Gieseker varieties, affine Springer fibers, and higher rank q, t-Catalan numbers

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MIT Infinite Dimensional Algebra Seminar

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Plan

1. Quantized Gieseker varieties (joint work with P. Etingof, V. Krylov, I. Losev)

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    - Gieseker varieties.
    - Quantizations.
    - Finite-dimensional representations.

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2. Geometric construction of representations (joint work with E. Gorsky, M. Vazirani)

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2. Geometric construction of representations (joint work with E. Gorsky, M. Vazirani)
   - Coulomb branches.
   - BFN Springer theory.
   - Hilbert schemes of points on singular curves.

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   - Finite-dimensional representations.
2. Geometric construction of representations (joint work with E. Gorsky, M. Vazirani)
   - Coulomb branches.
   - BFN Springer theory.
   - Hilbert schemes of points on singular curves. in progress
3. Higher rank (q, t)-Catalan numbers (joint work with V. Krylov)

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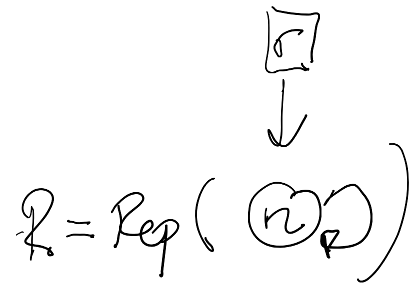
   - Gieseker varieties.
   - Quantizations.
   - Finite-dimensional representations.
2. Geometric construction of representations (joint work with E. Gorsky, M. Vazirani)
   - Coulomb branches.
   - BFN Springer theory.
   - Hilbert schemes of points on singular curves.
3. Higher rank (q, t)-Catalan numbers (joint work with V. Krylov)
   - Affine Springer fibers.
   - Affine pavings via the affine symmetric group.
   - Higher rank (q, t)-Catalan numbers.

Gieseker varieties



Fix positive integers $n, r > 0$. Define

- $R := \mathfrak{gl}_n(\mathbb{C}) \oplus \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$
- $\overline{R} := \mathfrak{sl}_n(\mathbb{C}) \oplus \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$.

The group $G := \mathrm{GL}_n(\mathbb{C})$ acts naturally both on R and $\overline{R}$, and the action lifts to the cotangent bundles $T^*R, T^*\overline{R}$. This action is Hamiltonian, with the same formula for the moment map

$\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)$ trace

$$\mu(A, B, i, j) = [A, B] - ij \in \mathfrak{gl}_n^* \cong \mathfrak{gl}_n.$$

$\mathfrak{gl}_n$ $\mathfrak{gl}_n$ $\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$

$$\mu^* : \mathfrak{gl}_n \to \mathbb{C}[T^*R]$$
$$\xi \longmapsto \xi_R$$

Gieseker varieties

Definition

We define the *Gieseker variety* $\mathcal{M}(n,r)$ (resp. $\overline{\mathcal{M}}(n,r)$) to be the Hamiltonian reduction (at level 0) of G acting on T^*R (resp. $T^*\overline{R}$).

$$\mathcal{M}(n,r) = \mu^{-1}(0)/\!\!/G = \operatorname{Spec}\left(\left[\frac{\mathbb{C}[T^*R]}{(\xi_R \mid \xi \in \mathfrak{gl}_n)}\right]^G\right)$$

- Affine
- Poisson
- Conical $\mathbb{C}^\times \curvearrowright \mathcal{M}(n,r)$ induced by $\mathbb{C}^\times \curvearrowright T^*R$ by dilations
- $\dim \mathcal{M}(n,r) = 2nr$, $\dim \overline{\mathcal{M}}(n,r) = 2nr-2$

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Note that

$$\mathcal{M}(n,r) = \overline{\mathcal{M}}(n,r) \times \mathbb{C}^2.$$

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Example

- $\mathcal{M}(n,1) = (\mathbb{C}^{2n})/S_n$, $\overline{\mathcal{M}}(n,1) = (\mathfrak{h} \oplus \mathfrak{h}^*)/S_n$, where $\mathfrak{h}$ is the reflection representation of S_n.
- $\overline{\mathcal{M}}(1,r) = \overline{\mathcal{O}}_{\min}$, the closure of the minimal nilpotent orbit in $\mathfrak{sl}_r$.

$\| \; \{x \in \mathfrak{sl}_r \mid x^2 = 0, \text{ rank } x \leq 1\}$

Gieseker varieties

Note that we have an action of the 1-dimensional torus $\mathbb{C}^\times$ on T^*R and $T^*\overline{R}$ by dilations. This action commutes with that of G, and descends to $\mathcal{M}(n,r)$, $\overline{\mathcal{M}}(n,r)$. These varieties are conical, singular, of dimension $2nr$ and $2nr-2$, respectively. Moreover, they carry a Poisson bracket of degree -2. We can construct symplectic resolutions $\mathcal{M}^\theta(n,r)$, $\overline{\mathcal{M}}^\theta(n,r)$ using GIT Hamiltonian reduction.

$\theta : GL_n \to \mathbb{C}^\times$

$$\mathcal{M}^\theta(n,r) = \mu^{-1}(0)^{\theta\text{-ss}} / G$$

μ is flat $\Rightarrow$ $\mathcal{M}^\theta(n,r)$ $(\theta \neq 1)$ is a resolution of $\mathcal{M}(n,r)$

Gieseker varieties

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Example

- $\mathcal{M}^\theta(n,1) = \mathrm{Hilb}^n(\mathbb{C}^2)$. $\xrightarrow{HC} \mathcal{M}(n,1)$
- $\overline{\mathcal{M}}^\theta(1,r) = T^*\mathbb{P}^{r-1}$.

$\downarrow$

$\overline{\mathcal{O}}_{min}$

Gieseker varieties

lifted from action on R (resp $\overline{R}$)

We have an action of the group $\mathbb{C}^\times \times \mathrm{GL}_r$ on T^*R (resp. $T^*\overline{R}$):

$$(t, g).(A, B, i, j) = (tA, t^{-1}B, ig^{-1}, gj)$$

Note that this descends to an action of $\mathbb{C}^\times \times \mathrm{PGL}_r$ on $\mathcal{M}(n, r)$ (resp. $\overline{\mathcal{M}}(n, r)$).

