sl}_n$, $\overline{H}_c(n)$, is the quotient of the semidirect product algebra $\mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle \rtimes S_n$ by the relations

- $x_1 + \cdots + x_n = y_1 + \cdots + y_n = 0$.
- $[x_i, x_j] = [y_i, y_j] = 0$.
- $[x_i, y_j] = \frac{1}{n} - cs_{ij} \ (i \neq j)$

**Theorem (Berest-Etingof-Ginzburg)**

The algebra $\overline{H}_c(n)$ admits a finite-dimensional representation if and only if $c = \frac{m}{n}$ with $\gcd(m; n) = 1$. In this case, $\overline{H}_c(n)$ has a unique irreducible finite-dimensional representation that we call $\overline{F}_{m/n}$. This representation doesn’t admit self-extensions.
Finite-dimensional representations

Since $\overline{A}_{m,n}(n,1)$ is the spherical Cherednik algebra, we have

$$\overline{L}_{m/n,1} = \overline{F}^{S_n}_{m/n}.$$  

This is, however, not the best way to think about $\overline{L}_{m/n,1}$.

Calaque-Enriquez-Engof

$$\overline{F}^{S_n}_{m/n} = \overline{F}^{S_m}_{n/n}$$

$$e_{\mathcal{H}^{n}(\mathfrak{g})e} \quad e_{\mathcal{H}^{m}(\mathfrak{g})e}$$
Finite-dimensional representations

Since $\mathcal{A}_{m/n}(n,1)$ is the spherical Cherednik algebra, we have

$$\overline{L}_{m/n,1} = \overline{F}^{S_n}_{m/n}.$$ 

This is, however, not the best way to think about $\overline{L}_{m/n,1}$.

**Theorem (Etingof-Krylov-Losev-S.)**

We have an isomorphism of $\mathbb{C}^\times \times \text{GL}_r$-modules:

$$\overline{L}_{m/n,r} \cong (\overline{F}_{n/m} \otimes (\mathbb{C}^{r*})^m)_{S_m}$$

```
\[ \overline{L}_{m/n,r} \cong (\overline{F}_{n/m} \otimes (\mathbb{C}^{r*})^m)_{S_m} \]
```
Finite-dimensional representations

Since $\overline{A}_{m/n}(n,1)$ is the spherical Cherednik algebra, we have

$$\overline{L}_{m/n,1} = \overline{F}_{m/n}^{S_n}.$$ 

This is, however, not the best way to think about $\overline{L}_{m/n,1}$.

**Theorem (Etingof-Krylov-Losev-S.)**

We have an isomorphism of $\mathbb{C}^\times \times \text{GL}_r$-modules:

$$\overline{L}_{m/n,r} \cong (\overline{F}_{n/m} \otimes (\mathbb{C}^{r*})^\otimes)^{S_m}$$

**Remark**

When $r = 1$, the isomorphism $\overline{F}_{m/n}^{S_n} \cong \overline{F}_{n/m}^{S_m}$ is due to Calaque-Enriquez-Etingof. Further ramifications are due to Gorsky-Oblomkov-Rasmussen-Shende, Etingof-Gorsky-Losev and more.
Finite-dimensional representations

The representation $\overline{F}_{n/m}$ has been intensively studied in recent years and character formulas are known for it. In particular, we can compute the $\mathbb{C}^\times \times \text{GL}_r$-character of $\overline{L}_{m/n,r}$:

$$
\text{ch}_{\mathbb{C}^\times \times \text{GL}_r}(\overline{L}_{m/n,r}) = \frac{1}{[n]_q} \sum_{\lambda \vdash m \atop r(\lambda) \leq \min(n;r)} s_{\lambda}(q^{1-n/2}, \ldots, q^{n-1})[W_r(\lambda)^*].
$$

$$
[n]_q = \begin{array}{c}
q^n - 1 \\
q^n - q^n \\
q^n - q^n \\
q^n - q^n
\end{array}
$$
Finite-dimensional representations

The representation $\overline{F}_{n/m}$ has been intensively studied in recent years and character formulas are known for it. In particular, we can compute the $\mathbb{C}^\times \times \text{GL}_r$-character of $\overline{L}_{m/n,r}$:

$$
\text{ch}_{\mathbb{C}^\times \times \text{GL}_r}(\overline{L}_{m/n,r}) = \frac{1}{[n]_q} \sum_{\lambda \vdash m, \text{ } r(\lambda) \leq \min(n; r)} s_\lambda(q^{\frac{1-n}{2}}, \ldots, q^{\frac{n-1}{2}})[W_r(\lambda)^*].
$$

\[ \text{dim}_{\mathbb{C}^\times \times \text{GL}_r}(\overline{L}_{m/n,r}) = \frac{r}{\text{gcd}(m, r)} \]

**Definition**

Let $m$ and $n$ be coprime positive integers and $r > 0$. We call

$$
C_{m/n}^r := \text{dim}(\overline{L}_{m/n,r}) = \frac{1}{n} \binom{nr + m - 1}{m} = \frac{1}{n} \dim \text{Sym}^m(\mathbb{C}^{nr})
$$

the rank $r$ rational $m/n$-Catalan number.
Finite-dimensional representations

\[ \gcd(m, n) = 1 \]

When \( r = 1 \):

\[
C_{m/n}^1 = \frac{1}{n} \binom{n + m - 1}{m} = \frac{1}{n + m} \binom{n + m}{m} = C_{n/m}^1
\]

counts the number of Dyck paths in an \( m \times n \)-rectangle.
Finite-dimensional representations

When \( r = 1 \):

\[
C_{m/n}^1 = \frac{1}{n} \binom{n + m - 1}{m} = \frac{1}{n + m} \binom{n + m}{m} = C_{n/m}^1
\]

counts the number of Dyck paths in an \( m \times n \)-rectangle.

