Lyndon words and quantum loop groups

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Quantum groups

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- The subalgebra $U_q(\mathfrak{n}^+) \subset U_q(\mathfrak{g})$ generated by the e_i's is:

$$U_q(\mathfrak{n}^+) = \mathbb{Q}(q)\Big\langle e_i \Big\rangle_{i\in I}$$

  modulo the relation $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} e_i^k e_j e_i^{1-a_{ij}-k} = 0, \ \forall i \neq j.$

The q-shuffle algebra

- An important viewpoint on $U_q(\mathfrak{n}^+)$ is given by comparing it to the q-shuffle algebra defined by Green, Rosso, and Schauenburg:

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- endowed with the following associative *shuffle product*:

$$[i_1 \dots i_k] * [j_1 \dots j_l] = \sum_{\{1,\dots,k+l\} = A \sqcup B, |A|=k, |B|=l} q^{\lambda_{A,B}} \cdot [s_1 \dots s_{k+l}]$$

where if $A = \{a_1 < \dots < a_k\}$ and $B = \{b_1 < \dots < b_l\}$, we write:

$$s_c = \begin{cases} i_\bullet & \text{if } c = a_\bullet \\ j_\bullet & \text{if } c = b_\bullet \end{cases} \qquad \text{and} \qquad \lambda_{A,B} = \sum_{A \ni a > b \in B} (\alpha_{s_a}, \alpha_{s_b})$$

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- This definition is designed so that there is an injective algebra homomorphism $\Phi : U_q(\mathfrak{n}^+) \hookrightarrow \mathcal{F}$ given by $e_i \mapsto [i]$, for all $i \in I$.

Standard Lyndon words

- By work of Lusztig, there exists a PBW basis:

$$U_q(\mathfrak{n}^+) = \bigoplus_{\beta_1 \geq \dots \geq \beta_n \in \Delta^+} \mathbb{Q}(q) \cdot e_{\beta_1} \dots e_{\beta_n}$$

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- Above, $\geq$ is any convex order on Δ^+, but there is a particularly interesting choice. Lalonde-Ram showed that there is a bijection:

$$\ell : \Delta^+ \xrightarrow{\sim} \Big\{ \text{standard Lyndon words} \Big\}$$

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- Leclerc showed that $\Phi(e_\beta)$ has the minimal largest word among all degree β elements of $\mathrm{Im}\ \Phi$, and this largest word is precisely $\ell(\beta)$.

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- The shuffle algebra still makes sense (using letters in $\widehat{I} = I \sqcup 0$ instead of in I) but the Lalonde-Ram bijection breaks down because of the imaginary roots. So does Leclerc's description of $\Phi(e_\beta)$.

Quantum loop/affine groups

- So we need a new viewpoint on $U_q(\widehat{\mathfrak{g}})$. Fortunately, we have an isomorphism (proposed by Drinfeld and proved by Beck, Damiani):

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- where the quantum loop group $U_q(L\mathfrak{g})$ has generators:

$$\{e_{i,d}, f_{i,d}, \varphi^{\pm}_{i,d'}\}_{i \in I, d \in \mathbb{Z}, d' \geq 0}$$

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- However, the subalgebra $U_q(L\mathfrak{n}^+) \subset U_q(L\mathfrak{g})$ generated by $\{e_{i,d}\}$ does not match the subalgebra $U_q(\widehat{\mathfrak{n}}^+) \subset U_q(\widehat{\mathfrak{g}})$ under the isomorphism in the box. The two subalgebras are "orthogonal".

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- **Our goal:** to do for $U_q(L\mathfrak{n}^+)$ what was done for $U_q(\mathfrak{n}^+)$: define a shuffle algebra model, and describe PBW bases via Lyndon words.

The loop q-shuffle algebra

- Instead of using $i \in I$ as letters, let us use the symbols $i^{(d)}$ as letters, for any $i \in I$ and $d \in \mathbb{Z}$. Consider the vector space:

$$\widehat{\mathcal{F}} = \bigoplus_{k \in \mathbb{N}, i_1, \dots, i_k \in I, d_1, \dots, d_k \in \mathbb{Z}} \mathbb{Q}(q) \cdot \left[i_1^{(d_1)} \dots i_k^{(d_k)} \right]$$

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- and make it into an algebra via the following shuffle product:

$$\left[i_1^{(d_1)} \dots i_k^{(d_k)} \right] * \left[j_1^{(e_1)} \dots j_l^{(e_l)} \right] =$$

$$\sum_{\substack{\{1, \dots, k+l\} = A \sqcup B \\ |A| = k, |B| = l}} \sum_{\substack{\pi_1 + \dots + \pi_{k+l} = 0 \\ \pi_1, \dots, \pi_{k+l} \in \mathbb{Z}}} \underbrace{\text{coefficient}}_{\in \mathbb{Q}(q)} \cdot \left[s_1^{(t_1 + \pi_1)} \dots s_{k+l}^{(t_{k+l} + \pi_{k+l})} \right]$$

where if $A = \{a_1 < \dots < a_k\}$ and $B = \{b_1 < \dots < b_l\}$, we write:

$$s_c = \begin{cases} i_\bullet & \text{if } c = a_\bullet \\ j_\bullet & \text{if } c = b_\bullet \end{cases}, \qquad t_c = \begin{cases} d_\bullet & \text{if } c = a_\bullet \\ e_\bullet & \text{if } c = b_\bullet \end{cases}$$

The coefficients have a reasonable, but rather lengthy definition.