Hamiltonian

$A, B \in \mathfrak{gl}_n \quad g \in GL_r$

Quantized Gieseker varieties

We will be interested in quantizations of $\mathcal{M}(n,r)$, $\overline{\mathcal{M}}(n,r)$ which can be produced using quantum Hamiltonian reduction, as follows.
Fix a parameter $c \in \mathbb{C}$. For each element $\xi \in \mathfrak{gl}_n$, let ξ_R denote the vector field given by the infinitesimal action on R. Note that we can see ξ_R as a differential operator, $\xi_R \in D(R)$. Then,

$$\mathcal{A}_c(n,r) := \left[\frac{D(R)}{D(R)\{\xi_R - c\,\mathrm{tr}(\xi) \mid \xi \in \mathfrak{gl}_n\}}\right]^G$$

and similarly, define $\overline{\mathcal{A}}_c(n,r)$ with $\overline{R}$ instead of R. Note that

$$\mathcal{A}_c(n,r) = \overline{\mathcal{A}}_c(n,r) \otimes D(\mathbb{C}).$$

$\mathcal{A}_c(n,r)$ has a filt. induced by Bernstein filtration on $D(R)$

$\mathrm{gr}\,\mathcal{A}_c(n,r) \xrightarrow{\sim} \mathbb{C}[\mathcal{M}(n,r)]$ as graded Poisson algebras

Quantized Gieseker varieties

The action of $\mathbb{C}^\times \times \mathrm{PGL}_r$ on $\mathcal{M}(n,r)$ is Hamiltonian, and gives a quantum comoment map $\Upsilon : \mathbb{C} \oplus \mathfrak{sl}_r \to \mathcal{A}_c(n,r)$.

Etingof-Ginzburg

Example

- (Gan-Ginzburg, Losev) When $r = 1$, $\mathcal{A}_c(n,1)$ (resp. $\overline{\mathcal{A}}_c(n,1)$) is the spherical rational Cherednik algebra of type $\mathfrak{gl}_n$ (resp. $\mathfrak{sl}_n$).
- When $n = 1$, $\overline{\mathcal{A}}_c(1,r) = D_c(\mathbb{P}^{r-1})$ and $\mathcal{A}_c(1,r) = D_c(\mathbb{P}^{r-1} \times \mathbb{C})$.

$c \in \mathbb{Z}$, $\overline{\mathcal{A}}_c(1,r) = \mathcal{D}(\mathcal{O}(c))$

Finite-dimensional representations

Theorem (Losev)

The algebra $\overline{\mathcal{A}}_c(n,r)$ admits a finite-dimensional representation if and only if $c = \frac{m}{n}$ with $\gcd(m;n) = 1$ and $c \notin (-r, 0)$. In this case, $\overline{\mathcal{A}}_c(n,r)$ admits a unique irreducible finite-dimensional representation that we call $\overline{L}_{m/n,r}$. This representation doesn't admit self-extensions.

We remark that we have an isomorphism $\overline{\mathcal{A}}_c(n,r) \cong \overline{\mathcal{A}}_{-c-r}(n,r)$. So we will focus on the case $c > 0$.

¿ $\dim_{\mathbb{C}} \overline{L}_{\frac{m}{n},r}$?

¿ $\mathrm{ch}_{\mathbb{C}^\times \times S_r} \overline{L}_{\frac{m}{n},r}$?

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Example Cherednik for DAHA

- When $r = 1$, this is a theorem of Berest-Etingof-Ginzburg.
- When $n = 1$, the theorem says that $D_c(\mathbb{P}^{r-1})$ admits a finite-dimensional representation if and only if we have an algebra of differential operators on a line bundle, $D(\mathcal{O}(m))$. In this case, $\overline{L}_{m/1,r} = \Gamma(\mathbb{P}^{r-1}, \mathcal{O}(m)) = Sym^m(\mathbb{C}^{r*})$.

Finite-dimensional representations

To describe the representation $\overline{L}_{m/n,r}$, we need finite-dimensional representations of the full rational Cherednik algebra.

Definition (Etingof-Ginzburg)

The rational Cherednik algebra of type $\mathfrak{sl}_n$, $\overline{H}_c(n)$, is the quotient of the semidirect product algebra $\mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle \rtimes S_n$ by the relations

- $x_1 + \cdots + x_n = y_1 + \cdots + y_n = 0$.
- $[x_i, x_j] = [y_i, y_j] = 0$.
- $[x_i, y_j] = \frac{1}{n} - cs_{ij}$ $(i \neq j)$

$$e := \frac{1}{n!} \sum_{w \in S_n} w \in \overline{H}_c(n)$$

Gan-Ginzburg: $\overline{A}_c(n,1) \cong e\overline{H}_c(n)e$

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The algebra $\overline{H}_c(n)$ admits a finite-dimensional representation if and only if $c = \frac{m}{n}$ with $\gcd(m; n) = 1$. In this case, $\overline{H}_c(n)$ has a unique irr. *finite-dimensional representation that we call $\overline{F}_{m/n}$. This representation doesn't admit self-extensions.*

Finite-dimensional representations

Since $\overline{\mathcal{A}}_{m/n}(n,1)$ is the spherical Cherednik algebra, we have

$$\overline{L}_{m/n,1} = \overline{F}^{S_n}_{m/n}.$$

This is, however, not the best way to think about $\overline{L}_{m/n,1}$.

Calaque-Enriquez-Etingof

$$\overline{F}^{S_n}_{\frac{m}{n}} \cong \overline{F}^{S_m}_{\frac{n}{m}}$$

$$\begin{array}{cc} \cap\!\!\!\!- & \cap \\ e\overline{H}_{\frac{m}{n}}(n)e & e\overline{H}_{\frac{n}{m}}(m)e \end{array}$$

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Theorem (Etingof-Krylov-Losev-S.)