**Theorem (Etingof-Krylov-Losev-S.)**

The number \( C_{m/n}^r \) counts the number of rank \( r \) semistandard parking functions on an \( m \times n \)-rectangle, that is, a Dyck path together with a function from its vertical steps to \( \{1, \ldots, r\} \) that is weakly increasing on consecutive vertical steps.
Example: $C_{3/2}^2$

$$C_{3/2}^2 = \frac{1}{2} \left( \frac{4 + 3 - 1}{3} \right) = 10.$$
Goal

Use geometry to:

1. Explain why the \( m, n \)-switch in the formula

\[
\overline{L}_{m/n,r} = (\overline{F}_{n/m} \otimes (\mathbb{C}^r) \otimes m) S_m
\]

is a natural thing to expect.

2. Produce a \( q, t \)-deformation \( C^r_{m/n}(q,t) \).
The relation with geometry is clearer for the algebra $\mathcal{A}_c(n, r)$ (as opposed to $\overline{\mathcal{A}}_c(n, r)$). Recall that we have a decomposition

$$\mathcal{A}_c(n, r) = \overline{\mathcal{A}}_c(n, r) \otimes D(\mathbb{C})$$

so the representation

$$L_{m/n,r} := \overline{L}_{m/n,r} \otimes \mathbb{C}[x]$$

is a representation of $\mathcal{A}_c(n, r)$. We will realize this and other representations geometrically.
We also have the $\mathfrak{gl}_n$-version of the rational Cherednik algebra $H_c(n)$, satisfying

$$H_c(n) = \overline{H}_c(n) \otimes D(\mathbb{C}) \overset{S_n \sim \text{monom}}{\longrightarrow}$$

and the representation $F_{m/n} = \overline{F}_{m/n} \otimes \mathbb{C}[x]$. It is still true that

$$L_{m/n,r} = (F_{n/m} \otimes (\mathbb{C}^r)^{\otimes m})S_m.$$
Coulomb branches

Let $G$ be a reductive group acting on a vector space $V$. We will denote:

- $\mathcal{O} := \mathbb{C}[[\epsilon]]$.
- $\mathcal{K} := \mathbb{C}((\epsilon))$.
- $G_{\mathcal{O}} = G[[\epsilon]]$, $G_{\mathcal{K}} = G((\epsilon))$.
- $V_{\mathcal{O}} := \mathcal{O} \otimes V = V[[\epsilon]]$, $V_{\mathcal{K}} := \mathcal{K} \otimes V = V((\epsilon))$.
- $I \subseteq G_{\mathcal{O}}$ consisting of elements $g \in G_{\mathcal{O}}$ so that $g|_{\epsilon=0} \in \mathcal{B}$ for a fixed Borel $\mathcal{B} \subseteq G$.
- $i \subseteq \mathfrak{g}[[\epsilon]]$ the Lie algebra of $I$, consisting of elements $X \in \mathfrak{g}[[\epsilon]]$ so that $X|_{\epsilon=0} \in \mathfrak{b}$.
Coulomb branches

Associated to these data, Braverman-Finkelberg-Nakajima construct a Poisson algebra $\mathcal{A}(G, V)$ and its quantization $\mathcal{A}_\hbar(G, V)$. We will not need the precise definition, but we will use the fact that it is supposed to model equivariant homology of the “Steinberg variety”

$$\mathcal{A}_\hbar(G, V) = H^G_{\bullet}(\mathfrak{X}_V \times_{V_K} \mathfrak{X}_V)$$

here, $\mathfrak{X}_V := (G_K \times V_\mathcal{O})/G_\mathcal{O}$ is a bundle over the affine Grassmannian. The quantization $\mathcal{A}_\hbar(G, V)$ appears when considering equivariant homology for the loop rotation action.
Coulomb branches

Associated to these data, Braverman-Finkelberg-Nakajima construct a Poisson algebra $\mathcal{A}(G, V)$ and its quantization $\mathcal{A}_\hbar(G, V)$. We will not need the precise definition, but we will use the fact that it is supposed to model equivariant homology of the “Steinberg variety”

$$\mathcal{A}(G, V) = H^G_{\ast} (\mathcal{F}_V \times_{V_{\mathcal{K}}} \mathcal{F}_V)$$

here, $\mathcal{F}_V := (G_{\mathcal{K}} \times V_{\mathcal{O}})/G_{\mathcal{O}}$ is a bundle over the affine Grassmannian. The quantization $\mathcal{A}_\hbar(G, V)$ appears when considering equivariant homology for the loop rotation action. Webster extended BFN’s construction to the parahoric setting, by considering now spaces of the form $\mathcal{F}_{N,P} := (G_{\mathcal{K}} \times N)/P$, where $P$ is a parahoric subgroup and $N \subseteq V_{\mathcal{O}}$ is a “nice” subspace stable under $P$. The resulting algebra $\mathcal{A}(G, N, P)$ may no longer be commutative.

$$\mathcal{A}(G, V_{\mathcal{O}}, I) = \text{Mat}_{\text{Hom}} (\mathcal{A}(G, V))$$
Coulomb branches

All the algebras we have seen are, in fact, examples of quantized Coulomb branches.

