Lyndon words and quantum loop groups

Andrei Neguț (joint work with Alexander Tsymbaliuk)

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Quantum groups

- Let \( \mathfrak{g} \) be a finite-dimensional simple Lie algebra over \( \mathbb{C} \), associated to a Cartan matrix \( (a_{ij})_{i,j \in I} \), and root system \( \Delta^+ \sqcup \Delta^- \).
Quantum groups

- Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$, associated to a Cartan matrix $\left(a_{ij}\right)_{i,j \in I}$, and root system $\Delta^+ \sqcup \Delta^-$. 

- The Drinfeld-Jimbo quantum group associated to $\mathfrak{g}$ is given by:

$$U_q(\mathfrak{g}) = \mathbb{Q}(q) \langle e_i, f_i, \varphi_i \rangle_{i \in I}$$

modulo certain relations that we will not recall.
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modulo certain relations that we will not recall.

- The subalgebra $U_q(\mathfrak{n}^+) \subset U_q(\mathfrak{g})$ generated by the $e_i$'s is:

$$U_q(\mathfrak{n}^+) = \mathbb{Q}(q) \langle e_i \rangle_{i \in I}$$

modulo the relation

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} q_i e_i^k e_j e_i^{1-a_{ij}-k} = 0, \quad \forall i \neq j.$$
The $q$-shuffle algebra

- An important viewpoint on $U_q(n^+)$ is given by comparing it to the $q$-shuffle algebra defined by Green, Rosso, and Schauenburg:

$$
\mathcal{F} = \bigoplus_{k \in \mathbb{N}, i_1, \ldots, i_k \in \mathcal{I}} \mathbb{Q}(q) \cdot [i_1 \ldots i_k]
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- endowed with the following associative shuffle product:

$$[i_1 \ldots i_k] * [j_1 \ldots j_l] = \sum_{\{1, \ldots, k+l\} = A \cup B, |A| = k, |B| = l} q^{\lambda_{A,B}} \cdot [s_1 \ldots s_{k+l}]$$

where if $A = \{a_1 < \cdots < a_k\}$ and $B = \{b_1 < \cdots < b_l\}$, we write:

$$s_c = \begin{cases} i & \text{if } c = a \\ j & \text{if } c = b \end{cases}$$

and

$$\lambda_{A,B} = \sum_{A \ni a > b \in B} (\alpha_{s_a}, \alpha_{s_b})$$
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\[
\lambda_{A,B} = \sum_{A \ni a > b \in B} (\alpha_{s_a}, \alpha_{s_b})
\]

- This definition is designed so that there is an injective algebra homomorphism \( \Phi : U_q(n^+) \hookrightarrow \mathcal{F} \) given by \( e_i \mapsto [i] \), for all \( i \in I \).
Standard Lyndon words

- By work of Lusztig, there exists a PBW basis:

\[ U_q(n^+) = \bigoplus_{\beta_1 \geq \cdots \geq \beta_n \in \Delta^+} \mathbb{Q}(q) \cdot e_{\beta_1} \cdots e_{\beta_n} \]

where \( e_\beta \in U_q(n^+) \) deform the root vectors of \( n^+ \), for all \( \beta \in \Delta^+ \).
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- Above, \( \geq \) is any convex order on \( \Delta^+ \), but there is a particularly interesting choice. Lalonde-Ram showed that there is a bijection:

\[
\ell : \Delta^+ \sim \rightarrow \{ \text{standard Lyndon words} \}
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where a word \([i_1 \ldots i_k] \in I^k\) is called Lyndon if it is lex smaller than all its suffixes. Thus, lexicographic order induces an order on \( \Delta^+ \).
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• Leclerc showed that \( \Phi(e_{\beta}) \) has the minimal largest word among all degree \( \beta \) elements of \( \text{Im } \Phi \), and this largest word is precisely \( \ell(\beta) \).
Example: words in type $A_n$

- Consider the Dynkin diagram:

```
  1  2  ...  n - 1  n
  O---O---...---O---O
```

- The positive roots are $\alpha_{ij} = \alpha_i + \cdots + \alpha_j$ for all $1 \leq i \leq j \leq n$.

- The bijection $\ell$ is given by: $\ell(\alpha_{ij}) = [i \ldots j]$ (somewhat predictably).