We have an isomorphism of $\mathbb{C}^\times \times \mathrm{GL}_r$-modules:

$\mathbb{C}^\times$ GL_r

$$\overline{L}_{m/n,r} \cong (\overline{F}_{n/m} \otimes (\mathbb{C}^{r*})^{\otimes m})^{S_m}$$

$\overline{\mathcal{A}}_{\frac{m}{n}}(n,r)$ -mod

S_m acts diag on $\overline{F}_{\frac{n}{m}} \otimes (\mathbb{C}^{r*})^{\otimes m}$

Finite-dimensional representations

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Remark

When $r = 1$, the isomorphism $\overline{F}^{S_n}_{m/n} \cong \overline{F}^{S_m}_{n/m}$ is due to Calaque-Enriquez-Etingof. Further ramifications are due to Gorsky-Oblomkov-Rasmussen-Shende, Etingof-Gorsky-Losev and more.

Finite-dimensional representations

The representation $\overline{F}_{n/m}$ has been intensively studied in recent years and character formulas are known for it. In particular, we can compute the $\mathbb{C}^\times \times \mathrm{GL}_r$-character of $\overline{L}_{m/n,r}$:

$$\mathrm{ch}_{\mathbb{C}^\times \times \mathrm{GL}_r}(\overline{L}_{m/n,r}) = \frac{1}{[n]_q} \sum_{\substack{\lambda \vdash m \\ r(\lambda) \leq \min(n;r)}} s_\lambda(q^{\frac{1-n}{2}}, \ldots, q^{\frac{n-1}{2}})[W_r(\lambda)^*].$$

Schur-function

$\mathfrak{gl}_n$-mod

$$[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$$

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$C^r_{\frac{1}{n}} = r$

divisible by $\frac{r}{\gcd(m;r)}$

Definition

Let m and n be coprime positive integers and $r > 0$. We call

$$C^r_{m/n} := \dim(\overline{L}_{m/n,r}) = \frac{1}{n}\binom{nr+m-1}{m} = \frac{1}{n}\dim Sym^m(\mathbb{C}^{nr})$$

the rank r rational m/n-Catalan number.

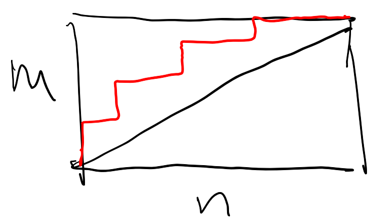
Finite-dimensional representations

gcd(m,n) = 1

When $r = 1$:

$$C^1_{m/n} = \frac{1}{n}\binom{n+m-1}{m} = \frac{1}{n+m}\binom{n+m}{m} = C^1_{n/m}$$

counts the number of Dyck paths in an $m \times n$-rectangle.



Finite-dimensional representations

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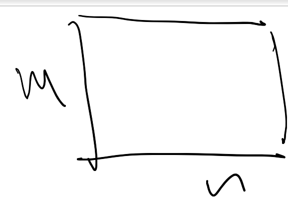
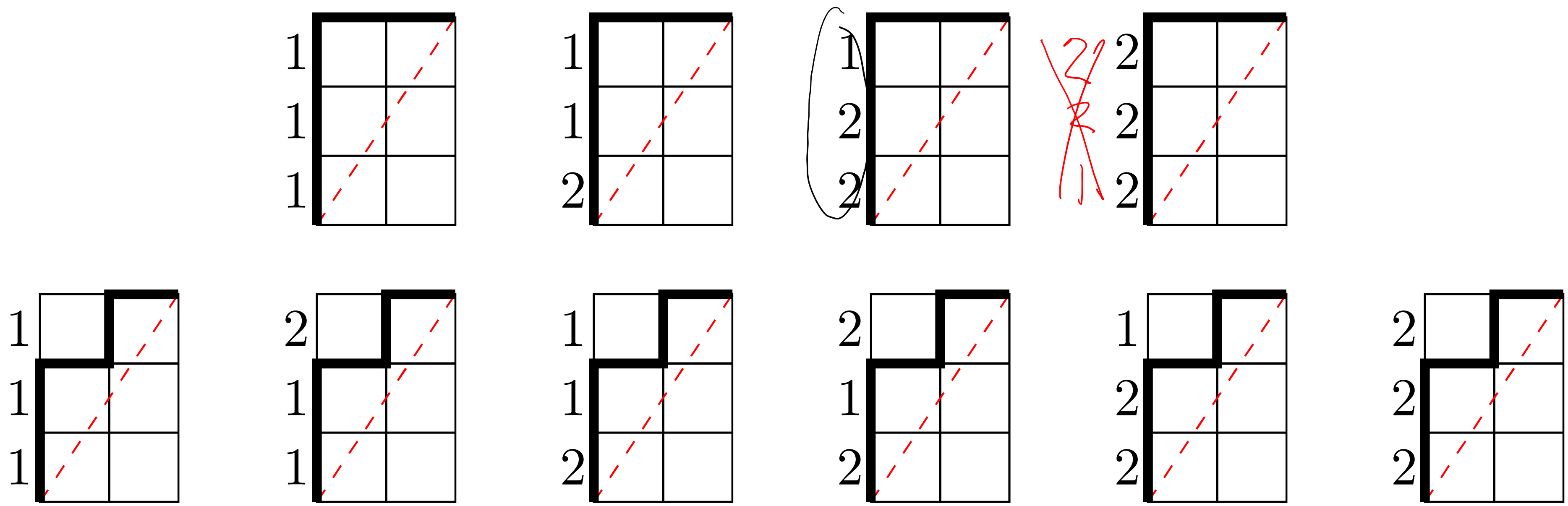
Theorem (Etingof-Krylov-Losev-S.)

The number $C^r_{m/n}$ counts the number of rank r semistandard parking functions on an $m \times n$-rectangle, that is, a Dyck path together with a function from its vertical steps to $\{1, \ldots, r\}$ that is weakly increasing on consecutive vertical steps.



Example: $C^2_{3/2}$

$$C^2_{3/2} = \frac{1}{2}\binom{4+3-1}{3} = 10.$$



Goal

Use geometry to:

1. Explain why the m, n-switch in the formula

$$\overline{L}_{m/n,r} = (\overline{F}_{n/m} \otimes (\mathbb{C}^r)^{\otimes m})^{S_m}$$

   is a natural thing to expect.