- The spherical rational Cherednik algebra $\mathcal{A}_c(n, 1)$ appears taking $G = \text{GL}_n$ and $V = \mathfrak{gl}_n \oplus \mathbb{C}^n$, the adjoint plus fundamental representation. (Kodera-Nakajima, Webster)

\[
\begin{array}{c}
\bigcirc \\
\downarrow \\
\bigcirc
\end{array}
\]
Coulomb branches

All the algebras we have seen are, in fact, examples of quantized Coulomb branches.

- The spherical rational Cherednik algebra $\mathcal{A}_c(n, 1)$ appears taking $G = \text{GL}_n$ and $V = \mathfrak{gl}_n \oplus \mathbb{C}^n$, the adjoint plus fundamental representation. (Kodera-Nakajima, Webster)

- The full rational Cherednik algebra $H_c(n)$ appears taking again $G = \text{GL}_n$ and $V = \mathfrak{gl}_n \oplus \mathbb{C}^n$, but using Webster’s construction with $P = I$ and $N = i \oplus \mathbb{C}^n[[\epsilon]] \subseteq V[[\epsilon]]$. (Braverman-Etingof-Finkelberg, Webster)
Coulomb branches

All the algebras we have seen are, in fact, examples of quantized Coulomb branches.

- The spherical rational Cherednik algebra $\mathcal{A}_c(n, 1)$ appears taking $G = \text{GL}_n$ and $V = \mathfrak{gl}_n \oplus \mathbb{C}^n$, the adjoint plus fundamental representation. (Kodera-Nakajima, Webster)

- The full rational Cherednik algebra $H_c(n)$ appears taking again $G = \text{GL}_n$ and $V = \mathfrak{gl}_n \oplus \mathbb{C}^n$, but using Webster’s construction with $P = I$ and $N = i \oplus \mathbb{C}[[\epsilon]] \subseteq V[[\epsilon]]$. (Braverman-Etingof-Finkelberg, Webster)

- For the quantized Gieseker variety $\mathcal{A}_c(n, r)$, we take $G = \text{GL}_n^{\times r}$ and $V = \mathfrak{gl}_n^{\oplus r} \oplus \mathbb{C}^n$, with action given by 
  $$(g_0, \ldots, g_{r-1}) \cdot (X_0, \ldots, X_{r-1}, v) = (g_1 X_1 g_0^{-1}, \ldots, g_0 X_{r-1} g_{r-1}^{-1}, g_0 v)$$
  (Nakajima-Takayama+Losev)
Hilburn-Kamnitzer-Weekes developed a Springer theory for Coulomb branches (extended to the parahoric setting by Garner-Kivinen). Consider an element $v \in V_{\mathcal{K}}$. The generalized affine Springer fiber is

$$Spr_{P,N}(v) := \{ [g] \in G_{\mathcal{K}}/P \mid gv \in N \}.$$ 

Under certain conditions on $\mathfrak{g}$, there is an action of the Coulomb branch algebra $A_{\hbar}(G, N, P)$ on

$$H_*^{L_v}(Spr_{P,N}(v))$$

where $L_v := \text{Stab}_{G_{\mathcal{K}} \ltimes \mathbb{C}^{\times}}(v)$. 
BFN Springer theory

Now consider the element

\[(Y, v) := \begin{pmatrix}
0 & 0 & \cdots & 0 & \varepsilon^n \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix},
\begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} \in \mathfrak{gl}_n[[\varepsilon]] \oplus \mathbb{C}^n[[\varepsilon]]\]
BFN Springer theory

Now consider the element

\[(Y, v) := \begin{pmatrix} 0 & 0 & \cdots & 0 & \varepsilon^m \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathfrak{gl}_n[[\varepsilon]] \oplus \mathbb{C}^n[[\varepsilon]]\]

Choosing a basis, we can identify

\[\mathbb{C}^n[[\varepsilon]] = \mathbb{C}[[\varepsilon]][1, y, \ldots, y^{n-1}]\]

Note that the action of the matrix $Y$ amounts to multiplication by $y$ in this basis, and the vector $v$ is identified with 1.
Hilbert schemes on singular curves

Let us specialize now to the rational Cherednik algebra case, so $G = \text{GL}_n$ and $V = \mathfrak{gl}_n \oplus \mathbb{C}^n$.