Standard Lyndon loop words

The results on this slide are joint work with Tsymbaliuk

- The algebra $\widehat{\mathcal{F}}$ is designed so that there is a homomorphism:

$$\widehat{\Phi} : U_q(L\mathfrak{n}^+) \hookrightarrow \widehat{\mathcal{F}}, \qquad e_{i,d} \mapsto \left[i^{(d)} \right], \quad \forall i \in I, d \in \mathbb{Z}$$

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- *Loop words* $\left[i_1^{(d_1)} \dots i_k^{(d_k)}\right]$ can be ordered lexicographically by:

$$i^{(d)} < j^{(e)} \quad \text{if} \quad \Big(d > e\Big) \text{ or } \Big(d = e \text{ and } i < j\Big)$$

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- Moreover, Im $\widehat{\Phi}$ consists of linear combinations of loop words, the largest words of which are concatenations of $\{\ell(\beta, d)\}_{\beta \in \Delta^+, d \in \mathbb{Z}}$.

On the bijection ℓ

- The bijection ℓ satisfies the property:

$$\ell(\beta, d) = \left[i_1^{(d_1)} \dots i_k^{(d_k)}\right] \quad \Rightarrow \quad \ell(\beta, d+\text{ht } \beta) = \left[i_1^{(d_1+1)} \dots i_k^{(d_k+1)}\right]$$

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- For example, in type A_n we have for all $d \in \{1, \dots, j - i + 1\}$:

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- **Theorem (N-T)**: The order on $\Delta^+ \times \mathbb{Z}$ induced by the bijection ℓ and lexicographic order on words is convex. This allows us to define root vectors $e_{(\beta,d)} \in U_q(L\mathfrak{n}^+)$ for all $\beta \in \Delta^+$ and $d \in \mathbb{Z}$, using the Beck-Damiani affine version of Lusztig's root vectors $e_\beta \in U_q(\mathfrak{n}^+)$.

Application: the Feigin-Odesskii shuffle algebra 1

- Our definition of the loop shuffle algebra $\widehat{\mathcal{F}}$ allows us to connect it with another shuffle algebra incarnation of $U_q(L\mathfrak{n}^+)$, this one due to Enriquez (inspired by the elliptic algebras of Feigin-Odesskii):

$$\mathcal{A}^+ \subset \bigoplus_{(k_i)_{i\in I}\in\mathbb{N}^I} \mathbb{Q}(q)(\dots, z_{i1}, \dots, z_{ik_i}, \dots)^{\text{symmetric in } z_{i1},\dots,z_{ik_i}, \forall i\in I}$$

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- consisting of rational functions of the form:

$$R(\dots, z_{i1}, \dots, z_{ik_i}, \dots) = \frac{r(\dots, z_{i1}, \dots, z_{ik_i}, \dots)}{\prod_{\{i \neq i'\} \subset I}^{\text{unordered}} \prod_{1 \leq a \leq k_i}^{1 \leq a' \leq k_{i'}} (z_{ia} - z_{i'a'})}$$

for r a Laurent polynomial, symmetric in $z_{i1}, \dots, z_{ik_i}$ $\forall i$, such that:

$$r(\dots, z_{ia}, \dots)\Big|_{(z_{i1}, z_{i2}, \dots, z_{i,1-a_{ij}}) \mapsto (w, wq_i^{-2}, \dots, wq_i^{2a_{ij}}),\ z_{j1} \mapsto wq_i^{a_{ij}}} = 0$$

for all $i \neq j$. The above vanishing of r is called a *wheel condition*.

Application: the Feigin-Odesskii shuffle algebra 2

- Let $\zeta_{ij}(x) = \frac{x - q^{-(\alpha_i, \alpha_j)}}{x-1}$. The multiplication on $\mathcal{A}^+$ is given by:

  $F(\dots, z_{i1}, \dots, z_{ik_i}, \dots) * G(\dots, z_{i1}, \dots, z_{il_i}, \dots) =$ symmetrization of

  $$F(\dots, z_{i1}, \dots, z_{ik_i}, \dots) G(\dots, z_{i,k_i+1}, \dots, z_{i,k_i+l_i}, \dots) \prod_{a \leq k_i, b > k_j}^{i,j \in I} \zeta_{ij}\left(\frac{z_{ia}}{z_{jb}}\right)$$

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- Varagnolo-Vasserot recently proved a result that implies the map Υ is injective in all finite types. So what about surjectivity?

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- As a technical step, we relate the shuffle algebras $\widehat{\mathcal{F}}$ and $\mathcal{A}^+$. To this end, we show that there exists an algebra homomorphism:

$$\iota : \mathcal{A}^+ \hookrightarrow \widehat{\mathcal{F}}$$

sending a rational function $R \in \mathcal{A}^+$ to:

$$\sum_{\substack{i_1,\dots,i_k \in I \\ d_1,\dots,d_k \in \mathbb{Z}}} \left[i_1^{(d_1)} \dots i_k^{(d_k)} \right] \int_{|z_1| \ll \dots \ll |z_k|} \frac{R(z_1,\dots,z_k) z_1^{-d_1} \dots z_k^{-d_k}}{\prod_{1 \leq a < b \leq k} \zeta_{i_a i_b}(z_a/z_b)} \prod_{a=1}^{k} \frac{dz_a}{2\pi i z_a}$$

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- Moreover, the following compositions are equal:

$$U_q(L\mathfrak{n}^+) \xrightarrow{\Upsilon} \mathcal{A}^+ \xrightarrow{\iota} \widehat{\mathcal{F}}, \qquad \widehat{\Phi} : U_q(L\mathfrak{n}^+) \to \widehat{\mathcal{F}}$$

which connects the two shuffle algebra realizations of $U_q(L\mathfrak{n}^+)$.