2. Produce a q, t-deformation $C^r_{m/n}(q, t)$.

The relation with geometry is clearer for the algebra $\mathcal{A}_c(n,r)$ (as opposed to $\overline{\mathcal{A}}_c(n,r)$). Recall that we have a decomposition

$$\mathcal{A}_c(n,r) = \overline{\mathcal{A}}_c(n,r) \otimes D(\mathbb{C})$$

so the representation

$$L_{m/n,r} := \overline{L}_{m/n,r} \otimes \mathbb{C}[x]$$

is a representation of $\mathcal{A}_c(n,r)$. We will realize this and other representations geometrically.

We also have the $\mathfrak{gl}_n$-version of the rational Cherednik algebra $H_c(n)$, satisfying

$$H_c(n) = \overline{H}_c(n) \otimes D(\mathbb{C})$$

S_n-invariants

and the representation $F_{m/n} = \overline{F}_{m/n} \otimes \mathbb{C}[x]$. It is still true that

$$L_{m/n,r} = (F_{n/m} \otimes (\mathbb{C}^r)^{\otimes m})^{S_m}.$$

$\Uparrow$

Multiply $\mathfrak{sl}_n$-case by $\mathbb{C}[x]$

Coulomb branches

Let G be a reductive group acting on a vector space V. We will denote:

- $\mathcal{O} := \mathbb{C}[[\epsilon]]$.
- $\mathcal{K} := \mathbb{C}((\epsilon))$.
- $G_{\mathcal{O}} = G[[\epsilon]]$, $G_{\mathcal{K}} = G((\epsilon))$.
- $V_{\mathcal{O}} := \mathcal{O} \otimes V = V[[\epsilon]]$, $V_{\mathcal{K}} := \mathcal{K} \otimes V = V((\epsilon))$.
- $I \subseteq G_{\mathcal{O}}$ consisting of elements $g \in G_{\mathcal{O}}$ so that $g|_{\epsilon=0} \in \mathcal{B}$ for a fixed Borel $\mathcal{B} \subseteq G$.
- $\mathfrak{i} \subseteq \mathfrak{g}[[\epsilon]]$ the Lie algebra of I, consisting of elements $X \in \mathfrak{g}[[\epsilon]]$ so that $X|_{\epsilon=0} \in \mathfrak{b}$.

Coulomb branches

Associated to these data, Braverman-Finkelberg-Nakajima construct a Poisson algebra $\mathcal{A}(G, V)$ and its quantization $\mathcal{A}_\hbar(G, V)$. We will not need the precise definition, but we will use the fact that it is supposed to model equivariant homology of the "Steinberg variety"

$$\mathcal{A}_{\hbar}(G, V) = H_*^{G_\mathcal{K} \rtimes \mathbb{C}^\times}(\mathfrak{X}_V \times_{V_\mathcal{K}} \mathfrak{X}_V)$$

here, $\mathfrak{X}_V := (G_\mathcal{K} \times V_\mathcal{O})/G_\mathcal{O}$ is a bundle over the affine Grassmannian. The quantization $\mathcal{A}_\hbar(G, V)$ appears when considering equivariant homology for the loop rotation action.

$$\begin{tikzcd} & \mathfrak{X}_V = (G_\mathcal{K} \times V_\mathcal{O})/G_\mathcal{O} \arrow[dl] \arrow[dr] & \\ G_\mathcal{K}/G_\mathcal{O} = Gr & & V_\mathcal{K} \end{tikzcd}$$

$[g, v] \mapsto gv$

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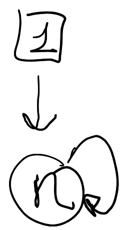
$\mathcal{A}(G, V_\mathcal{O}, I) = \mathrm{Mat}_{|W|}(\mathcal{A}(G,V))$

Coulomb branches

All the algebras we have seen are, in fact, examples of quantized Coulomb branches.

- The spherical rational Cherednik algebra $\mathcal{A}_c(n,1)$ appears taking $G = \mathrm{GL}_n$ and $V = \mathfrak{gl}_n \oplus \mathbb{C}^n$, the adjoint plus fundamental representation. (Kodera-Nakajima, Webster)



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- The full rational Cherednik algebra $H_c(n)$ appears taking again $G = \mathrm{GL}_n$ and $V = \mathfrak{gl}_n \oplus \mathbb{C}^n$, but using Webster's construction with $P = I$ and $N = \mathfrak{i} \oplus \mathbb{C}^n[[\epsilon]] \subseteq V[[\epsilon]]$. (Braveman-Etingof-Finkelberg, Webster)

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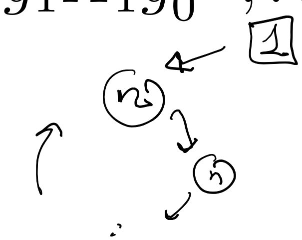
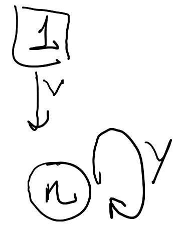
- For the quantized Gieseker variety $\mathcal{A}_c(n,r)$, we take $G = \mathrm{GL}_n^{\times r}$ and $V = \mathfrak{gl}_n^{\oplus r} \oplus \mathbb{C}^n$, with action given by $(g_0, \dots, g_{r-1}) \cdot (X_0, \dots, X_{r-1}, v) = (g_1 X_1 g_0^{-1}, \dots, g_0 X_{r-1} g_{r-1}^{-1}, g_0 v)$ (Nakajima-Takayama+Losev)



r vertices

BFN Springer theory



Hilburn-Kamnitzer-Weekes developed a Springer theory for Coulomb branches (extended to the parahoric setting by Garner-Kivinen). Consider an element $v \in V_{\mathcal{K}}$. The *generalized affine Springer fiber* is

$$Spr_{P,N}(v) := \{[g] \in G_{\mathcal{K}}/P \mid gv \in N\}.$$

Under certain conditions on $\cancel{x}$ v, there is an action of the Coulomb branch algebra $\mathcal{A}_{\hbar}(G, N, P)$ on

$$H_*^{L_v}(Spr_{P,N}(v))$$

where $L_v := \mathrm{Stab}_{G_{\mathcal{K}} \rtimes \mathbb{C}^\times_{\mathrm{rot}}}(v)$.