- Now suppose you wanted an affine version of all of this business: $U_q(g) \to U_q(\hat{g})$.

- The shuffle algebra still makes sense (using letters in $\hat{I} = I \sqcup \{0\}$ instead of in $I$) but the Lalonde-Ram bijection breaks down because of the imaginary roots. So does Leclerc's description of $\Phi(e^{\beta})$.

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Lyndon words and quantum loop groups
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```
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  ₁  ₂  ...  n⁻¹  n
```

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Lyndon words and quantum loop groups
So we need a new viewpoint on $U_q(\hat{g})$. Fortunately, we have an isomorphism (proposed by Drinfeld and proved by Beck, Damiani):

$$U_q(\hat{g}) \cong U_q(Lg)$$
So we need a new viewpoint on $U_q(\hat{\mathfrak{g}})$. Fortunately, we have an isomorphism (proposed by Drinfeld and proved by Beck, Damiani):

$$U_q(\hat{\mathfrak{g}}) \cong U_q(L\mathfrak{g})$$

where the quantum loop group $U_q(L\mathfrak{g})$ has generators:

$$\{e_i,d, f_i,d, \varphi_{i,d'} \mid i \in I, d \in \mathbb{Z}, d' \geq 0\}$$
Quantum loop/affine groups

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- where the quantum loop group $U_q(Lg)$ has generators:

\[
\{ e_i, d, f_i, d, \varphi_i^{\pm} \}_{i \in I, d \in \mathbb{Z}, d' \geq 0}
\]

- However, the subalgebra $U_q(Ln^+) \subset U_q(Lg)$ generated by $\{ e_i, d \}$ does not match the subalgebra $U_q(\hat{n}^+) \subset U_q(\hat{g})$ under the isomorphism in the box. The two subalgebras are “orthogonal”.

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- **Our goal:** to do for $U_q(Ln^+)$ what was done for $U_q(n^+)$: define a shuffle algebra model, and describe PBW bases via Lyndon words.
The loop $q$-shuffle algebra

- Instead of using $i \in I$ as letters, let us use the symbols $i^{(d)}$ as letters, for any $i \in I$ and $d \in \mathbb{Z}$. Consider the vector space:

$$\hat{\mathcal{F}} = \bigoplus_{k \in \mathbb{N}, i_1, \ldots, i_k \in I, d_1, \ldots, d_k \in \mathbb{Z}} \mathbb{Q}(q) \cdot \left[ i_1^{(d_1)} \ldots i_k^{(d_k)} \right]$$

The coefficients have a reasonable, but rather lengthy definition.

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Lyndon words and quantum loop groups
The loop \( q \)-shuffle algebra

- Instead of using \( i \in l \) as letters, let us use the symbols \( i^{(d)} \) as letters, for any \( i \in l \) and \( d \in \mathbb{Z} \). Consider the vector space:

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\hat{\mathcal{F}} = \bigoplus_{k \in \mathbb{N}, i_1, \ldots, i_k \in l, d_1, \ldots, d_k \in \mathbb{Z}} \mathbb{Q}(q) \cdot \left[ i_1^{(d_1)} \cdots i_k^{(d_k)} \right]
\]

- and make it into an algebra via the following shuffle product:

\[
\left[ i_1^{(d_1)} \cdots i_k^{(d_k)} \right] \ast \left[ j_1^{(e_1)} \cdots j_l^{(e_l)} \right] = \sum_{\text{coefficient}} \cdot \left[ s_1^{(t_1+\pi_1)} \cdots s_{k+l}^{(t_{k+l}+\pi_{k+l})} \right]
\]

where if \( A = \{ a_1 < \ldots < a_k \} \) and \( B = \{ b_1 < \ldots < b_l \} \), we write:

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s_c = \begin{cases} 
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\end{cases}, \quad 
t_c = \begin{cases} 
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The coefficients have a reasonable, but rather lengthy definition.
Standard Lyndon loop words

The results on this slide are joint work with Tsymbaliuk