Theorem (Garner-Kivinen, Gorsky-S.-Vazirani)

Assume $\gcd(m; n) = 1$ and let $C = \{y^n = x^m\} \subseteq \mathbb{C}^2$. Then:

1. $\text{Spr}_{G_\mathcal{O}, V_\mathcal{O}}(Y, \nu) = \text{Hilb}(C, 0) \cong \bigcup_{k \geq 0} \text{Hilb}^k(C, \mathcal{O})$
Hilbert schemes on singular curves

Let us specialize now to the rational Cherednik algebra case, so $G = \text{GL}_n$ and $V = \mathfrak{gl}_n \oplus \mathbb{C}^n$.

Theorem (Garner-Kivinen, Gorsky-S.-Vazirani)

Assume $\gcd(m; n) = 1$ and let $C = \{y^n = x^m\} \subseteq \mathbb{C}^2$. Then:

1. $Spr_{G, V}(Y, v) = \text{Hilb}(C, 0)$.

2. $Spr_{I, \mathbb{C}^n[[\varepsilon]]} = \text{PHilb}(C, 0)$, where $\text{PHilb}(C, 0)$ is the moduli space of flags of ideals $\chi \sim \varepsilon$

$$I_k \supseteq I_{k+1} \supseteq \cdots \supseteq I_{k+n} = xI_k$$

where $I_j \in \text{Hilb}(C, 0)$ and $\dim(I_j/I_{j+1}) = 1$.

$$Spr_{I, V}(\gamma, v) = \bigvee_{I_k \in \text{Hilb}(C, 0)} \gamma_{I_k} \quad \text{s.t.} \quad I_k = xI_{k+1}$$

$$= \text{Hilb}(C, 0) \times \mathcal{I} \text{Inn}$$
Hilbert schemes on singular curves

Let us specialize now to the rational Cherednik algebra case, so \( G = \text{GL}_n \) and \( V = \mathfrak{gl}_n \oplus \mathbb{C}^n \).

**Theorem (Garner-Kivinen, Gorsky-S.-Vazirani)**

Assume \( \gcd(m; n) = 1 \) and let \( C = \{ y^n = x^m \} \subseteq \mathbb{C}^2 \). Then:

1. \( \text{Spr}_{G\mathcal{O}, V\mathcal{O}}(Y, \nu) = \text{Hilb}(C, 0) \).
2. \( \text{Spr}_{I, \mathcal{O}[\mathbb{C}^n[[\epsilon]]]} = \text{PHilb}(C, 0) \), where \( \text{PHilb}(C, 0) \) is the moduli space of flags of ideals

\[
I_k \supseteq I_{k+1} \supseteq \cdots \supseteq I_{k+n} = xI_k
\]

where \( I_j \in \text{Hilb}(C, 0) \) and \( \dim(I_j/I_{j+1}) = 1 \).

**Remark**

More generally, Garner and Kivinen show that if \( C \) is any planar curve and \( 0 \in C \), then \( \text{Hilb}(C, 0) \) can be realized as a generalized affine Springer fiber for the same quiver gauge theory.
Hilbert schemes on singular curves

Garner-Kivinen
Parahoric affine Springer theory + G.K.-dim computation

We still take $C = \{x^m = y^n\}$ with $\gcd(m;n) = 1$.

**Theorem (Garner-Kivinen, Gorsky-S.-Vazirani)**

*We have an action of $\mathcal{A}_{m/n}(n,1)$ on $H_{\ast}^{\mathbb{C}^\times}(\text{Hilb}(C,0))$ and of $H_{m/n}(n)$ on $H_{\ast}^{\mathbb{C}^\times}(\text{PHilb}(C,0))$. Moreover,*

- $H_{\ast}^{\mathbb{C}^\times}(\text{Hilb}(C,0)) = L_{m/n,1}$.
- $H_{\ast}^{\mathbb{C}^\times}(\text{PHilb}(C,0)) = F_{m/n}$.

**Construct geometric operator on $\mathfrak{h}_{2,1}^\vee (\text{PHilb}(C,0))$**

$\chi_{1(2\cdots n)}(I_2 \cdots I_\ell, \chi I_k \geq I_{k+1}) \Rightarrow I_{k+1} \preceq \cdots \preceq \chi I_k \geq \chi I_{k+1}$

($12 \cdots n, \text{"inv/e"}$

$S_n$ - Springer-type action
What about $L_{m/n}(n, r)$?

We now consider the following element in $V[[\epsilon]]$ where, recall, $V$ is the space of representations of the cyclic quiver with dimension vector $(n, \ldots, n)$ and framing $(1, 0, \ldots, 0)$.

Its affine Springer fiber $Spr_{G_{\mathcal{O}}, V_{\mathcal{O}}}((Y, \text{Id}, \ldots, \text{Id}, \nu))$ is a subset of the product of affine Grassmannians $(\text{GL}_{n, \mathcal{K}} / \text{GL}_{n, \mathcal{O}})^r$. 
Theorem (Gorsky-S.-Vazirani)

Assume $\gcd(m; n) = 1$ and let $C = \{y^n = x^m\}$. Then $\text{Spr}_{\mathcal{O}, \mathcal{V}}(Y, \text{Id}, \ldots, \text{Id}, v)$ is the moduli space of flags

$$J_0 \supseteq \cdots \supseteq J_r = yJ_\circ$$

where $J_k \in \text{Hilb}(C, 0)$ for every $k$. 
Theorem (Gorsky-S.-Vazirani) \( \Lambda \subseteq \text{PHilb}^y(C,0) \)

Assume \( \gcd(m;n) = 1 \) and let \( C = \{ y^n = x^m \} \). Then \( \text{Spr}_{G_{\mathcal{O}}, V_{\mathcal{O}}}(Y, \text{Id}, \ldots, \text{Id}, v) \) is the moduli space of flags

\[
\bigcup J_0 \supseteq \cdots \supseteq J_r = yJ_1
\]

where \( J_k \in \text{Hilb}(C,0) \) for every \( k \).