BFN Springer theory

Now consider the element

$$(Y, v) := \begin{pmatrix} 0 & 0 & \cdots & 0 & \epsilon^m \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathfrak{gl}_n[[\epsilon]] \oplus \mathbb{C}^n[[\epsilon]]$$

BFN Springer theory

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$Y^n = \varepsilon^m$

Choosing a basis, we can identify

$$\mathbb{C}^n[[\epsilon]] = \mathbb{C}[[\epsilon]][1, y, \ldots, y^{n-1}]$$

Note that the action of the matrix Y amounts to multiplication by y in this basis, and the vector v is identified with 1.

Hilbert schemes on singular curves

Let us specialize now to the rational Cherednik algebra case, so $G = \mathrm{GL}_n$ and $V = \mathfrak{gl}_n \oplus \mathbb{C}^n$.

Theorem (Garner-Kivinen, Gorsky-S.-Vazirani)

Assume $\gcd(m;n) = 1$ *and let* $C = \{y^n = x^m\} \subseteq \mathbb{C}^2$. *Then:*

1. $Spr_{G_{\mathcal{O}}, V_{\mathcal{O}}}(Y, v) = \mathrm{Hilb}(C, 0). = \bigsqcup_{k \geq 0} \mathrm{Hilb}^k(C, 0)$

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2. $Spr_{I, \mathfrak{i} \oplus \mathbb{C}^n[[\epsilon]]} = \mathrm{PHilb}(C, 0)$, *where* $\mathrm{PHilb}(C, 0)$ *is the moduli space of flags of ideals*

$x \sim \varepsilon$

$$I_k \supseteq I_{k+1} \supseteq \cdots \supseteq I_{k+n} = xI_k$$

where $I_j \in \mathrm{Hilb}(C, 0)$ *and* $\dim(I_j/I_{j+1}) = 1$.

$Spr_{I, V_{\mathcal{O}}}(Y, v) = I_k \supseteq \cdots \supseteq I_{k+n} = xI_k$ s.t. $I_k \in \mathrm{Hilb}(C, 0)$

$= \mathrm{Hilb}(C, 0) \times \mathcal{F}l_n$

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Remark

More generally, Garner and Kivinen show that if C is any planar curve and $0 \in C$, then $\mathrm{Hilb}(C, 0)$ *can be realized as a generalized affine Springer fiber for the same quiver gauge theory.*

Hilbert schemes on singular curves

Garner-Kivinen

Parahoric affine Springer theory + GK-dim computation

elliptic? Ella?

We still take $C = \{x^m = y^n\}$ with $\gcd(m;n) = 1$.

Theorem (Garner-Kivinen, Gorsky-S.-Vazirani)

We have an action of $\mathcal{A}_{m/n}(n,1)$ *on* $H_*^{\mathbb{C}^\times}(\mathrm{Hilb}(C,0))$ *and of* $H_{m/n}(n)$ *on* $H_*^{\mathbb{C}^\times}(\mathrm{PHilb}(C,0))$*. Moreover,*

- $H_*^{\mathbb{C}^\times}(\mathrm{Hilb}(C,0)) = L_{m/n,1}$.
- $H_*^{\mathbb{C}^\times}(\mathrm{PHilb}(C,0)) = F_{m/n}$.

GSV Construct geometric operator on $H_*^{\mathbb{C}^\times}(\mathrm{PHilb}(\mathbb{C},0))$

$x_1(12\cdots n): I_k \supseteq \cdots \supseteq xI_k = I_{k+n} \mapsto I_{k+1} \supseteq \cdots \supseteq xI_k \supseteq xI_{k+1}$

$(12\cdots n)^{-1}y$, "inverse"

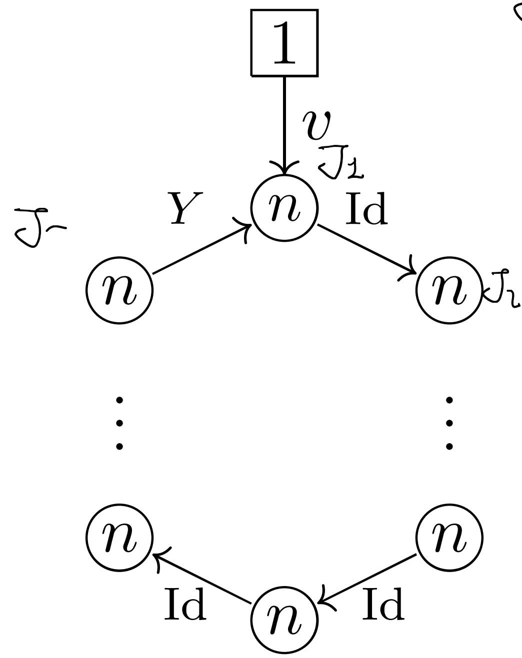
S_n - Springer-type action

What about $L_{m/n}(n, r)$?

We now consider the following element in $V[[\epsilon]]$ where, recall, V is the space of representations of the cyclic quiver with dimension vector $(n, \ldots, n)$ and framing $(1, 0, \ldots, 0)$.

$J_1 \supseteq J_2 \supseteq \ldots \supseteq J_r \supseteq YJ_1$

$Y = \begin{pmatrix} 0 & & \varepsilon^m \\ 1 & \ddots & 0 \\ & & 1 \end{pmatrix}$



$\mathcal{O}^n$

Its affine Springer fiber $Spr_{G_{\mathcal{O}}, V_{\mathcal{O}}}((Y, \mathrm{Id}, \ldots, \mathrm{Id}, v))$ is a subset of the product of affine Grassmannians $(\mathrm{GL}_{n,\mathcal{K}} / \mathrm{GL}_{n,\mathcal{O}})^r$.