- The algebra $\hat{F}$ is designed so that there is a homomorphism:

$$\hat{\Phi} : U_q(Ln^+) \hookrightarrow \hat{F}, \quad e_{i,d} \mapsto [i^{(d)}], \quad \forall i \in I, d \in \mathbb{Z}$$
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• Loop words $[i_1^{(d_1)} \ldots i_k^{(d_k)}]$ can be ordered lexicographically by:

$$i^{(d)} < j^{(e)} \quad \text{if} \quad (d > e) \text{ or } (d = e \text{ and } i < j)$$
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$$\hat{\Phi} : U_q(L\mathfrak{n}^+) \hookrightarrow \hat{F}, \quad e_i, d \mapsto [i^{(d)}], \quad \forall i \in I, d \in \mathbb{Z}$$

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- This yields a notion of Lyndon loop words, and we have a bijection:

$$\ell : \Delta^+ \times \mathbb{Z} \sim \rightarrow \left\{ \text{standard Lyndon loop words} \right\}$$
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- This yields a notion of Lyndon loop words, and we have a bijection:
  $$\ell : \Delta^+ \times \mathbb{Z} \sim \rightarrow \{\text{standard Lyndon loop words}\}$$

- Moreover, $\text{Im} \hat{\Phi}$ consists of linear combinations of loop words, the largest words of which are concatenations of $\{\ell(\beta, d)\}_{\beta \in \Delta^+, d \in \mathbb{Z}}$. 

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On the bijection $\ell$

- The bijection $\ell$ satisfies the property:

$$\ell(\beta, d) = \left[i_1^{(d_1)} \ldots i_k^{(d_k)}\right] \Rightarrow \ell(\beta, d + \text{ht } \beta) = \left[i_1^{(d_1+1)} \ldots i_k^{(d_k+1)}\right]$$

so to prescribe $\ell$, it suffices to give $\ell(\beta, d)$ for $d \in \{1, \ldots, \text{ht } \beta\}$.
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- Moreover, $\ell(\beta, d)$ only has letters $i^{(*)}$ with $* \in \left\{ \left\lfloor \frac{d}{\text{ht} \beta} \right\rfloor, \left\lceil \frac{d}{\text{ht} \beta} \right\rceil \right\}$. 

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- For example, in type $A_n$ we have for all $d \in \{1, \ldots, j - i + 1\}$:

$$\ell(\alpha_{ij}, d) = \left[ (j - d + 1)^{(1)}(j - d)^{(0)} \ldots i^{(0)}(j - d + 2)^{(1)} \ldots j^{(1)} \right]$$
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- **Theorem (N-T):** The order on $\Delta^+ \times \mathbb{Z}$ induced by the bijection $\ell$ and lexicographic order on words is convex. This allows us to define root vectors $e_{(\beta, d)} \in U_q(Ln^+)$ for all $\beta \in \Delta^+$ and $d \in \mathbb{Z}$, using the Beck-Damiani affine version of Lusztig’s root vectors $e_\beta \in U_q(n^+)$. 

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Our definition of the loop shuffle algebra $\widehat{\mathcal{F}}$ allows us to connect it with another shuffle algebra incarnation of $U_q(L\mathfrak{n}^\perp)$, this one due to Enriquez (inspired by the elliptic algebras of Feigin-Odesskii):

$$\mathcal{A}^+ \subset \bigoplus_{(k_i)_{i \in I} \in \mathbb{N}^I} \mathbb{Q}(q)(\ldots, z_{i1}, \ldots, z_{ik_i}, \ldots)$$

symmetric in $z_{i1}, \ldots, z_{ik_i}, \forall i \in I$
Our definition of the loop shuffle algebra $\hat{F}$ allows us to connect it with another shuffle algebra incarnation of $U_q(Ln^+)$, this one due to Enriquez (inspired by the elliptic algebras of Feigin-Odesskii):

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consisting of rational functions of the form:

$$R(\ldots, z_{i1}, \ldots, z_{ik_i}, \ldots) = \frac{r(\ldots, z_{i1}, \ldots, z_{ik_i}, \ldots)}{\prod_{\text{unordered}} \prod_{1 \leq a' \leq k_i} (z_{ia} - z_{i'a'})}$$