Let us call \( \text{CPHilb}^y(C,0) \) this moduli space. We have

\[
\text{LM}_{\eta, r} = H^*(\text{CPHilb}^y(C,0)) = (H^*(\text{PHilb}^y(C,0))) \otimes (\mathbb{C}^r)^\otimes m S_m
\]

which is a geometric manifestation of the \( m,n \)-switch.

If \((a_1, \ldots, a_r) = m \quad a_i > 0, \quad \sum a_i = m\)

\[
\bigcup \text{PHilb}^y_{(a_1, \ldots, a_r)}(C,0) = \text{CPHilb}^y(C,0)
\]

\[
H^*(\text{PHilb}^y_{(a_1, \ldots, a_r)}(C,0)) \text{ is a glr-weight space in } \text{LM}_{\eta, r}.
\]
Theorem (Gorsky-S.-Vazirani)

Assume \( \gcd(m; n) = 1 \) and let \( C = \{ y^n = x^m \} \). Then
\[
\text{Spr}_{G_0, V_0}(Y, \text{Id}, \ldots, \text{Id}, v)
\]
is the moduli space of flags
\[
J_0 \supseteq \cdots \supseteq J_r = yJ_1
\]
where \( J_k \in \text{Hilb}(C, 0) \) for every \( k \).

Let us call \( \text{CPHilb}^y(C, 0) \) this moduli space. We have
\[
H^\C_*(\text{CPHilb}^y(C, 0)) = (H^\C_*(\text{PHilb}^y(C, 0)) \otimes (\C^r)^\otimes m)_S \cong m
\]
which is a geometric manifestation of the \( m, n \)-switch.

Remark

Using results of Garner-Kivinen, one can show that given any curve \( C \),
a similar moduli space is a generalized affine Springer fiber for the
cyclic quiver. Algebraically, this should be manifested by the
correspondence between minimally supported representations of
\( \mathcal{A}_{m/n}(n, r) \) and \( H_{n/m}(m) \) given in joint work with Etingof, Krylov and Losev.
Back to the finite case

For the representation $\overline{L}_{m/n,r}$, Oblomkov-Yun realize the representation $\overline{F}_{n/m}$ geometrically as the associated graded (with respect to the perverse filtration) of the cohomology of a certain affine Springer fiber. We can interpret these Springer fibers as a certain moduli space of $\mathbb{C}[[z^m, z^n]]$-invariant subsets of $\mathbb{C}((z))$. Consider:

$$X_{m/n}^r := \{L_0 \supseteq \cdots \supseteq L_r = z^m L_0\}$$

where

- $L_i$ is a $\mathbb{C}[[z^m, z^n]]$-submodule of $\mathbb{C}((z))$.
- $\dim(L_0/(\mathbb{C}[[z]] \cap L_0)) - \dim(\mathbb{C}[[z]]/(\mathbb{C}[[z]] \cap L_0)) = 0$. 
The space $X^r_{m/n}$ is obviously disconnected. For each composition

$a := (a_1, \ldots, a_r) \vdash_r m$, we may consider the space

\[ X^a_{m/n} \subseteq X^r_{m/n} \]

given by the conditions that $\dim(L_{i-1}/L_i) = a_i$. It is then clear that

\[ X^r_{m/n} = \bigsqcup_{a \vdash_r m} X^a_{m/n} \]

Each $X^a_{m/n}$ is an affine Springer fiber on a partial affine flag variety. Each of these admits an affine paving (Lusztig-Smel, Goresky-Kottwitz-MacPherson...) that can be described using the combinatorics of the affine symmetric group $\hat{S}_m$. 
We think of $\hat{S}_m$ as the space of $m$-periodic bijections $\sigma : \mathbb{Z} \rightarrow \mathbb{Z}$, subject to the condition that $\sum_{i=1}^{m} (\sigma(i) - i) = 0$.

**Definition**

We say that an element $\sigma \in \hat{S}_m$ is $n$-stable if $\sigma(x + n) \supseteq \sigma(x)$ for every $x \in \mathbb{Z}$. Let us denote by $\hat{S}_m^n$ the space of $n$-stable elements of $\hat{S}_m$.