Theorem (Gorsky-S.-Vazirani)

Assume $\gcd(m;n) = 1$ *and let* $C = \{y^n = x^m\}$. *Then* $Spr_{G_{\mathcal{O}}, V_{\mathcal{O}}}(Y, \mathrm{Id}, \dots, \mathrm{Id}, v)$ *is the moduli space of flags*

$$J_0 \supseteq \cdots \supseteq J_r = yJ_0$$

where $J_k \in \mathrm{Hilb}(C, 0)$ *for every* k.

Theorem (Gorsky-S.-Vazirani) $H \quad PHilb^{x}(C,0)$

Assume $\gcd(m;n)=1$ *and let* $C=\{y^n=x^m\}$. *Then* $Spr_{G_{\mathcal{O}},V_{\mathcal{O}}}(Y,\mathrm{Id},\dots,\mathrm{Id},v)$ *is the moduli space of flags*

$$J_0 \overset{a_1}{\supseteq} \cdots \overset{a_r}{\supseteq} J_r = yJ_1$$

where $J_k \in \mathrm{Hilb}(C,0)$ *for every* k.

Let us call $\mathrm{CPHilb}^y(C,0)$ this moduli space. We have

$$L_{\frac{m}{n},r} = H_*^{\mathbb{C}^\times}(\mathrm{CPHilb}^y(C,0)) = (H_*^{\mathbb{C}^\times}(\mathrm{PHilb}^y(C,0)) \otimes (\mathbb{C}^r)^{\otimes m})^{S_m}$$

which is a geometric manifestation of the m,n-switch.

If $(a_1,\dots,a_r) \vDash m \quad a_i \geq 0, \ \sum a_i = m$

$$\bigsqcup_{(a_1,\dots,a_r)\vDash m} PHilb^y_{(a_1,\dots,a_r)}(\mathbb{C},0) = CPHilb^y(\mathbb{C},0)$$

$H_*^{\mathbb{C}^\times}(CPHilb^y_{(a_1,\dots,a_r)}(\mathbb{C},0))$ is a $\mathfrak{gl}_r$-weight space in $L_{\frac{m}{n},r}$.

Theorem (Gorsky-S.-Vazirani)

Assume $\gcd(m;n) = 1$ *and let* $C = \{y^n = x^m\}$. *Then* $Spr_{G_{\mathcal{O}}, V_{\mathcal{O}}}(Y, \mathrm{Id}, \dots, \mathrm{Id}, v)$ *is the moduli space of flags*

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Remark

Using results of Garner-Kivinen, one can show that given any curve C*, a similar moduli space is a generalized affine Springer fiber for the cyclic quiver. Algebraically, this should be manifested by the correspondence between minimally supported representations of* $\mathcal{A}_{m/n}(n, r)$ *and* $H_{n/m}(m)$ *given in joint work with Etingof, Krylov and Losev.*

Back to the finite case

For the representation $\overline{L}_{m/n,r}$, Oblomkov-Yun realize the representation $\overline{F}_{n/m}$ geometrically as the associated graded (with respect to the perverse filtration) of the cohomology of a certain affine Springer fiber. We can interpret these Springer fibers as a certain moduli space of $\mathbb{C}[[z^m, z^n]]$-invariant subsets of $\mathbb{C}((z))$. Consider:

$$X^r_{m/n} := \{L_0 \supseteq \cdots \supseteq L_r = z^m L_0\}$$

where

- L_i is a $\mathbb{C}[[z^m, z^n]]$-submodule of $\mathbb{C}((z))$.
- $\dim(L_0/(\mathbb{C}[[z]] \cap L_0)) - \dim(\mathbb{C}[[z]]/(\mathbb{C}[[z]] \cap L_0)) = 0$.

Oblomkov-Yun: $\underline{a} = (1, \dots, 1)$

$\underline{g} = (m)$

The space $X^r_{m/n}$ is obviously disconnected. For each composition $\mathbf{a} := (a_1, \dots, a_r) \vDash_r m$, we may consider the space

$a_i \geqslant 0$

$\sum a_i = m$

$$X^{\mathbf{a}}_{m/n} \subseteq X^r_{m/n}$$

given by the conditions that $\dim(L_{i-1}/L_i) = a_i$. It is then clear that

$$X^r_{m/n} = \bigsqcup_{\mathbf{a} \vDash_r m} X^{\mathbf{a}}_{m/n}$$

Each $X^{\mathbf{a}}_{m/n}$ is an affine Springer fiber on a partial affine flag variety. Each of these admits an affine paving (Lusztig-Smelt, Goresky-Kottwitz-MacPherson...) that can be described using the combinatorics of the affine symmetric group $\widehat{S}_m$.

We think of $\widehat{S}_m$ as the space of m-periodic bijections $\sigma : \mathbb{Z} \to \mathbb{Z}$, subject to the condition that $\sum_{i=1}^{m}(\sigma(i) - i) = 0$.

Definition

We say that an element $\sigma \in \widehat{S}_m$ is n-stable if $\sigma(x+n) \gneq \sigma(x)$ for every $x \in \mathbb{Z}$. Let us denote by $\widehat{S}_m^n$ the space of n-stable elements of $\widehat{S}_m$.

If $\mathbf{a} = (1, 1, \dots, 1)$, a theorem of Gorsky-Mazin-Vazirani (after Lusztig-Smelt and Hikita) tells us that $X^{\mathbf{a}}_{m/n}$ admits an affine paving with cells indexed by $\widehat{S}_m^n$. We now do a parabolic version of this construction.