for $r$ a Laurent polynomial, symmetric in $z_{i1}, \ldots, z_{ik_i}, \forall i$, such that:

$$r(\ldots, z_{ia}, \ldots)\big|_{(z_{i1}, z_{i2}, \ldots, z_{i, 1-a_{ij}}) \mapsto (w, wq_{i}^{-2}, \ldots, wq_{i}^{2a_{ij}}), z_{j1} \mapsto wq_{i}^{a_{ij}}} = 0$$

for all $i \neq j$. The above vanishing of $r$ is called a \textit{wheel condition}. 

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Let $\zeta_{ij}(x) = \frac{x^{-q^{-(\alpha_i, \alpha_j)}}}{x-1}$. The multiplication on $\mathcal{A}^+$ is given by:

$$F(\ldots, z_{i1}, \ldots, z_{ik_i}, \ldots) \ast G(\ldots, z_{i1}, \ldots, z_{il_i}, \ldots) = \text{symmetrization of}$$

$$F(\ldots, z_{i1}, \ldots, z_{ik_i}, \ldots) G(\ldots, z_{i,k_i+1}, \ldots, z_{i,k_i+l_i}, \ldots) \prod_{a \leq k_i, b > k_j} \zeta_{ij} \left( \frac{Z_{ia}}{Z_{jb}} \right)$$
Let $\zeta_{ij}(x) = \frac{x - q^{-(\alpha_i, \alpha_j)}}{x - 1}$. The multiplication on $A^+$ is given by:

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This is designed so that there is an algebra homomorphism:

$$\Upsilon: U_q(Ln^+) \longrightarrow A^+, \quad e_i, d \mapsto z_{i1}^d, \quad \forall i \in I, d \in \mathbb{Z}$$
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$$F(\ldots, z_{i1}, \ldots, z_{ik}, \ldots) G(\ldots, z_{i,k+1}, \ldots, z_{i,k+l}, \ldots) \prod_{a \leq k, b > k_j}^{i,j \in I} \zeta_{ij} \left( \frac{Z_{ia}}{Z_{jb}} \right)$$

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I showed that the map $\Upsilon$ is an isomorphism in affine type $A$, although those methods do not readily generalize to other types.
Application: the Feigin-Odesskii shuffle algebra 2

- Let $\zeta_{ij}(x) = \frac{x - q^{-(\alpha_i, \alpha_j)}}{x - 1}$. The multiplication on $\mathcal{A}^+$ is given by:

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- I showed that the map $\Upsilon$ is an isomorphism in affine type $A$, although those methods do not readily generalize to other types.

- Varagnolo-Vasserott recently proved a result that implies the map $\Upsilon$ is injective in all finite types. So what about surjectivity?
• **Theorem (N-T)** The map $\Upsilon$ is surjective, hence an isomorphism.
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• As a technical step, we relate the shuffle algebras $\hat{F}$ and $A^+$. To this end, we show that there exists an algebra homomorphism:

$$\iota : A^+ \hookrightarrow \hat{F}$$

sending a rational function $R \in A^+$ to:

$$\sum_{i_1, \ldots, i_k \in I} \left[ \prod_{i=1}^{k} i^{(d_i)} \right] \int_{|z_1| \ll \cdots \ll |z_k|} \frac{R(z_1, \ldots, z_k)z_1^{-d_1} \cdots z_k^{-d_k}}{\prod_{1 \leq a < b \leq k} \zeta_{i_ai_b}(z_a/z_b)} \prod_{a=1}^{k} \frac{dz_a}{2\pi iz_a}$$
Theorem (N-T) The map $\Upsilon$ is surjective, hence an isomorphism.

As a technical step, we relate the shuffle algebras $\hat{F}$ and $A^+$. To this end, we show that there exists an algebra homomorphism:

$$\iota : A^+ \to \hat{F}$$

sending a rational function $R \in A^+$ to:

$$\sum_{i_1, \ldots, i_k \in \Gamma} \binom{d_1}{i_1} \cdots \binom{d_k}{i_k} \int_{|z_1| \ll \cdots \ll |z_k|} \frac{R(z_1, \ldots, z_k)z_1^{-d_1} \cdots z_k^{-d_k}}{\prod_{1 \leq a < b \leq k} \zeta_{i_ia_ib}(z_a/z_b)} \prod_{a=1}^{k} \frac{dz_a}{2\pi i z_a}$$

Moreover, the following compositions are equal:

$$U_q(Ln^+) \xrightarrow{\Upsilon} A^+ \xrightarrow{\iota} \hat{F}$$

which connects the two shuffle algebra realizations of $U_q(Ln^+)$. 

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Lyndon words and quantum loop groups