If $a = (1, 1, \ldots, 1)$, a theorem of Gorsky-Mazin-Vazirani (after Lusztig-Smelt and Hikita) tells us that $X_{m/n}^a$ admits an affine paving with cells indexed by $\hat{S}_m^n$. We now do a parabolic version of this construction.
For $a \models r m$, let $S_a = S_{a_1} \times S_{a_2} \times \cdots \times S_{a_r} \subseteq S_m \subseteq \hat{S}_m$ be the corresponding parabolic subgroup. We denote by $(S_a \setminus \hat{S}_m) \subseteq \hat{S}_m$ the set of minimal length right coset representatives. More precisely, $(S_a \setminus \hat{S}_m)$ consists of elements $\sigma \in \hat{S}_m$ satisfying:

- $\sigma^{-1}(1) < \cdots < \sigma^{-1}(a_1)$.
- $\sigma^{-1}(a_1 + 1) < \cdots < \sigma^{-1}(a_1 + a_2)$.

\[ \vdots \]

- $\sigma^{-1}(a_1 + \cdots + a_{r-1} + 1) < \cdots < \sigma^{-1}(m)$

$\Theta = (1, \ldots, 1)$ \quad $(S_a \setminus \hat{S}_m) \cong \hat{S}_m$

$\Theta = (m)$ \quad $S_a \setminus \hat{S}_m = \text{affine Grassmann permutation}$
Lemma

The space $X_{m/n}^a$ has an affine paving with cells in bijection with

$$\mathcal{S}_m^\circ \cap (S_a \setminus \mathcal{S}_m).$$

The dimension of the cell $C_\sigma$ associated with $\sigma \in \mathcal{S}_m^\circ \cap (S_a \setminus \mathcal{S}_m)$ is given by

$$\dim(C_\sigma) = \#\{(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n - 1\} \mid \sigma(i + j) < \sigma(i)\}$$

Geometrically, we have a projection $X_{m/n}^{(1, \ldots, 1)} \to X_{m/n}^a$. This projection maps cells to cells, but it may decrease the dimension of a cell. If a cell in $X_{m/n}^{(1, \ldots, 1)}$ is indexed by $\sigma \in \mathcal{S}_m^\circ \cap (S_a \setminus \mathcal{S}_m)$, its dimension is preserved.

$$\mathcal{S}_m^\circ \cap (S_a \setminus \mathcal{S}_m) \Rightarrow \text{SSPF of type } \alpha$$

1. appear a timer
2. appear a timer

José Simental (MPIM)  Gieseker, Springer, Catalan  March 12, 2021
We define the higher rank \( m/n \) \((q, t)\)-Catalan number as:

\[
C^{r}_{m/n}(q, t; q_1, \ldots, q_r) := \sum_{a \models_r m} q_1^{a_1} \cdots q_r^{a_r} \sum_{\sigma \in \hat{S}_m \cap (S_a \setminus \hat{S}_m)} q^{\text{coarea}(\sigma)} t^{\dim_{\mathbb{C}}(C_{\sigma})}
\]
We define the higher rank $m/n \ (q, t)$-Catalan number as:

$$C^{r}_{m/n}(q, t; q_1, \ldots, q_r) := \sum_{a \subseteq r \cdot m} q_1^{a_1} \cdots q_r^{a_r} \sum_{\sigma \in \hat{S}_m \cap (S_a \setminus \hat{S}_m)} q^{\text{coarea}(\sigma)} t^{\text{dim}_C(C_\sigma)}$$

- $C^{r}_{m/n}(1, 1; 1, \ldots, 1) = C^{r}_{m/n}$.
We define the higher rank $m/n$ $(q, t)$-Catalan number as:

$$C^r_{m/n}(q, t; q_1, \ldots, q_r) := \sum_{a \subseteq_r m} q_1^{a_1} \cdots q_r^{a_r} \sum_{\sigma \in \hat{S}_m \cap (S_a \setminus \hat{S}_m)} q^{\coarea(\sigma)} t^{\dim_{\mathbb{C}}(C_\sigma)}$$

- $C^r_{m/n}(1, 1; 1, \ldots, 1) = C^r_{m/n}$.
- $C^r_{m/n}(1, t^2; 1, \ldots, 1) = \mathcal{H}(X^r_{m/n}; t)$. 
We define the higher rank $m/n$ $(q, t)$-Catalan number as:

\[
C^r_{m/n}(q, t; q_1, \ldots, q_r) := \sum_{\mathbf{a} \equiv_r m} q_a^{a_1} \cdots q_a^{a_r} \sum_{\sigma \in \hat{S}_m \cap (S_a \setminus \hat{S}_m)} q^{\text{coarea} (\sigma)} t^\dim \mathbb{C}(C_\sigma)
\]

- $C^r_{m/n}(1, 1; 1, \ldots, 1) = C^r_{m/n}$.
- $C^r_{m/n}(1, t^2; 1, \ldots, 1) = \mathcal{H}(X^r_{m/n}; t)$.
- $C^r_{m/n}(q, q^{-1}; q_1^{-1}, \ldots, q_r^{-1})$ is the $\mathbb{C}^\times \times \text{GL}_r$-character of $\overline{L}_{m/n,r}$. 
We define the higher rank $m/n$ $(q, t)$-Catalan number as:

$$C_{m/n}^{r}(q, t; q_1, \ldots, q_r) := \sum_{a \subseteq_r m} q_1^{a_1} \cdots q_r^{a_r} \sum_{\sigma \in \hat{S}_m \cap (S_a \setminus \hat{S}_m)} q^{\coarea(\sigma)} t^{\dim_{\mathbb{C}}(C_{\sigma})}$$