For $\mathbf{a} \vDash_r m$, let $S_{\mathbf{a}} = S_{a_1} \times S_{a_2} \times \cdots \times S_{a_r} \subseteq S_m \subseteq \widehat{S}_m$ be the correspoding parabolic subgroup. We denote by $(S_{\mathbf{a}} \backslash \widehat{S}_m) \subseteq \widehat{S}_m$ the set of minimal length right coset representatives. More precisely, $(S_{\mathbf{a}} \backslash \widehat{S}_m)$ consists of elements $\sigma \in \widehat{S}_m$ satisfying:

- $\sigma^{-1}(1) < \cdots < \sigma^{-1}(a_1)$.
- $\sigma^{-1}(a_1 + 1) < \cdots < \sigma^{-1}(a_1 + a_2)$.
- $\vdots$
- $\sigma^{-1}(a_1 + \cdots + a_{r-1} + 1) < \cdots < \sigma^{-1}(m)$

$\underline{a} = (1, \dots, 1) \quad (S_{\underline{a}} \backslash \widehat{S}_m) = \widehat{S}_m$

$\underline{a} = (m) \quad S_{\underline{a}} \backslash \widehat{S}_m =$ affine Grassmannian permutations

Lemma

The space $X^{\mathbf{a}}_{m/n}$ has an affine paving with cells in bijection with

$$\widehat{S}^n_m \cap (S_{\mathbf{a}} \backslash \widehat{S}_m).$$

The dimension of the cell C_σ associated with $\sigma \in \widehat{S}^n_m \cap (S_{\mathbf{a}} \backslash \widehat{S}_m)$ is given by

$$\dim(C_\sigma) = \#\{(i,j) \in \{1,\ldots,m\} \times \{1,\ldots,n-1\} \mid \sigma(i+j) < \sigma(i)\}$$

Geometrically, we have a projection $X^{(1,\ldots,1)}_{m/n} \to X^{\mathbf{a}}_{m/n}$. This projection maps cells to cells, but it may decrease the dimension of a cell. If a cell in $X^{(1,\ldots,1)}_{m/n}$ is indexed by $\sigma \in \widehat{S}^n_m \cap (S_{\mathbf{a}} \backslash \widehat{S}_m)$, its dimension is preserved.

$\widehat{S}^n_m \cap (S_a \backslash \widehat{S}_m) \leftrightarrows$ SSPF of type $\underline{a}$

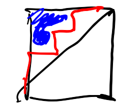
1 appears a_1 times

2 appears a_2 times

$\vdots$

n appears a_n times

We define the higher rank m/n (q,t)-Catalan number as:



$$C^r_{m/n}(q,t;q_1,\dots,q_r) := \sum_{\mathbf{a}\vDash_r m} q_1^{a_1}\cdots q_r^{a_r} \sum_{\sigma\in\widehat{S}^n_m\cap(S_{\mathbf{a}}\setminus\widehat{S}_m)} q^{\mathrm{coarea}(\sigma)} t^{\dim_{\mathbb{C}}(C_\sigma)}$$

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- $C^r_{m/n}(1,1;1,\dots,1) = C^r_{m/n}$.

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- $C^r_{m/n}(1,1;1,\dots,1) = C^r_{m/n}$.
- $C^r_{m/n}(1,t^2;1,\dots,1) = \mathcal{H}(X^r_{m/n};t)$.

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- $C^r_{m/n}(q,q^{-1};q_1^{-1},\dots,q_r^{-1})$ is the $\mathbb{C}^\times\times\mathrm{GL}_r$-character of $\overline{L}_{m/n,r}$.

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- $C^r_{m/n}(q,t;q_1,\dots,q_r) = C^r_{m/n}(t,q;q_1,\dots,q_r)$. $\Leftarrow$ Shuffle thm (Mellit, Carlsson-Mellit)

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$\overline{F}_{\frac{n}{m}}$

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- $C^r_{m/n}(q,t;q_1,\dots,q_r) = C^r_{m/n}(t,q;q_1,\dots,q_r)$.
- $C^r_{m/n}(q,t;q_1,\dots,q_r)$ is symmetric in $q_1,\dots,q_r$.

$$C^r_{\frac{1}{n}}(q,t;q_1,\dots,q_r) = q_1+\dots+q_r$$

$$C^r_{\frac{m}{1}}(q,t;q_1,\dots,q_r) = h_m(q_1,\dots,q_r)$$

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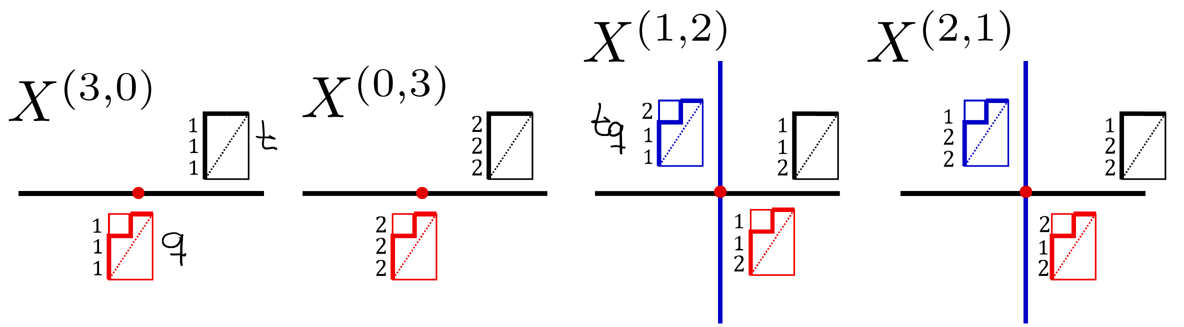
- $C^r_{m/n}(q,t;q_1,\dots,q_r) = C^r_{m/n}(t,q;q_1,\dots,q_r)$. $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$
- $C^r_{m/n}(q,t;q_1,\dots,q_r)$ is symmetric in $q_1,\dots,q_r$.
- Let $r = kd$ and $q_1,\dots,q_r\in q^{\mathbb{Z}}$ so that $q_1+\cdots+q_r = [d]_q[k]_{q^{nd}}$. Then,

$$C^r_{m/n}(q^{2d},q^{-2d};q_1,\dots,q_r) = \frac{[d]}{[nd]}\begin{bmatrix} nr+m-1 \\ m\end{bmatrix}$$

Example: $\overline{L}_{3/2,2}$

We have $L_{3/2,2} = W^*_{(2,1)} + (q+q^{-1})W^*_{(3)}$, where W_λ is a representation of $\mathfrak{gl}_2$.