- $C_{m/n}^{r}(1, 1; 1, \ldots, 1) = C_{m/n}^{r}$.
- $C_{m/n}^{r}(1, t^2; 1, \ldots, 1) = \mathcal{H}(X_{m/n}^{r}; t)$.
- $C_{m/n}^{r}(q, q^{-1}; q_1^{-1}, \ldots, q_r^{-1})$ is the $\mathbb{C}^\times \times \text{GL}_r$-character of $\overline{L}_{m/n, r}$.
- $C_{m/n}^{r}(q, t; q_1, \ldots, q_r) = C_{m/n}^{r}(t, q; q_1, \ldots, q_r)$. $\Leftarrow$ Shuffle $\oplus_{\mu, \nu}$ (Mellit, Carlsson-Mellit).
We define the higher rank $m/n$ $(q, t)$-Catalan number as:

$$C_{m/n}^r(q, t; q_1, \ldots, q_r) := \sum_{a \in \mathbb{N}_m} q_1^{a_1} \cdots q_r^{a_r} \sum_{\sigma \in \hat{S}_m \cap (S_a \setminus \hat{S}_m)} q^{\text{coarea}(\sigma)} t^\dim_{\mathbb{C}}(C_\sigma)$$

- $C_{m/n}^r(1, 1; 1, \ldots, 1) = C_{m/n}^r$.
- $C_{m/n}^r(1, t^2; 1, \ldots, 1) = \mathcal{H}(X_{m/n}^r; t)$.
- $C_{m/n}^r(q, q^{-1}; q_1^{-1}, \ldots, q_r^{-1})$ is the $\mathbb{C}^\times \times \text{GL}_r$-character of $\bar{L}_{m/n,r}$.
- $C_{m/n}^r(q, t; q_1, \ldots, q_r) = C_{m/n}^r(t, q; q_1, \ldots, q_r)$.
- $C_{m/n}^r(q, t; q_1, \ldots, q_r)$ is symmetric in $q_1, \ldots, q_r$.

$$C_{m/n}^r(q, t; q_1, \ldots, q_r) = q_1 + \ldots + q_r$$
$$C_{m/n}^r(q, t; q_1, \ldots, q_r) = h_m(q_1, \ldots, q_r)$$
We define the higher rank $m/n$ $(q, t)$-Catalan number as:

\[ C_{m/n}^r (q, t; q_1, \ldots, q_r) := \sum_{a \subseteq_r m} q_1^{a_1} \cdots q_r^{a_r} \sum_{\sigma \in \hat{S}_m \cap (S_a \setminus \hat{S}_m)} q^{\operatorname{coarea}(\sigma)} t^{\dim_{\mathbb{C}}(C_{\sigma})} \]

- $C_{m/n}^r (1, 1; 1, \ldots, 1) = C_{m/n}^r$.
- $C_{m/n}^r (1, t^2; 1, \ldots, 1) = \mathcal{H}(X_{m/n}^r; t)$.
- $C_{m/n}^r (q, q^{-1}; q_1^{-1}, \ldots, q_r^{-1})$ is the $\mathbb{C}^\times \times \operatorname{GL}_r$-character of $\overline{L}_{m/n,r}$.
- $C_{m/n}^r (q, t; q_1, \ldots, q_r) = C_{m/n}^r (t, q; q_1, \ldots, q_r)$.
- $C_{m/n}^r (q, t; q_1, \ldots, q_r)$ is symmetric in $q_1, \ldots, q_r$.

Let $r = kd$ and $q_1, \ldots, q_r \in q^{\mathbb{Z}}$ so that $q_1 + \cdots + q_r = [d]_q [k]_q^{nd}$. Then,

\[ C_{m/n}^r (q^{2d}, q^{-2d}; q_1, \ldots, q_r) = \frac{[d]}{[nd]} \binom{nr + m - 1}{m} \]
Example: $\overline{L_{3/2,2}}$

We have $L_{3/2,2} = W^*_{(2,1)} + (q + q^{-1})W^*_{(3)}$, where $W_{\lambda}$ is a representation of $\mathfrak{gl}_2$.

- $X^2_{3/2} = \square X_{3/2}^{(3,0)} \sqcup X_{3/2}^{(0,3)} \sqcup X_{3/2}^{(1,2)} \sqcup X_{3/2}^{(2,1)} = \mathbb{P}^1 \sqcup \mathbb{P}^1 \sqcup (\mathbb{P}^1 \sqcup \mathbb{P}^1) \sqcup (\mathbb{P}^1 \sqcup \mathbb{P}^1)$.

\[
\begin{align*}
X_{3/2}^{(3,0)} & \quad X_{3/2}^{(0,3)} & \quad X_{3/2}^{(1,2)} & \quad X_{3/2}^{(2,1)} \\
1 & 1 & 2 & 2 & q + t & q + t & q + qt + t & q + qt + t
\end{align*}
\]

so $C_{3/2}^2(q, t; q_1, q_2) = (q_1^3 + q_2^3)(q + t) + (q_1 q_2 + q_1 q_2^2)(q + qt + t)$. 
Example: $\bar{L}_{3/2,2}$

We have $L_{3/2,2} = W_{(2,1)}^* + (q + q^{-1})W_{(3)}^*$, where $W_{\lambda}$ is a representation of $\mathfrak{gl}_2$.