- $X^2_{3/2} = X^{(3,0)}_{3/2} \sqcup X^{(0,3)}_{3/2} \sqcup X^{(1,2)}_{3/2} \sqcup X^{(2,1)}_{3/2} = \mathbb{P}^1 \sqcup \mathbb{P}^1 \sqcup (\mathbb{P}^1 \cup \mathbb{P}^1) \sqcup (\mathbb{P}^1 \cup \mathbb{P}^1)$.



$q+t$ $\quad$ $q+t$ $\quad$ $q+qt+t$ $\quad$ $q+qt+t$

so $C^2_{3/2}(q,t;q_1,q_2) = (q_1^3+q_2^3)(q+t) + (q_1^2q_2+q_1q_2^2)(q+qt+t)$.

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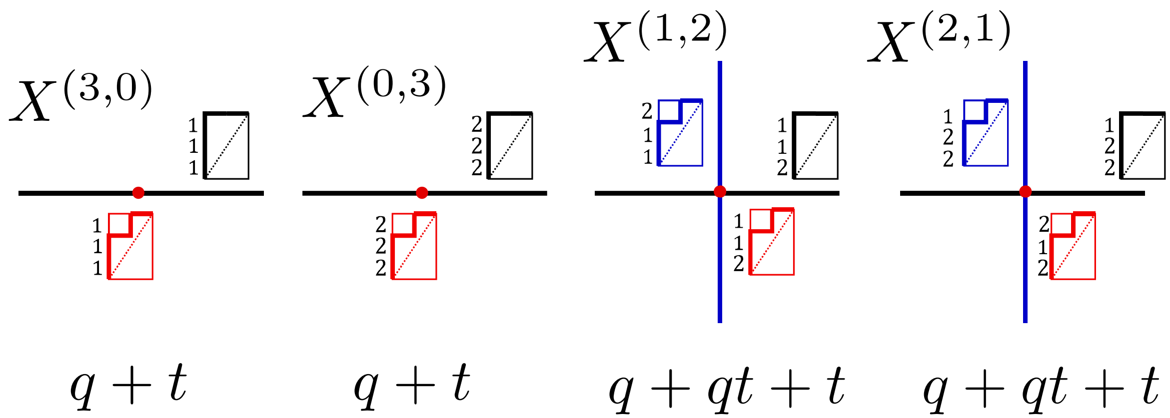

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$(q_1^{-2}q_2^{-1} + q_1^{-1}q_2^{-2}) + (q+q^{-1})(q_1^{-3}+q_2^{-3}+q_1^{-2}q_2^{-1}+q_1^{-1}q_2^{-2})$.

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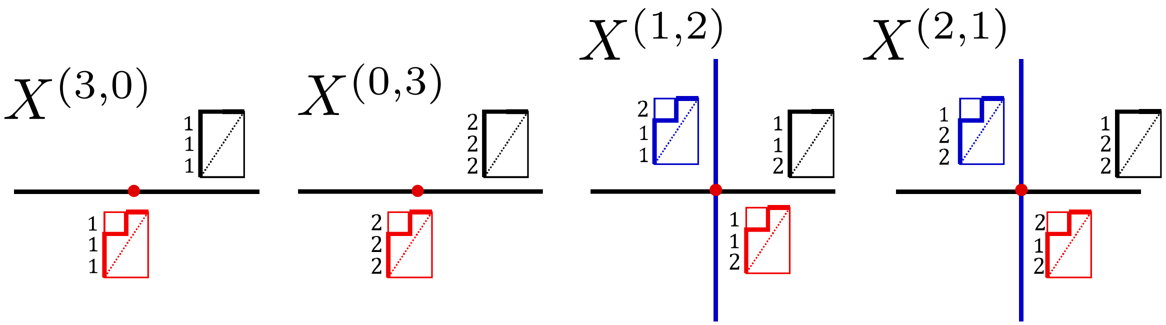

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- For $d=1, k=2$,
$C^2_{3/2}(q^2,q^{-2};q^2,q^{-2}) = q^8+2q^4+q^2+2+q^{-2}+2q^{-4}+q^{-8}$.

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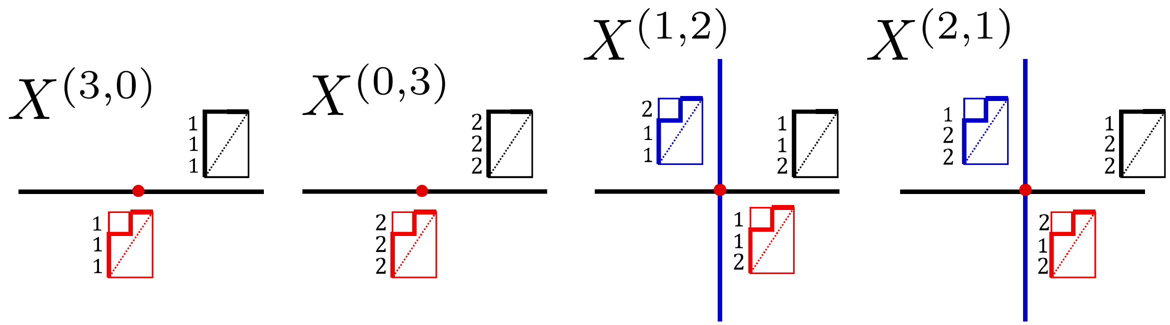

$q + t$ $\quad$ $q + t$ $\quad$ $q + qt + t$ $\quad$ $q + qt + t$

so $C^2_{3/2}(q, t; q_1, q_2) = (q_1^3 + q_2^3)(q + t) + (q_1^2 q_2 + q_1 q_2^2)(q + qt + t)$.

- For $d = 2, k = 1$,
$C^2_{3/2}(q^4, q^{-4}; q, q^{-1}) = q^7 + q^5 + q^3 + 2q + 2q^{-1} + q^{-3} + q^{-5} + q^{-7}$.

Future directions

- Relate $C^r_{m/n}(q,t;q_1,\dots,q_r)$ to Neguț's EHA action on K-theory of $\mathcal{M}^\theta(n,r)$.

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- K-theoretic analogue of $L_{m/n,r} = (F_{n/m} \otimes (\mathbb{C}^r)^{\otimes m})^{S_m}$.
- Higher rank Catalan numbers in other types.

Thank you!