- $X_{3/2}^2 = X_{3/2}^{(3,0)} \sqcup X_{3/2}^{(0,3)} \sqcup X_{3/2}^{(1,2)} \sqcup X_{3/2}^{(2,1)} = \mathbb{P}^1 \sqcup \mathbb{P}^1 \sqcup (\mathbb{P}^1 \sqcup \mathbb{P}^1) \sqcup (\mathbb{P}^1 \sqcup \mathbb{P}^1)$.

\[ X^{(3,0)} \quad X^{(0,3)} \quad X^{(1,2)} \quad X^{(2,1)} \]

\[ q + t \quad q + t \quad q + qt + t \quad q + qt + t \]

so $C_{3/2}^2(q, t; q_1, q_2) = (q_1^3 + q_2^3)(q + t) + (q_1^2 q_2 + q_1 q_2^2)(q + qt + t)$.

- $C_{3/2}^2(q, q^{-1}; q_1^{-1}, q_2^{-1}) = (q_1^{-2} q_2^{-1} + q_1^{-1} q_2^{-2}) + (q + q^{-1})(q_1^{-3} + q_2^{-3} + q_1^{-2} q_2^{-1} + q_1^{-1} q_2^{-2})$. 
Example: $\overline{L}_{3/2,2}$

We have $L_{3/2,2} = W_{(2,1)}^* + (q + q^{-1})W_{(3)}^*$, where $W_{\lambda}$ is a representation of $\mathfrak{gl}_2$.

- $X_{3/2}^2 = X_{3/2}^{(3,0)} \sqcup X_{3/2}^{(0,3)} \sqcup X_{3/2}^{(1,2)} \sqcup X_{3/2}^{(2,1)} = \mathbb{P}^1 \sqcup \mathbb{P}^1 \sqcup (\mathbb{P}^1 \cup \mathbb{P}^1) \sqcup (\mathbb{P}^1 \cup \mathbb{P}^1)$.

$$
\begin{array}{cccc}
X^{(3,0)} & X^{(0,3)} & X^{(1,2)} & X^{(2,1)} \\
1 \quad 1 & 2 \quad 2 & 1 \quad 2 & 2 \quad 2 \\
\end{array}
$$

- $q + t \quad q + t \quad q + qt + t \quad q + qt + t$

so $C_{3/2}^2(q,t;q_1,q_2) = (q_1^3 + q_2^3)(q + t) + (q_1^2q_2 + q_1q_2^2)(q + qt + t)$.

- For $d = 1, \ k = 2$,

\[ C_{3/2}^2(q^2,q^{-2};q^2,q^{-2}) = q^8 + 2q^4 + q^2 + 2 + q^{-2} + 2q^{-4} + q^{-8}. \]
Example: $\bar{L}_{3/2,2}$

We have $L_{3/2,2} = W_{(2,1)}^* + (q + q^{-1})W_{(3)}^*$, where $W_\lambda$ is a representation of $\mathfrak{gl}_2$.

- $X_{3/2}^2 = X_{3/2}^{(3,0)} \sqcup X_{3/2}^{(0,3)} \sqcup X_{3/2}^{(1,2)} \sqcup X_{3/2}^{(2,1)} = \mathbb{P}^1 \sqcup \mathbb{P}^1 \sqcup (\mathbb{P}^1 \sqcup \mathbb{P}^1) \sqcup (\mathbb{P}^1 \sqcup \mathbb{P}^1)$.

\[ q + t \quad q + t \quad q + qt + t \quad q + qt + t \]

so $C_{3/2}^2(q, t; q_1, q_2) = (q_1^3 + q_2^3)(q + t) + (q_1^2q_2 + q_1q_2^2)(q + qt + t)$.

- For $d = 2, k = 1$,

$C_{3/2}^2(q^4, q^{-4}; q, q^{-1}) = q^7 + q^5 + q^3 + 2q + 2q^{-1} + q^{-3} + q^{-5} + q^{-7}$. 
Future directions

- Relate $C^r_{m/n}(q, t; q_1, \ldots, q_r)$ to Neguţ’s EHA action on $K$-theory of $\mathcal{M}^\theta(n, r)$. 

Future directions

- Relate $C^r_{m/n}(q, t; q_1, \ldots, q_r)$ to Neguț’s EHA action on $K$-theory of $M^\theta(n, r)$.
- $K$-theoretic analogue of $L_{m/n,r} = (F_{n/m} \otimes (C^r)^{\otimes m})S_m$.
Future directions

- Relate $C^r_{m/n}(q, t; q_1, \ldots, q_r)$ to Neguț’s EHA action on $K$-theory of $\mathcal{M}^\theta(n, r)$.
- $K$-theoretic analogue of $L_{m/n,r} = (F_{n/m} \otimes (\mathbb{C}^r)^{\otimes m})^{S_m}$.
- Higher rank Catalan numbers in other types.
Thank you!